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ON COMMON FIXED POINT THEOREMS IN S-METRIC SPACES USING C-CLASS FUNCTIONS

Gurucharan Singh Saluja

Abstract. In this paper, we prove some common fixed point theorems on S-metric spaces for two pairs of weakly compatible mappings by using C-class functions and give some consequences as corollaries of the established results. We also give an example in support of the result. The results presented in this paper generalize, extend and enrich various results in the existing literature.

1. Introduction

Fixed point theory is one of the most important topic in the development of nonlinear analysis. As it is well known, one of the most useful theorem in nonlinear analysis is the Banach contraction principle [**6**]. In 1922, Banach proved the celebrated fixed point theorem, which assures the existence and uniqueness of a fixed point under certain conditions.

There are many extensions of the famous Banach contraction principle in the literature, which states that every self mapping T defined on a complete metric space (X, d) satisfying the condition:

$$d(T(x), T(y)) \leqslant c\, d(x, y), \tag{1.1}$$

for all $x, y \in X$, where $c \in (0, 1)$ is a constant, has a unique fixed point and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\geqslant 1}$ is convergent to the fixed point.

Generalizing the Banach contraction principle, Jungck [**10**] initiated the study of common fixed point for a pair of commuting mappings satisfying contractive type conditions. In 1982, Sessa [**40**] introduced a weaker concept of commutativity, which is generally known as weak commutativity and proved some interesting

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results on the existence of common fixed points for a pair of self maps. He also showed that weak commuting mappings are commuting but the converse need not to be true. Later, Jungck [**11**] generalized the concept of weak commutativity by introducing the notion of compatible mappings which is more general than weakly commuting mappings and showed that weak commuting maps are compatible but converse need not be true. In 1996, Jungck [**12**] generalized the concept of compatibility by introducing weakly compatible mappings.

Mustafa and Sims [**16**] introduced a new notion of generalized metric space called G-metric space and gave a modification to the Banach contraction principle. After then, several authors studied various fixed point and common fixed point problems for many classes of contractive mappings in generalized metric spaces (see, [**1, 2, 7, 17, 18, 19, 20, 41**] and many others).

Sedghi et al. [**38**] introduced the concept of S-metric spaces which generalized G-metric spaces and D^*-metric spaces. In [**38**] the authors proved some properties of S-metric spaces. Also, they obtained some fixed point theorems in the setting of S-metric spaces for a self-map.

Gupta [**8**] in 2013, introduced the notion of cyclic contraction in S-metric spaces and proved some fixed theorems which are proper generalizations of the results of Sedghi et al. [**38**].

Recently, a large number of authors have published many papers on S-metric spaces in different ways (see, e.g.,[**14, 15, 21, 22, 23, 24, 25, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 42, 43**] and many others).

In 2014, the notion of C-class function was introduced by Ansari [**5**] that is pivotal result in fixed point theory.

In this paper, we prove some common fixed point theorems on S-metric spaces for two pairs of weakly compatible mappings by using C-class functions and give some corollaries of our results. Our results generalize, extend and enrich several results in the existing literature.

2. Preliminaries

In this section, we recall some definitions and lemmas that will be used to prove our main results.

Definition 2.1. ([**38**]) Let E be a nonempty set and let $S\colon E^3 \to [0,\infty)$ be a function satisfying the following conditions for all u, v, w, $t \in E$:

$(S1)$ $S(u,v,w) = 0$ if and only if $u = v = w$;

$(S2)$ $S(u,v,w) \leqslant S(u,u,t) + S(v,v,t) + S(w,w,t)$.

Then the function S is called an S-metric on E and the pair (E,S) is called an S-metric space or simply SMS.

Example 2.1. ([**38**]) Let $E = \mathbb{R}^n$ and $\|\cdot\|$ a norm on E, then $S(u,v,w) = \|v + w - 2u\| + \|v - w\|$ is an S-metric on E.

Example 2.2. ([**38**]) Let E be a nonempty set and d be an ordinary metric on E. Then $S(u,v,w) = d(u,w) + d(v,w)$ for all $u,v,w \in E$ is an S-metric on E.

EXAMPLE 2.3. ([**39**]) Let $E = \mathbb{R}$ be the real line. Then $S(u, v, w) = |u - w| + |v - w|$ for all $u, v, w \in \mathbb{R}$ is an S-metric on E. This S-metric on E is called the usual S-metric on E.

DEFINITION 2.2. Let (E, S) be an S-metric space. For $r > 0$ and $u \in E$ we define the open ball $B_S(u, r)$ and closed ball $B_S[u, r]$ with center u and radius r as follows, respectively:

$$B_S(u, r) = \{v \in E : S(v, v, u) < r\},$$

$$B_S[u, r] = \{v \in E : S(v, v, u) \leqslant r\}.$$

EXAMPLE 2.4. ([**39**]) Let $E = \mathbb{R}$. Denote $S(u, v, w) = |v + w - 2u| + |v - w|$ for all $u, v, w \in \mathbb{R}$. Then

$$\begin{aligned} B_S(1, 2) &= \{v \in \mathbb{R} : S(v, v, 1) < 2\} = \{v \in \mathbb{R} : |v - 1| < 1\} \\ &= \{v \in \mathbb{R} : 0 < v < 2\} = (0, 2), \end{aligned}$$

and

$$\begin{aligned} B_S[2, 4] &= \{v \in \mathbb{R} : S(v, v, 2) \leqslant 4\} = \{v \in \mathbb{R} : |v - 2| \leqslant 2\} \\ &= \{v \in \mathbb{R} : 0 \leqslant v \leqslant 4\} = [0, 4]. \end{aligned}$$

DEFINITION 2.3. ([**38**], [**39**]) Let (E, S) be an S-metric space and $A \subset E$.

• The subset A is said to be an open subset of E, if for every $u \in A$ there exists $r > 0$ such that $B_S(u, r) \subset A$.

• A sequence $\{u_n\}$ in E converges to $u \in E$ if $S(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $S(u_n, u_n, u) < \varepsilon$. We denote this by $\lim_{n\to\infty} u_n = u$ or $u_n \to u$ as $n \to \infty$.

• A sequence $\{u_n\}$ in E is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$ we have $S(u_n, u_n, u_m) < \varepsilon$.

• The S-metric space (E, S) is called complete if every Cauchy sequence in E is convergent.

• Let τ be the set of all $A \subset E$ having the property that for every $u \in A$, A contains an open ball centered in u. Then τ is a topology on E (induced by the S-metric space).

• A nonempty subset A of E is S-closed if closure of A coincides with A.

DEFINITION 2.4. ([**38**]) Let (E, S) be an S-metric space. A mapping $\mathcal{R}\colon E \to E$ is said to be a contraction if there exists a constant $0 \leqslant q < 1$ such that

$$S(\mathcal{R}u, \mathcal{R}v, \mathcal{R}w) \leqslant q\, S(u, v, w), \tag{2.1}$$

for all $u, v, w \in E$.

REMARK 2.1. ([**38**]) If the S-metric space (E, S) is complete and $\mathcal{R}\colon E \to E$ is a contraction mapping, then $\mathcal{R}$ has a unique fixed point in E.

Definition 2.5. ([**38**]) Let (E, S) and (E', S') be two S-metric spaces. A function $g\colon E \to E'$ is said to be continuous at a point $y_0 \in E$ if for every sequence $\{y_n\}$ in E with $S(y_n, y_n, y_0) \to 0$, $S'(g(y_n), g(y_n), g(y_0)) \to 0$ as $n \to \infty$. We say that g is continuous on E if g is continuous at every point $y_0 \in E$.

