JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
J. Int. Math. Virtual Inst., **13**(1)(2023), 37-53
DOI: 10.7251/JIMVI2301037M

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

DERIVATIONS OF FUZZY IDEALS IN ORDERED SEMIRINGS

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Abstract. In this paper, we introduce the notion of a derivation of ordered semirings, study some properties of derivations of ordered semirings, derivations of fuzzy ideals and relations between derivation and homomorphism of ordered semirings. We prove that if μ is a fuzzy prime ideal and d is an onto derivation of idempotent ordered semiring M with identity then $d(\mu)$ and $d^{-1}(\mu)$ are fuzzy prime ideals of M.

1. Introduction

The notion of derivation is useful in studying the structures, properties of algebraic systems and has important role in characterizing algebraic structures. Bresar and Vukman established that a prime ring must be commutative if it admits a nonzero left derivation[**2**] in 1990. The first result in this direction is due to Posner[**19**] in 1957. Kim studied right derivation and generalized derivation of incline algebra[**5, 6**]. The notion of derivation is generalized in various directions such as right derivation, left derivation, f-derivation, reverse derivation, orthogonal derivation, (f, g)-derivation, generalized right derivation, etc. Murali Krishna Rao and Venkateswarlu introduced the notion of generalized right derivation of $\Gamma-$ incline and right derivation of ordered $\Gamma-$semiring[**15, 16**]. The notion of a semiring was first introduced by Vandiver[**23**] in 1934. A universal algebra $S = (S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b + c) = ab + ac,\ (a + b)c = ac + bc,$ for all $a, b, c \in S$. A natural example of semiring is the set of all natural numbers under usual addition

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2010 *Mathematics Subject Classification.* Primary 06D99, 16Y60, 08A72; Secondary 16Y60, 08A72.

Key words and phrases. Semiring, ordered semiring, fuzzy ideal, derivation of fuzzy ideal.
Communicated by Dusko Bogdanic.

and multiplication of numbers. The theory of rings and semigroups considerably impacts the development of the theory of semirings. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring.

Zadeh developed the fuzzy set theory in 1965[**22**]. Many papers on fuzzy sets appeared, showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. The fuzzification of algebraic structure was introduced by Rosenfeld and he introduced the notion of fuzzy subgroups[**20**] in 1971. Fuzzy subrings and fuzzy ideals in rings were studied by Liu. Applying the concept of fuzzy sets to the theory of $\Gamma-$ring, Jun and Lee introduced the notion of fuzzy ideals in $\Gamma-$ring and studied the properties of fuzzy ideals of $\Gamma-$ring[**4**]. Mandal studied fuzzy ideals and fuzzy interior ideals in an ordered semiring[**9**]. Dutta studied fuzzy ideals of $\Gamma-$semirings.

In this paper, our main aim is to study the properties of derivations of fuzzy ideals in an ordered semiring. We prove that if d is a derivation on an ordered semiring M in which $(M,+)$ is an idempotent, $b+d(b)=d(b)$ and $d(d(b))=d(b)$, for all $b \in M$ then d is a homomorphism.

2. Preliminaries

In this section we recall some of the fundamental concepts and definitions necessary for this paper.

Definition 2.1. A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x+0=x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2. An element a of a semiring S is called a regular element if there exists an element b of S such that $a = aba$.

Definition 2.3. A semiring S is called a regular semiring if every element of S is a regular element.

Definition 2.4. An element a of a semiring S is called a multiplicatively idempotent (an additively idempotent) element if $aa = a (a+a=a)$.

Definition 2.5. An element b of a semiring M is called an inverse element of a of M if $ab = ba = 1$.

Definition 2.6. A non-empty subset A of semiring M is called

(i) a subsemiring of M if A is an additive subsemigroup of M and $AA \subseteq A$.
(ii) a left(right) ideal of M if A is an additive subsemigroup of M and $MA \subseteq A(AM \subseteq A)$.
(iii) an ideal if A is an additive subsemigroup of M, $MA \subseteq A$ and $AM \subseteq A$.