Definition 2.6. ([**5**]) A mapping $F\colon [0, \infty) \times [0, \infty) \to R$ is called a C-class function if it is continuous and satisfies the following axioms:
(i) $F(s, t) \leqslant s$,
(ii) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, \infty)$.

Note that for some F, we have that $F(0, 0) = 0$. The letter $\mathcal{C}$ denotes the set of all C-class functions. The following example shows that $\mathcal{C}$ is nonempty.

Example 2.5. ([**5**]) Each of the functions $F\colon [0, \infty) \times [0, \infty) \to R$ defined below are elements of $\mathcal{C}$.
(i) $F(s, t) = s - t$,
(ii) $F(s, t) = m\, s$, $0 < m < 1$,
(iii) $F(s, t) = \frac{s}{(1+t)^r}$, $r \in (0, \infty)$,
(iv) $F(s, t) = \frac{log(t+a^s)}{1+t}$, $a > 1$,
(v) $F(s, t) = \frac{ln(1+a^s)}{2}$, $a > e$,
(vi) $F(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0, \infty)$,
(vii) $F(s, t) = s\, log_{t+a}\, a$, $a > 1$,
$(viii)$ $F(s, t) = s - \big(\frac{1+s}{2+s}\big)\big(\frac{t}{1+t}\big)$,
(ix) $F(s, t) = s\beta(s)$, where $\beta\colon [0, \infty) \to [0, \infty)$ and is continuous,
(x) $F(s, t) = s - \big(\frac{t}{k+t}\big)$, $F(s, t) = s \Rightarrow t = 0$,
(xi) $F(s, t) = \frac{s}{(1+s)^r}$, $r \in (0, \infty)$.

Definition 2.7. ([**5**]) A function $\varphi\colon [0, \infty) \to [0, \infty)$ is said to be an ultra altering distance function, if it is continuous, non-decreasing such that $\varphi(t) > 0$ for $t > 0$.

Remark 2.2. ([**5**]) We denote by Φ_u the class of all ultra altering distance functions.

Definition 2.8. ([**13**]) Consider the class of functions $\Psi = \{\psi | \psi\colon [0, \infty) \to [0, \infty)\}$, which satisfy the following assertions:
$(\mathbf{\Psi_1})$ ψ is nondecreasing and continuous;
$(\mathbf{\Psi_2})$ $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 for all $t > 0$;
$(\mathbf{\Psi_3})$ $\sum_{n=1}^{\infty} \psi^n(t)$ is convergent for all $t > 0$.

Remark 2.3. ([**13**]) If $\psi^n(t)) \to 0$ as $n \to \infty$ for all $t > 0$, then we have $\psi(t) < t$ for all $t > 0$.

Definition 2.9. Let E be a non-empty set and let $P, Q\colon E \to E$ be two self mappings of E. Then a point $u \in E$ is called a
$(\mathbf{\Lambda_1})$ fixed point of operator P if $P(u) = u$;
$(\mathbf{\Lambda_2})$ common fixed point of P and Q if $P(u) = Q(u) = u$.

DEFINITION 2.10. ([**2**]) Let F and G be single valued self-mappings on a set E. If $z = Gv = Hv$ for some $v \in E$, then v is called a coincidence point point of G and H, and z is called a point of coincidence of G and H.

DEFINITION 2.11. ([**11**]) Let R and T be single valued self-mappings on a set E. Mappings R and T are said to be commuting if $RTq = TRq$ for all $q \in E$.

DEFINITION 2.12. ([**12**]) Let L and M be single valued self-mappings on a set E. Mappings L and M are said to be weakly compatible if they commute at their coincidence points, i.e., if $Lu = Mu$ for some $u \in E$ implies $LMu = MLu$.

LEMMA 2.1. *([**38**], Lemma 2.5) In an S-metric space, we have $S(u,u,v) = S(v,v,u)$ for all $u,v \in E$.*

LEMMA 2.2. *([**38**], Lemma 2.12) Let (E,S) be an S-metric space. If $u_n \to u$ and $v_n \to v$ as $n \to \infty$ then $S(u_n,u_n,v_n) \to S(u,u,v)$ as $n \to \infty$.*

LEMMA 2.3. *([**8**], Lemma 8) Let (E,S) be an S-metric space and A be a non-empty subset of E. Then A is S-closed if and only if for any sequence $\{u_n\}$ in A such that $u_n \to u$ as $n \to \infty$, then $u \in A$.*

LEMMA 2.4. *([**38**]) Let (E,S) be an S-metric space. If $r > 0$ and $u \in E$, then the ball $B_S(u,r)$ is an open subset of E.*

LEMMA 2.5. *([**39**]) The limit of a convergent sequence in a S-metric space (E,S) is unique.*

LEMMA 2.6. *([**38**]) In a S-metric space (E,S), any convergent sequence is Cauchy.*

3. Main results

In this section, we shall prove some common fixed point theorems in the setting of S-metric spaces for two pairs of weakly compatible mappings using C-class functions.

THEOREM 3.1. *Let (E,S) be a complete S-metric space and let $A,B,P,Q\colon E \to E$ be four mappings satisfying the following conditions:*

(i)

$$S(Au,Av,Bw) \leqslant F\Big(\psi(\Delta(u,v,w)),\varphi(\Delta(u,v,w))\Big), \tag{3.1}$$

where

$$\Delta(u,v,w) = \max\Big\{S(Pu,Pv,Qw), S(Bw,Bw,Av), S(Bw,Bw,Qw), \frac{S(Bw,Bw,Aw)S(Pu,Pu,Qw)}{[1+S(Bw,Bw,Pu)]}\Big\},$$

for all $u,v,w \in E$, $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$;

(ii) $A(E) \subseteq Q(E)$ and $B(E) \subseteq P(E)$;

(iii) the pairs (A,P) and (B,Q) are weakly compatible.

If one of the range $A(E)$ or $B(E)$ or $P(E)$ or $Q(E)$ is closed in E, then A, B, P and Q have a unique common fixed point in E.

Proof. Let $u_0 \in E$. Since $A(E) \subseteq Q(E)$, we can choose $u_1 \in E$ such that $v_0 = Au_0 = Qu_1$. Similarly, since $B(E) \subseteq P(E)$, there exists $u_2 \in E$ such that $v_1 = Bu_1 = Pu_2$. Continuing in this manner, we obtain a sequence $\{v_n\}_{n=0}^{\infty}$ in E such that

$$(*)\begin{cases} v_{2n} = Au_{2n} = Qu_{2n+1}, \\ v_{2n+1} = Bu_{2n+1} = Pu_{2n+2}. \end{cases}$$

We shall now show that $\{v_n\}_{n=0}^{\infty}$ is a Cauchy sequence in E.

Now, we consider the following cases.