(iv) a $k-$ideal if A is a subsemiring of $M, AM \subseteq A, MA \subseteq A$ and $x \in M,\ x+y \in A, y \in A$ then $x \in A$.

DEFINITION 2.7. An ordered semiring M is said to be commutative semiring if $xy = yx$, for all $x, y \in M$.

DEFINITION 2.8. A non zero element a in an ordered semiring M is said to be a zero divisor if there exists a non zero element $b \in M$, such that $ab = ba = 0$.

DEFINITION 2.9. An ordered semiring M with unity 1 and zero element 0 is called an integral ordered semiring if it has no zero divisors.

DEFINITION 2.10. A non-empty subset A of an ordered semiring M is called a subsemiring M if $(A, +)$ is a subsemigroup of $(M, +)$ and $ab \in A$, for all $a, b \in A$.

DEFINITION 2.11. Let M be an ordered semiring. A non-empty subset I of M is called a left (right) ideal of an ordered semiring M if I is closed under addition, $MI \subseteq I$ $(IM \subseteq I)$ and if for any $a \in M,\ b \in I,\ a \leqslant b \Rightarrow a \in I$. I is called an ideal of M if it is both a left ideal and a right ideal of M.

DEFINITION 2.12. A non-empty subset A of ordered semiring M is called a $k-$ideal if A is an ideal and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.

DEFINITION 2.13. Let M be an ordered semiring. A mapping $f : M \to M$ is called an endomorphism if

(i) f is an onto ,
(ii) $f(a + b) = f(a) + f(b)$,
(iii) $f(ab) = f(a)f(b)$, for all $a, b \in M$.

DEFINITION 2.14. Let M be an ordered semiring. A mapping $d : M \to M$ is called a derivation if it satisfies
(i) $d(x + y) = d(x) + d(y)$
(ii) $d(xy) = d(x)y + xd(y)$ for all $x, y \in M$.

DEFINITION 2.15. Let M be a non-empty set. Then a mapping $f : M \to [0, 1]$ is called a fuzzy subset of M.

DEFINITION 2.16. Let f be a fuzzy subset of a non-empty set M. For $t \in [0, 1]$, the set $f_t = \{x \in M \mid f(x) \geqslant t\}$ is called a level subset of M with respect to f.

DEFINITION 2.17. Let M be a semiring. A fuzzy subset μ of M is said to be fuzzy subsemiring of M if it satisfies the following conditions
(i) $\mu(x + y) \geqslant min\ \{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geqslant min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$.

DEFINITION 2.18. A fuzzy subset μ of a semiring M is called a fuzzy left (right) ideal of M if it satisfies the following conditions
(i) $\mu(x + y) \geqslant min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geqslant \mu(y)\ (\mu(x))$, for all $x, y \in M$.

Definition 2.19. A fuzzy subset μ of a semiring M is called a fuzzy ideal of M if it satisfies the following conditions
(i) $\mu(x+y) \geqslant min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geqslant max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$

Definition 2.20. Let A be non-empty subset of a semiring M. The characteristic function of A is a fuzzy subset of M and it is defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.21. Let S and T be non empty sets and $\phi : S \to T$ be a any function. A fuzzy subset f of S is called a $\phi-$invariant if $\phi(x) = \phi(y) \Rightarrow f(x) = f(y)$.

3. Derivations of ordered semirings

In this section, we introduce the notion of a derivation of an ordered semiring and study some of their properties.

Definition 3.1. A semiring M is called an ordered semiring if it admits a compatible relation $\leqslant$. i.e. $\leqslant$ is a partial ordering on M satisfies the following conditions. If $a \leqslant b$ and $c \leqslant d$ then

(i) $a + c \leqslant b + d$
(ii) $ac \leqslant bd$
(iii) $ca \leqslant db$, for all $a, b, c, d \in M$.

Definition 3.2. An ordered semiring M is said to be totally ordered semiring M if any two elements of M are comparable.