Case I. If $v_n = v_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all positive integers, then $v_{n+1} = v_{n+2}$. For if, $v_{n+1} \neq v_{n+2}$, then, for $n = 2m$, where $m \in \mathbb{N}$ and using Lemma 2.1, we get

$$\begin{aligned}
&\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1}) \\
&= \max\Big\{S(Pu_{2m+2}, Pu_{2m+2}, Qu_{2m+1}), S(Bu_{2m+1}, Bu_{2m+1}, Au_{2m+2}), \\
&\qquad S(Bu_{2m+1}, Bu_{2m+1}, Qu_{2m+1}), \\
&\qquad \frac{S(Bu_{2m+1}, Bu_{2m+1}, Au_{2m+1})S(Pu_{2m+2}, Pu_{2m+2}, Qu_{2m+1})}{[1 + S(Bu_{2m+1}, Bu_{2m+1}, Pu_{2m+2})]}\Big\} \qquad (3.2) \\
&= \max\Big\{S(v_{2m+1}, v_{2m+1}, v_{2m}), S(v_{2m+1}, v_{2m+1}, v_{2m+2}), S(v_{2m+1}, v_{2m+1}, v_{2m}), \\
&\qquad \frac{S(v_{2m+1}, v_{2m+1}, v_{2m+1})S(v_{2m+1}, v_{2m+1}, v_{2m})}{[1 + S(v_{2m+1}, v_{2m+1}, v_{2m+1})]}\Big\} \\
&= \max\Big\{S(v_{2m+1}, v_{2m+1}, v_{2m}), S(v_{2m+2}, v_{2m+2}, v_{2m+1}), S(v_{2m+1}, v_{2m+1}, v_{2m}), 0\Big\} \\
&= \max\Big\{S(v_{2m+1}, v_{2m+1}, v_{2m}), S(v_{2m+2}, v_{2m+2}, v_{2m+1})\Big\}.
\end{aligned}$$

If $\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1}) = S(v_{2m+2}, v_{2m+2}, v_{2m+1})$, then from equation (3.1) and using the property of F, we obtain

$$\begin{aligned}
S(v_{n+2}, v_{n+2}, v_{n+1}) &= S(v_{2m+2}, v_{2m+2}, v_{2m+1}) \\
&= S(Au_{2m+2}, Au_{2m+2}, Bu_{2m+1}) \\
&\leqslant F\Big(\psi(\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1})), \varphi(\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1}))\Big) \\
&= F\Big(\psi(S(v_{2m+2}, v_{2m+2}, v_{2m+1})), \varphi(S(v_{2m+2}, v_{2m+2}, v_{2m+1}))\Big) \\
&= F\Big(\psi(S(v_{n+2}, v_{n+2}, v_{n+1})), \varphi(S(v_{n+2}, v_{n+2}, v_{n+1}))\Big) \\
&\leqslant \psi(S(v_{n+2}, v_{n+2}, v_{n+1})) \\
&< S(v_{n+2}, v_{n+2}, v_{n+1}), \qquad (3.3)
\end{aligned}$$

which is a contradiction. Hence we must have $v_{n+1} = v_{n+2}$ when n is even. Using a similar argument equality also holds when n is odd. Thus in any case, whenever $y_n = y_{n+1}$ holds for some n, $y_{n+1} = y_{n+2}$. By repeating this process inductively, one can obtain $y_n = y_{n+r}$ for all $r \geqslant 1$ and $\{v_n\}_{n=0}^{\infty}$ will be eventually a constant sequence and hence is Cauchy.

Case II. If $v_n \neq v_{n+1}$ for every $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers, then for $n = 2m + 1$, where $m \in \mathbb{N}$ and using Lemma 2.1, we have

$$\begin{aligned}
&\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1}) \\
&= \max\Big\{S(Pu_{2m+2}, Pu_{2m+2}, Qu_{2m+1}), S(Bu_{2m+1}, Bu_{2m+1}, Au_{2m+2}), \\
&\qquad S(Bu_{2m+1}, Bu_{2m+1}, Qu_{2m+1}), \\
&\qquad \frac{S(Bu_{2m+1}, Bu_{2m+1}, Au_{2m+1})S(Pu_{2m+2}, Pu_{2m+2}, Qu_{2m+1})}{[1 + S(Au_{2m+1}, Au_{2m+1}, Pu_{2m+2})]}\Big\} \\
&= \max\Big\{S(v_{2m+1}, v_{2m+1}, v_{2m}), S(v_{2m+1}, v_{2m+1}, v_{2m+2}), S(v_{2m+1}, v_{2m+1}, v_{2m}), \\
&\qquad \frac{S(v_{2m+1}, v_{2m+1}, v_{2m+1})S(v_{2m+1}, v_{2m+1}, v_{2m})}{[1 + S(v_{2m+1}, v_{2m+1}, v_{2m+1})]}\Big\} \\
&= \max\Big\{S(v_{2m+1}, v_{2m+1}, v_{2m}), S(v_{2m+2}, v_{2m+2}, v_{2m+1}), S(v_{2m+1}, v_{2m+1}, v_{2m}), 0\Big\}.
\end{aligned}$$

Now, if $\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1}) = S(v_{2m+2}, v_{2m+2}, v_{2m+1})$, then from equation (3.1) and using the property of F, we obtain

$$\begin{aligned}
S(v_{n+1}, v_{n+1}, v_n) &= S(v_{2m+2}, v_{2m+2}, v_{2m+1}) \\
&= S(Au_{2m+2}, Au_{2m+2}, Bu_{2m+1}) \\
&\leqslant F\Big(\psi(\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1})), \varphi(\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1}))\Big) \\
&= F\Big(\psi(S(v_{2m+2}, v_{2m+2}, v_{2m+1})), \varphi(S(v_{2m+2}, v_{2m+2}, v_{2m+1}))\Big) \\
&= F\Big(\psi(S(v_{n+1}, v_{n+1}, v_n)), \varphi(S(v_{n+1}, v_{n+1}, v_n))\Big) \\
&\leqslant \psi(S(v_{n+1}, v_{n+1}, v_n)) \\
&< S(v_{n+1}, v_{n+1}, v_n),
\end{aligned}$$

which is a contradiction.

Therefore, $S(v_{2m+2}, v_{2m+2}, v_{2m+1}) \leqslant S(v_{2m+1}, v_{2m+1}, v_{2m})$ and consequently

$$\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1}) = S(v_{2m+1}, v_{2m+1}, v_{2m}).$$

Using equation (3.1) and the property of F, we have

$$\begin{aligned} S(v_{n+1}, v_{n+1}, v_n) &= S(v_{2m+2}, v_{2m+2}, v_{2m+1}) \\ &= S(Au_{2m+2}, Au_{2m+2}, Bu_{2m+1}) \\ &\leqslant F\Big(\psi(\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1})), \varphi(\Delta(u_{2m+2}, u_{2m+2}, u_{2m+1}))\Big) \\ &= F\Big(\psi(S(v_{2m+1}, v_{2m+1}, v_{2m})), \varphi(S(v_{2m+1}, v_{2m+1}, v_{2m}))\Big) \\ &= F\Big(\psi(S(v_n, v_n, v_{n-1})), \varphi(S(v_n, v_n, v_{n-1}))\Big) \\ &\leqslant \psi(S(v_n, v_n, v_{n-1})) \\ &< S(v_n, v_n, v_{n-1}). \qquad (3.4) \end{aligned}$$

Using a similar arguments, one can obtain the same inequality when n is an even integer. Thus, we have $S(v_{n+1}, v_{n+1}, v_n) < S(v_n, v_n, v_{n-1})$ for all $n \in \mathbb{N}$, and the sequence $\{S(v_{n+1}, v_{n+1}, v_n)\}$ is decreasing and bounded below by zero. Hence, there exists $d \geqslant 0$ such that $S(v_{n+1}, v_{n+1}, v_n) \to d$ as $n \to \infty$.

Let $H_n = S(v_{n+1}, v_{n+1}, v_n)$. Now using (3.1) for $u = v_{n+1}$, $v = v_n$, $\Delta(v_n, v_n, v_{n-1}) = S(v_{n+1}, v_{n+1}, v_n)$ for every $n \in \mathbb{N}$ and using the property of F, we obtain

$$\begin{aligned} H_n &= S(v_{n+1}, v_{n+1}, v_n) = S(Av_n, Av_n, Bv_{n-1}) \\ &\leqslant F\Big(\psi(\Delta(v_n, v_n, v_{n-1})), \varphi(\Delta(v_n, v_n, v_{n-1}))\Big) \\ &\leqslant F\Big(\psi(S(v_{n+1}, v_{n+1}, v_n)), \varphi(S(v_{n+1}, v_{n+1}, v_n))\Big) \\ &\leqslant \psi(S(v_{n+1}, v_{n+1}, v_n)). \qquad (3.5) \end{aligned}$$

Since ψ is continuous, so taking limit as $n \to \infty$ in equation (3.5) and using the property of ψ, we obtain

$$d \leqslant \psi(d) < d,$$

which is a contradiction. Thus, $d = 0$.

Hence

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty} S(v_{n+1}, v_{n+1}, v_n) = 0. \qquad (3.6)$$

Now, we show that $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence in E. Because of (3.6) it is sufficient to show that $\{v_{2n}\}_{n=1}^{\infty}$ is a Cauchy sequence. If otherwise, then there exists $\varepsilon > 0$ for which we can find subsequences $\{v_{2m(k)}\}$ and $\{v_{2n(k)}\}$ of $\{v_{2n}\}$ and increasing sequences of integers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k)$ is smallest index for which,

$$n(k) > m(k) > k, \qquad (3.7)$$

$$S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)}) \geqslant \varepsilon. \qquad (3.8)$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (3.7). Then

$$S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)-1}) < \varepsilon. \qquad (3.9)$$

Now, using (3.8), ($S2$) and Lemma 2.1, we have

$$\begin{aligned} \varepsilon &\leqslant S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)}) \\ &= S(v_{2n(k)}, v_{2n(k)}, v_{2m(k)}) \\ &\leqslant 2S(v_{2n(k)}, v_{2n(k)}, v_{2n(k)-1}) \\ &\quad +S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)-1}) \\ &\leqslant \varepsilon + 2S(v_{2n(k)}, v_{2n(k)}, v_{2n(k)-1}). \text{ (by (3.7))} \end{aligned} \tag{3.10}$$

Letting $k \to \infty$ in equation (3.10) and using (3.6), we get

$$\lim_{k\to\infty} S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)}) = \varepsilon. \tag{3.11}$$

Again, using ($S2$) and Lemma 2.1, we have

$$\begin{aligned} S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)}) &\leqslant 2S(v_{2m(k)}, v_{2m(k)}, v_{2m(k)-1}) \\ &\quad +S(v_{2n(k)}, v_{2n(k)}, v_{2m(k)-1}) \\ &= 2S(v_{2m(k)}, v_{2m(k)}, v_{2m(k)-1}) \\ &\quad +S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)}). \end{aligned} \tag{3.12}$$

Also, by using ($S2$) and Lemma 2.1, we have

$$\begin{aligned} S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)}) &\leqslant 2S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2m(k)}) \\ &\quad +S(v_{2n(k)}, v_{2n(k)}, v_{2m(k)}) \\ &= 2S(v_{2m(k)}, v_{2m(k)}, v_{2m(k)-1}) \\ &\quad +S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)}). \end{aligned} \tag{3.13}$$

Letting $k \to \infty$ in equation (3.13) and using (3.6), (3.9) and (3.11), we get

$$\lim_{k\to\infty} S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)}) = \varepsilon. \tag{3.14}$$

Again, note that with the help of ($S2$) and Lemma 2.1, we have

$$\begin{aligned} S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)+1}) &\leqslant 2S(v_{2m(k)}, v_{2m(k)}, v_{2m(k)-1}) \\ &\quad +S(v_{2n(k)+1}, v_{2n(k)+1}, v_{2m(k)-1}) \\ &\leqslant 2S(v_{2m(k)}, v_{2m(k)}, v_{2m(k)-1}) \\ &\quad +2S(v_{2n(k)+1}, v_{2n(k)+1}, v_{2n(k)}) \\ &\quad +S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)}). \end{aligned} \tag{3.15}$$

Also, with the help of $(S2)$ and Lemma 2.1, we have

$$\begin{aligned} S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)}) &= S(v_{2n(k)}, v_{2n(k)}, v_{2m(k)-1}) \\ &\leqslant 2S(v_{2n(k)}, v_{2n(k)}, v_{2n(k)+1}) \\ &\quad +S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)+1}) \\ &= 2S(v_{2n(k)+1}, v_{2n(k)+1}, v_{2n(k)}) \\ &\quad +S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)+1}) \\ &\leqslant 2S(v_{2n(k)+1}, v_{2n(k)+1}, v_{2n(k)}) \\ &\quad +2S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2m(k)}) \\ &\quad +S(v_{2n(k)+1}, v_{2n(k)+1}, v_{2m(k)}) \\ &= 2S(v_{2n(k)+1}, v_{2n(k)+1}, v_{2n(k)}) \\ &\quad +2S(v_{2m(k)}, v_{2m(k)}, v_{2m(k)-1}) \\ &\quad +S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)+1}). \end{aligned} \tag{3.16}$$

Letting $k \to \infty$ in equation (3.16) and using (3.6), (3.11),(3.14) and (3.15), we get

$$\lim_{k\to\infty} S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)+1}) = \varepsilon. \tag{3.17}$$

Again, we notice that by using Lemma 2.1

$$\begin{aligned} S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)+1}) &\leqslant 2S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2m(k)}) \\ &\quad +S(v_{2n(k)+1}, v_{2n(k)+1}, v_{2m(k)}) \\ &= 2S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2m(k)}) \\ &\quad +S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)+1}), \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} S(v_{2m(k)}, v_{2m(k)}, v_{2n(k)+1}) &\leqslant 2S(v_{2m(k)}, v_{2m(k)}, v_{2m(k)-1}) \\ &\quad +S(v_{2n(k)+1}, v_{2n(k)+1}, v_{2m(k)-1}) \\ &= 2S(v_{2m(k)}, v_{2m(k)}, v_{2m(k)-1}) \\ &\quad +S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)+1}). \end{aligned} \tag{3.19}$$

Letting $k \to \infty$ in equation (3.18) and (3.19) and using (3.6) and (3.17), we get

$$\lim_{k\to\infty} S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)+1}) = \varepsilon. \tag{3.20}$$

Now consider inequality (3.1) and putting $u = v = u_{2m(k)-1}$, $w = u_{2n(k)}$, we obtain

$$\begin{aligned} S(Au_{2m(k)-1}, Au_{2m(k)-1}, Bu_{2n(k)}) &= S(v_{2m(k)-1}, v_{2m(k)-1}, v_{2n(k)}) \\ &\leqslant F\Big(\psi(\Delta(u_{2m(k)-1}, u_{2m(k)-1}, u_{2n(k)})), \\ &\qquad \varphi(\Delta(u_{2m(k)-1}, u_{2m(k)-1}, u_{2n(k)}))\Big), \end{aligned} \tag{3.21}$$

where

$$\Delta(u_{2m(k)-1}, u_{2m(k)-1}, u_{2n(k)})$$

$$= \max\Big\{S(Pu_{2m(k)-1}, Pu_{2m(k)-1}, Qu_{2n(k)}), S(Bu_{2n(k)}, Bu_{2n(k)}, Au_{2m(k)-1}), \\ S(Bu_{2n(k)}, Bu_{2n(k)}, Qu_{2n(k)}), \\ \frac{S(Bu_{2n(k)}, Bu_{2n(k)}, Au_{2n(k)})S(Pu_{2m(k)-1}, Pu_{2m(k)-1}, Qu_{2n(k)})}{[1+S(Bu_{2n(k)}, Bu_{2n(k)}, Pu_{2m(k)-1})]}\Big\}$$

$$= \max\Big\{S(v_{2m(k)-2}, v_{2m(k)-2}, v_{2n(k)-1}), S(v_{2n(k)}, v_{2n(k)}, v_{2m(k)-1}), \\ S(v_{2n(k)}, v_{2n(k)}, v_{2n(k)-1}), \\ \frac{S(v_{2n(k)}, v_{2n(k)}, v_{2n(k)})S(v_{2m(k)-2}, v_{2m(k)-2}, v_{2n(k)-1})}{[1+S(v_{2n(k)}, v_{2n(k)}, v_{2m(k)-2})]}\Big\}.$$

Passing to the limit as $k \to \infty$ in the above inequality and using equations (3.6) and (3.14), we obtain

$$\Delta(u_{2m(k)-1}, u_{2m(k)-1}, u_{2n(k)}) = \max\{\varepsilon, \varepsilon, 0, 0\} = \varepsilon. \tag{3.22}$$

Now, passing to the limit as $k \to \infty$ in (3.21), using the equations (3.14), (3.22) the properties of F and ψ, we have

$$\begin{aligned} \varepsilon &\leqslant F\Big(\psi(\varepsilon), \varphi(\varepsilon)\Big) \\ &\leqslant \psi(\varepsilon) < \varepsilon, \end{aligned} \tag{3.23}$$

which is a contradiction. Hence $\{v_{2n}\}_{n=1}^{\infty}$ is a Cauchy sequence in E. Thus, in both the cases, it has been shown that $\{v_n\}_{n=0}^{\infty}$ is a Cauchy sequence in E. Since (E, S) is a complete S-metric space, so $\{v_n\}_{n=0}^{\infty}$ is convergent in E. Suppose $v_n \to p$ as $n \to \infty$. We shall now show that p is a common fixed point for the mappings A and Q. It is clear that

$$\lim_{n\to\infty} v_{2n} = \lim_{n\to\infty} Au_{2n} = \lim_{n\to\infty} Qu_{2n+1} = p, \tag{3.24}$$

and

$$\lim_{n\to\infty} v_{2n+1} = \lim_{n\to\infty} Bu_{2n+1} = \lim_{n\to\infty} Pu_{2n+2} = p. \tag{3.25}$$

Assume that $P(E)$ is closed, there exists a $z \in E$ such that $p = Pz$. We claim that $Az = p$. If not, then

$$\Delta(z, z, u_{2n+1})$$

$$= \max\Big\{S(Pz, Pz, Qu_{2n+1}), S(Bu_{2n+1}, Bu_{2n+1}, Az), S(Bu_{2n+1}, Bu_{2n+1}, Qu_{2n+1}), \\ \frac{S(Bu_{2n+1}, Bu_{2n+1}, Au_{2n+1})S(Pz, Pz, Qu_{2n+1})}{[1+S(Bu_{2n+1}, Bu_{2n+1}, Pz)]}\Big\}. \tag{3.26}$$

Passing to the limit as $n \to \infty$ in equation (3.26) and using $(S1)$, we get

$$\begin{aligned} \lim_{n\to\infty} \Delta(z, z, u_{2n+1}) &= \max\{0, S(p, p, Az), 0, 0\} \\ &= S(p, p, Az). \end{aligned} \tag{3.27}$$

Now, from equation (3.1), we have

$$S(Az,Az,Bu_{2n+1}) \leqslant F\Big(\psi(\Delta(z,z,u_{2n+1})),\varphi(\Delta(z,z,u_{2n+1}))\Big), \tag{3.28}$$

on taking the limit as $n\to\infty$ in equation (3.28), which implies that

$$S(Az,Az,p) \leqslant F\Big(\psi(S(p,p,Az)),\varphi(S(p,p,Az))\Big), \tag{3.29}$$

using Lemma 2.1, the property of F and the property of ψ, we obtain

$$S(p,p,Az) \leqslant \psi(S(p,p,Az)) < S(p,p,Az),$$

which is a contradiction as $S(p,p,Az) > 0$. Hence $S(p,p,Az) = 0$, that is, $Az = p$ and $Az = Pz = p$. Since the mappings A and P are weakly compatible, so $Ap = APz = PAz = Pp$. Next, we claim that $Ap = p$. If not, then

$$\begin{aligned}
&\Delta(p,p,u_{2n+1})\\
&= \max\Big\{S(Pp,Pp,Qu_{2n+1}), S(Bu_{2n+1},Bu_{2n+1},Ap), S(Bu_{2n+1},Bu_{2n+1},Qu_{2n+1}),\\
&\qquad \frac{S(Bu_{2n+1},Bu_{2n+1},Au_{2n+1})S(Pp,Pp,Qu_{2n+1})}{[1+S(Bu_{2n+1},Bu_{2n+1},Pp)]}\Big\}.
\end{aligned} \tag{3.30}$$

Passing to the limit as $n\to\infty$ in equation (3.30) and using $(S1)$, $Ap = Pp$ and Lemma 2.1, we get

$$\begin{aligned}
\lim_{n\to\infty}\Delta(p,p,u_{2n+1}) &= \max\{S(Ap,Ap,p),S(Ap,Ap,p),0,0\}\\
&= S(Ap,Ap,p).
\end{aligned} \tag{3.31}$$

Now, from equation (3.1), we have

$$S(Ap,Ap,Bu_{2n+1}) \leqslant F\Big(\psi(\Delta(p,p,u_{2n+1})),\varphi(\Delta(p,p,u_{2n+1}))\Big), \tag{3.32}$$

on taking the limit as $n\to\infty$ in equation (3.32), which implies that

$$S(Ap,Ap,p) \leqslant F\Big(\psi(S(Ap,Ap,p)),\varphi(S(Ap,Ap,p))\Big), \tag{3.33}$$

using the property of F and the property of ψ, we obtain

$$S(Ap,Ap,p) \leqslant \psi(S(Ap,Ap,p)) < S(Ap,Ap,p),$$

which is a contradiction as $S(Ap,Ap,p) > 0$. Hence $S(Ap,Ap,p) = 0$, that is, $Ap = p$ and hence $Ap = Pp = p$.

Moreover, we show that p is a common fixed point of the mappings B and Q. Since $A(E) \subseteq Q(E)$, there is some $r \in E$ such that $Ap = Qr$. Then $Ap = Qr = Pp = p$. We claim that $Br = p$. If not, then from equation (3.1) and using Lemma 2.1 and $(S1)$, we have

$$\begin{aligned}
S(p,p,Br) &= S(Ap,Ap,Br)\\
&\leqslant F\Big(\psi(\Delta(p,p,r)),\varphi(\Delta(p,p,r))\Big),
\end{aligned} \tag{3.34}$$

where

$$
\begin{aligned}
\Delta(p,p,r) &= \max\Big\{S(Pp,Pp,Qr), S(Br,Br,Ap), S(Br,Br,Qr), \\
&\qquad \frac{S(Br,Br,Ar)S(Pp,Pp,Qr)}{[1+S(Br,Br,Pp)]}\Big\} \\
&= \max\Big\{S(p,p,p), S(Br,Br,p), S(Br,Br,p), \\
&\qquad \frac{S(Br,Br,Ar)S(p,p,p)}{[1+S(Br,Br,p)]}\Big\} \\
&= \max\Big\{0, S(p,p,Br), S(p,p,Br), 0\Big\} \\
&= S(p,p,Br).
\end{aligned}
$$

Using this value in equation (3.34) and using the property of F and ψ, we have

$$
\begin{aligned}
S(p,p,Br) &\leqslant F\Big(\psi(S(p,p,Br)), \varphi(S(p,p,Br))\Big) \\
&\leqslant \psi(S(p,p,Br)) < S(p,p,Br),
\end{aligned}
$$

which is a contradiction as $S(p,p,Br) > 0$. Hence $S(p,p,Br) = 0$, that is, $Br = p$. Thus, $Br = Qr = p$ and by weak compatibility of the mappings B and Q, we have $Bp = BQr = QBr = Qp$. Now, we show that p is a fixed point of B. If $Bp \neq p$, then by equation (3.1) and using Lemma 2.1 and ($S1$), we have

$$
\begin{aligned}
S(p,p,Bp) &= S(Ap,Ap,Bp) \\
&\leqslant F\Big(\psi(\Delta(p,p,p)), \varphi(\Delta(p,p,p))\Big), \qquad (3.35)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(p,p,p) &= \max\Big\{S(Pp,Pp,Qp), S(Bp,Bp,Ap), S(Bp,Bp,Qp), \\
&\qquad \frac{S(Bp,Bp,Ap)S(Pp,Pp,Qp)}{[1+S(Bp,Bp,Pp)]}\Big\} \\
&= \max\Big\{S(p,p,Bp), S(Bp,Bp,p), S(Bp,Bp,Bp), \\
&\qquad \frac{S(Bp,Bp,p)S(p,p,Bp)]}{[1+S(Bp,Bp,p)]}\Big\} \\
&\leqslant \max\Big\{S(p,p,Bp), S(p,p,Bp), 0, S(p,p,Bp)\Big\} \\
&= S(p,p,Bp).
\end{aligned}
$$

Using this value in equation (3.35) and using the property of F and ψ, we have

$$
\begin{aligned}
S(p,p,Bp) &\leqslant F\Big(\psi(S(p,p,Bp)), \varphi(S(p,p,Bp))\Big) \\
&\leqslant \psi(S(p,p,Bp)) < S(p,p,Bp),
\end{aligned}
$$

which is a contradiction as $S(p,p,Bp) > 0$. Hence $S(p,p,Bp) = 0$, that is, $Bp = p$. Hence $Ap = Bp = Pp = Qp = p$. This shows that p is a common fixed point of the mappings A, B, P and Q. A similar fashion is also valid for the case in which

$Q(E)$ or $A(E)$ or $B(E)$ is closed. Also, the uniqueness of the common fixed point p follows from Theorem 3.1. This completes the proof. $\square$

From Theorem 3.1, we obtain the following corollaries.

COROLLARY 3.1. *Let (E,S) be a complete S-metric space and let $A,B,P,Q\colon E\to E$ be four mappings satisfying the following conditions:*

(i)

$$S(Au,Av,Bw)\leqslant F\Big(\psi(S(Pu,Pv,Qw)),\varphi(S(Pu,Pv,Qw))\Big),$$

for all $u,v,w\in E$, $\psi\in\Psi$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$;

(ii) *$A(E)\subseteq Q(E)$ and $B(E)\subseteq P(E)$;*

(iii) *the pairs (A,P) and (B,Q) are weakly compatible.*

If one of the range $A(E)$ or $B(E)$ or $P(E)$ or $Q(E)$ is closed in E, then A, B, P and Q have a unique common fixed point in E.

COROLLARY 3.2. *Let (E,S) be a complete S-metric space and let $A,B,P,Q\colon E\to E$ be four mappings satisfying the following conditions:*

(i)

$$S(Au,Av,Bw)\leqslant F\Big(\psi(\Delta_1(u,v,w)),\varphi(\Delta_1(u,v,w))\Big),$$

where

$$\Delta_1(u,v,w) \;=\; \max\Big\{S(Pu,Pv,Qw),S(Bw,Bw,Av),S(Bw,Bw,Qw)\Big\},$$

for all $u,v,w\in E$, $\psi\in\Psi$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$;

(ii) *$A(E)\subseteq Q(E)$ and $B(E)\subseteq P(E)$;*

(iii) *the pairs (A,P) and (B,Q) are weakly compatible.*

If one of the range $A(E)$ or $B(E)$ or $P(E)$ or $Q(E)$ is closed in E, then A, B, P and Q have a unique common fixed point in E.

COROLLARY 3.3. *Let (E,S) be a complete S-metric space and let $A,B,P,Q\colon E\to E$ be four mappings satisfying the following conditions:*

(i)

$$S(Au,Av,Bw)\leqslant F\Big(\psi(\Delta_2(u,v,w)),\varphi(\Delta_2(u,v,w))\Big),$$

where

$$\begin{aligned}\Delta_2(u,v,w) \;=\; \max\Big\{&S(Pu,Pv,Qw),S(Bw,Bw,Av),\\ &\frac{S(Bw,Bw,Aw)S(Pu,Pu,Qw)}{[1+S(Bw,Bw,Pu)]}\Big\},\end{aligned}$$

for all $u,v,w\in E$, $\psi\in\Psi$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$;

(ii) *$A(E)\subseteq Q(E)$ and $B(E)\subseteq P(E)$;*

(iii) *the pairs (A,P) and (B,Q) are weakly compatible.*

If one of the range $A(E)$ or $B(E)$ or $P(E)$ or $Q(E)$ is closed in E, then A, B, P and Q have a unique common fixed point in E.

Corollary 3.4. *Let (E,S) be a complete S-metric space and let $A,B,P,Q\colon E\to E$ be four mappings satisfying the following conditions:*

(i)

$$S(Au,Av,Bw)\leqslant F\Big(\psi(\Delta_3(u,v,w)),\varphi(\Delta_3(u,v,w))\Big),$$

where

$$\Delta_3(u,v,w) = \max\Big\{S(Bw,Bw,Av),S(Bw,Bw,Qw),\frac{S(Bw,Bw,Aw)S(Pu,Pu,Qw)}{[1+S(Bw,Bw,Pu)]}\Big\},$$

for all $u,v,w\in E$, $\psi\in\Psi$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$;

(ii) $A(E)\subseteq Q(E)$ and $B(E)\subseteq P(E)$;

(iii) the pairs (A,P) and (B,Q) are weakly compatible.

If one of the range $A(E)$ or $B(E)$ or $P(E)$ or $Q(E)$ is closed in E, then A, B, P and Q have a unique common fixed point in E.

If we take $P=Q=I$ (where I is an identity mapping) in Theorem 3.1, then we have the following result.

Corollary 3.5. *Let (E,S) be a complete S-metric space and let $A,B\colon E\to E$ be two mappings satisfying the following condition:*

$$S(Au,Av,Bw)\leqslant F\Big(\psi(\Gamma(u,v,w)),\varphi(\Gamma(u,v,w))\Big),$$

where

$$\Gamma(u,v,w) = \max\Big\{S(u,v,w),S(Bw,Bw,Av),S(Bw,Bw,w),\frac{S(Bw,Bw,Aw)S(u,u,w)}{[1+S(Bw,Bw,u)]}\Big\},$$

for all $u,v,w\in E$, $\psi\in\Psi$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$. Then A and B have a unique common fixed point in E.

If we take $F(s,t)=h\,s$ where $h\in[0,1)$ is a constant, in Theorem 3.1, then we have the following result.

Corollary 3.6. *Let (E,S) be a complete S-metric space and let $A,B,P,Q\colon E\to E$ be four mappings satisfying the following conditions:*

(i)

$$S(Au,Av,Bw) \leqslant h\,\psi\Big(\max\Big\{S(Pu,Pv,Qw),S(Bw,Bw,Av),S(Bw,Bw,Qw),\frac{S(Bw,Bw,Aw)S(Pu,Pu,Qw)}{[1+S(Bw,Bw,Pu)]}\Big\}\Big),$$

for all $u,v,w\in E$, where $h\in[0,1)$ is a constant and $\psi\in\Psi$;

(ii) $A(E)\subseteq Q(E)$ and $B(E)\subseteq P(E)$;

(iii) the pairs (A,P) and (B,Q) are weakly compatible.

If one of the range $A(E)$ or $B(E)$ or $P(E)$ or $Q(E)$ is closed in E, then A, B, P and Q have a unique common fixed point in E.

If we take $F(s,t)=s$ and $\psi(t)=q\,t$, where $q\in[0,1)$ is a constant, in Theorem 3.1, then we have the following result.

Corollary 3.7. *Let (E,S) be a complete S-metric space and let $A,B,P,Q\colon E\to E$ be four mappings satisfying the following conditions:*

(i)

$$\begin{aligned} S(Au,Av,Bw) \quad &\leqslant \quad q\,\max\Big\{S(Pu,Pv,Qw),S(Bw,Bw,Av),S(Bw,Bw,Qw),\\ &\qquad\qquad \frac{S(Bw,Bw,Aw)S(Pu,Pu,Qw)}{[1+S(Bw,Bw,Pu)]}\Big\},\end{aligned}$$

for all $u,v,w\in E$, where $q\in[0,1)$ is a constant;

(ii) *$A(E)\subseteq Q(E)$ and $B(E)\subseteq P(E)$;*

(iii) *the pairs (A,P) and (B,Q) are weakly compatible.*

If one of the range $A(E)$ or $B(E)$ or $P(E)$ or $Q(E)$ is closed in E, then A, B, P and Q have a unique common fixed point in E.

Corollary 3.8. *Let (E,S) be a complete S-metric space and let $A,B,P,Q\colon E\to E$ be four mappings satisfying the following conditions:*

(i)

$$\begin{aligned} S(Au,Av,Bw) \quad &\leqslant \quad p_1\,S(Pu,Pv,Qw)+p_2\,S(Bw,Bw,Av)+p_3\,S(Bw,Bw,Qw)\\ &\qquad +p_4\,\frac{S(Bw,Bw,Aw)S(Pu,Pu,Qw)}{[1+S(Bw,Bw,Pu)]},\end{aligned}$$

for all $u,v,w\in E$, where p_1,p_2,p_3,p_4 are nonnegative reals such that $p_1+p_2+p_3+p_4<1$;

(ii) *$A(E)\subseteq Q(E)$ and $B(E)\subseteq P(E)$;*

(iii) *the pairs (A,P) and (B,Q) are weakly compatible.*

If one of the range $A(E)$ or $B(E)$ or $P(E)$ or $Q(E)$ is closed in E, then A, B, P and Q have a unique common fixed point in E.

Proof. Follows from Corollary 3.7, by noting that

$$p_1\,S(Pu,Pv,Qw)+p_2\,S(Bw,Bw,Av)+p_3\,S(Bw,Bw,Qw)$$

$$+p_4\,\frac{S(Bw,Bw,Aw)S(Pu,Pu,Qw)}{[1+S(Bw,Bw,Pu)]}$$

$$\begin{aligned}\leqslant \quad &(p_1+p_2+p_3+p_4)\max\Big\{S(Pu,Pu,Qw),S(Bw,Bw,Av),S(Bw,Bw,Qw),\\ &\qquad\qquad \frac{S(Bw,Bw,Aw)S(Pu,Pu,Qw)}{[1+S(Bw,Bw,Pu)]}\Big\}.\end{aligned}$$

$\square$

If we take $A=B$ and $P=Q=I$ (where I is an identity mapping) in Theorem 3.1, then we have the following result.

COROLLARY 3.9. *Let (E, S) be a complete S-metric space and let $A\colon E \to E$ be a mapping satisfying the inequality:*

$$S(Au, Av, Aw) \leqslant F\Big(\psi(\Gamma_1(u, v, w)), \varphi(\Gamma_1(u, v, w))\Big),$$

where

$$\Gamma_1(u, v, w) \quad = \quad \max\Big\{S(u, v, w), S(Aw, Aw, Av), S(Aw, Aw, w)\Big\},$$

for all $u, v, w \in E$, $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. Then A has a unique fixed point in E.

If we take $\max\Big\{S(u, v, w), S(Aw, Aw, Av), S(Aw, Aw, w)\Big\} = S(u, v, w)$, $F(s, t) = s$ and $\psi(t) = k\,t$, where $k \in [0, 1)$ is a constant, in Corollary 3.9, then we have the following result.

COROLLARY 3.10. *Let (E, S) be a complete S-metric space and let $A\colon E \to E$ be a mapping satisfying the inequality:*

$$S(Au, Av, Aw) \leqslant k\,S(u, v, w),$$

for all $u, v, w \in E$, where $k \in [0, 1)$ is a constant. Then A has a unique fixed point in E.

REMARK 3.1. Corollary 3.10 extends the well-known Banach fixed point theorem [**6**] from complete metric space to the setting of complete S-metric space.

COROLLARY 3.11. *Let (E, S) be a complete S-metric space and let $A\colon E \to E$ be a self-mapping of E satisfying the contractive condition:*

$$S(A^n u, A^n v, A^n w) \leqslant k\,S(u, v, w),$$

for all $u, v, w \in E$, where n is some positive integer and $k \in [0, 1)$ is a constant. Then A has a unique fixed point in E.

PROOF. By Corollary 3.10, there exists $z \in E$ such that $A^n z = z$. Then

$$\begin{aligned} S(Az, Az, z) &= S(AA^n z, AA^n z, A^n z) \\ &= S(A^n Az, A^n Az, A^n z) \\ &\leqslant k\,S(Az, Az, z), \end{aligned}$$

which is a contradiction, since $0 \leqslant k < 1$ and so $S(Az, Az, z) = 0$, that is, $Az = z$. This shows that A has a unique fixed point in E. This completes the proof. ☐

REMARK 3.2. Corollary 3.10 is a special case of Corollary 3.11 for $n = 1$.

Now, we give an example in support of the results.

EXAMPLE 3.1. Let $E = [0, 1]$. We define the function $S\colon E^3 \to [0, \infty)$ by

$$S(u, v, w) = \begin{cases} 0, & \text{if } u = v = w, \\ \max\{u, v, w\}, & \text{if otherwise,} \end{cases}$$

for all $u, v, w \in E$, then S is an S-metric on E. Define four self-maps $A, B, P, Q\colon E \to E$ on E by $A(u) = \frac{u}{4}$, $B(u) = \frac{u}{4}$, $P(u) = u$ and $Q(u) = \frac{u}{2}$ for all $u \in E$. We also define $\psi\colon [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. Clearly ψ is continuous on $[0,\infty)$ satisfying $0 < \psi(t) < t$ for all $t > 0$. Now consider the following cases:

Case I. (1) Let $u > v > w$. Then we have

$$S(Au, Av, Bw) = S\Big(\frac{u}{4}, \frac{v}{4}, \frac{w}{4}\Big) = \max\Big\{\frac{u}{4}, \frac{v}{4}, \frac{w}{4}\Big\} = \frac{u}{4},$$
$$S(Pu, Pv, Qw) = S\Big(u, v, \frac{w}{2}\Big) = \max\Big\{u, v, \frac{w}{2}\Big\} = u,$$
$$S(Pu, Pu, Qw) = S\Big(u, u, \frac{w}{2}\Big) = \max\Big\{u, u, \frac{w}{2}\Big\} = u,$$
$$S(Bw, Bw, Av) = S\Big(\frac{w}{4}, \frac{w}{4}, \frac{v}{4}\Big) = \max\Big\{\frac{w}{4}, \frac{w}{4}, \frac{v}{4}\Big\} = \frac{v}{4},$$
$$S(Bw, Bw, Qw) = S\Big(\frac{w}{4}, \frac{w}{4}, \frac{w}{2}\Big) = \max\Big\{\frac{w}{4}, \frac{w}{4}, \frac{w}{2}\Big\} = \frac{w}{2},$$
$$S(Bw, Bw, Aw) = S\Big(\frac{w}{4}, \frac{w}{4}, \frac{w}{4}\Big) = \max\Big\{\frac{w}{4}, \frac{w}{4}, \frac{w}{4}\Big\} = \frac{w}{4},$$
$$S(Bw, Bw, Pu) = S\Big(\frac{w}{4}, \frac{w}{4}, u\Big) = \max\Big\{\frac{w}{4}, \frac{w}{4}, u\Big\} = u,$$
$$\frac{S(Bw, Bw, Aw)S(Pu, Pu, Qw)]}{[1 + S(Bw, Bw, Pu)]} = \frac{\frac{w}{4}.u}{[1+u]} = \frac{uw}{4(1+u)}.$$

Now using the inequality of Corollary 3.6 and the property of ψ, we have

$$\begin{aligned} S(Au, Av, Bw) &= \frac{u}{4} \\ &\leqslant h\,\psi\Big(\max\Big\{u, \frac{v}{4}, \frac{w}{2}, \frac{uw}{4(1+u)}\Big\}\Big) \\ &= h\,\psi(u) = h\,\frac{u}{2}, \end{aligned}$$

or,

$$\frac{1}{4} \leqslant \frac{h}{2},$$

or,

$$h \geqslant \frac{1}{2}.$$

If we take $0 \leqslant h < 1$, then the inequality of Corollary 3.6 is satisfied. Hence we conclude that

$$\begin{aligned} S(Au, Av, Bw) \leqslant\; & h\,\psi\Big(\max\Big\{S(Pu, Pv, Qw), S(Bw, Bw, Av), S(Bw, Bw, Qw), \\ & \frac{S(Bw, Bw, Aw)S(Pu, Pu, Qw)}{[1 + S(Bw, Bw, Pu)]}\Big\}\Big). \end{aligned}$$

(2) Now using inequality of Corollary 3.7, we have

$$\begin{aligned} S(Au, Av, Bw) &= \frac{u}{4} \\ &\leqslant q\max\Big\{u, \frac{v}{4}, \frac{w}{2}, \frac{uw}{4(1+u)}\Big\} \\ &= q\,u, \end{aligned}$$

or,

$$\frac{1}{4} \leqslant q,$$

or,

$$q \geqslant \frac{1}{4}.$$

If we take $0 \leqslant q < 1$, then the inequality of Corollary 3.7 is satisfied. Hence we conclude that

$$\begin{aligned} S(Au, Av, Bw) &\leqslant q \max\Big\{S(Pu, Pv, Qw), S(Bw, Bw, Av), S(Bw, Bw, Qw), \\ &\qquad \frac{S(Bw, Bw, Aw)S(Pu, Pu, Qw)}{[1 + S(Bw, Bw, Pu)]}\Big\}. \end{aligned}$$

(3) Now using inequality of Corollary 3.8, we have

$$\begin{aligned} S(Au, Av, Bw) &= \frac{u}{4} \\ &\leqslant p_1\, u + p_2\, \frac{v}{4} + p_3\, \frac{w}{2} + p_4\, \frac{uw}{4(1+u)}. \end{aligned}$$

If we take $u = 1$, $v = \frac{1}{2}$ and $w = \frac{1}{4}$, then we have

$$\frac{1}{4} \leqslant p_1 + \frac{p_2}{4} + \frac{p_3}{8} + \frac{p_4}{32},$$

or,

$$8 \leqslant 32\, p_1 + 8\, p_2 + 4\, p_3 + p_4.$$

The above inequality is satisfied for: (i) $p_1 = \frac{1}{8}$, $p_2 = \frac{1}{4}$, $p_3 = \frac{1}{2}$ and $p_4 = 0$, (ii) $p_1 = \frac{1}{4}$ and $p_2 = p_3 = p_4 = 0$ and (iii) $p_1 = \frac{1}{8}$, $p_2 = \frac{1}{2}$ and $p_3 = p_4 = 0$ with $p_1 + p_2 + p_3 + p_4 < 1$. Thus we conclude that

$$\begin{aligned} S(Au, Av, Bw) &\leqslant p_1\, S(Pu, Pv, Qw) + p_2\, S(Bw, Bw, Av) + p_3\, S(Bw, Bw, Qw) \\ &\quad + p_4\, \frac{S(Bw, Bw, Aw)S(Pu, Pu, Qw)}{[1 + S(Bw, Bw, Pu)]}. \end{aligned}$$

(4) Now using inequality of Corollary 3.10, we have

$$S(Au, Av, Aw) = \frac{u}{4} \leqslant k\, u,$$

or,

$$\frac{1}{4} \leqslant k,$$

or,

$$k \geqslant \frac{1}{4}.$$

If we take $0 \leqslant k < 1$, then the inequality of Corollary 3.10 is satisfied. Hence we conclude that

$$S(Au, Av, Aw) \leqslant k\, S(u, v, w).$$

Case II. Now we show that the pairs (A, P) and (B, Q) are weakly compatible. For this, suppose that $Qz = Bz$ for $z \in E$. Then $\frac{z}{2} = \frac{z}{4}$. It follows that $z = 0$. Now, we consider $BQ(z) = B(Q(z)) = B(0) = 0$ and $QB(z) = Q(B(z)) = Q(0) = 0$. Thus, the pair (B, Q) is weakly compatible. Now, let $Az = Pz$ for $z \in E$. This implies that $\frac{z}{4} = z$ and hence $z = 0$. Now, we consider $AP(z) = A(P(z)) = A(0) = 0$ and $PA(z) = P(A(z)) = P(0) = 0$. It follows that the pair (A, P) is also weakly compatible.

Case III. Now we show that $A(E) \subseteq Q(E)$ and $B(E) \subseteq P(E)$. Since $E = [0, 1]$, so it is easy to see that $A(E) = [0, \frac{1}{4}]$, $B(E) = [0, \frac{1}{4}]$, $P(E) = [0, 1]$ and $Q(E) = [0, \frac{1}{2}]$. Hence $A(E) \subseteq Q(E)$ and $B(E) \subseteq P(E)$.

Thus all the conditions of Corollary 3.6, Corollary 3.7, Corollary 3.8 and Corollary 3.10 are satisfied and hence the mappings A, B, P and Q have a unique common fixed point, namely $u = 0 \in E$.

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Gurucharan Singh Saluja, H.N. 3/1005, Geeta Nagar, Raipur, Raipur-492001 (Chhattisgarh), India

Email address: saluja1963@gmail.com