Definition 3.3. In an ordered semiring M

(i) $(M, +)$ is a positively ordered if $a + b \geqslant a, b$ for all $a, b \in M$.
(ii) $(M, +)$ is a negatively ordered if $a + b \leqslant a, b$ for all , $a, b \in M$.
(iii) semigroup M is a positively ordered if $ab \geqslant a, b$ for all $a, b \in M$.
(iv) semigroup M is a negatively ordered if $ab \leqslant a, b,$ for all $a, b \in M$.

Definition 3.4. Let M be an ordered semiring. A mapping $d : M \to M$ is said to be derivation if

(i) $d(a + b) = d(a) + d(b)$
(ii) $d(ab) = d(a)b + ad(b)$
(iii) If $a \leqslant b$ then $d(a) \leqslant d(b)$, for all $a, b \in M$.

Definition 3.5. Let M be an ordered semiring. A mapping $f : M \to M$ is said to be homomorphism if

(i) $f(a + b) = f(a) + f(b)$
(ii) $f(ab) = f(a)f(b)$
(iii) If $a \leqslant b$ then $f(a) \leqslant f(b)$, for all $a, b \in M$.

Definition 3.6. Let d be a derivation of an ordered semiring M and μ be a fuzzy subset of M. Then μ is said to be d derivation invariant of M if $d(a) = d(b)$ then $\mu(a) = \mu(b)$, for all $a, b \in M$.

Example 3.1. Let N be a the set of all natural numbers.

(i). Let $M = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in N \cup \{0\} \right\}$. M is a semiring with respect to usual addition and multiplication of matrices. Define $[a_{ij}] \leqslant [b_{ij}]$ if and only if $a_{ij} \leqslant b_{ij}$, for all i, j, where $a_{ij}, b_{ij} \in M$. Then M is an ordered semiring.

Define $d : M \to M$ by $d\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & o \end{pmatrix}$. Then d is a derivation of M.

(ii). Let $M = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in N \cup \{0\} \right\}$ M is a semiring with respect to usual addition and multiplication of matrices. .
Define $[a_{ij}] \leqslant [b_{ij}]$ if and only if $a_{ij} \leqslant b_{ij}$, for all i, j, where $a_{ij}, b_{ij} \in M$. Then M is an ordered semiring.
Define $d : M \to M$ by $d\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ 0 & o \end{pmatrix}$.
Then d is not a derivation of M.

Example 3.2. Let N be the set of all natural numbers.
(i). Let $M = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in N \cup \{0\} \right\}$ M is a semiring with respect to usual addition and multiplication of matrices.
Define $[a_{ij}] \leqslant [b_{ij}]$ if and only if $a_{ij} \leqslant b_{ij}$, for all i, j, where $a_{ij}, b_{ij} \in M$. Then M is an ordered semiring.
Define $d : M \to M$ by $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & o \end{pmatrix}$.
Then d is a derivation of M.

Theorem 3.1. *Let M be an ordered semiring in which $(M, +)$ is a positively ordered with unity element and d be a derivation of M.*

(i) *If $x \in M$ then $xd(1) \leqslant d(x)$*
(ii) *If $d(1) = 1$ then $x \leqslant d(x)$.*

Proof. Let M be an ordered semiring in which $(M, +)$ is positively ordered with unity element and d be a derivation of M.

(1). Let $x \in M$. Then $x1 = x$.

$$\begin{aligned} d(x) &= d(x1) \\ &= d(x)1 + xd(1) \\ \Rightarrow xd(1) &\leqslant d(x1) \\ &= d(x). \end{aligned}$$

(2). Suppose $d(1) = 1$. By (1), we have $xd(1) \leqslant d(x)$.

$$\begin{aligned} &\Rightarrow x1 \leqslant d(x) \\ &\Rightarrow x \leqslant d(x). \end{aligned}$$

□

Theorem 3.2. *Let d be a derivation of an ordered semiring M with a zero element. Then $d(0) = 0$.*

Proof. Let d be a derivation of an ordered semiring $M,\ x \in M$. Then

$$\begin{aligned} d(x0) &= d(x)0 + 0d(x) \\ \Rightarrow d(0) &= 0 + 0 = 0. \end{aligned}$$

Therefore $d(0) = 0$. □

Theorem 3.3. *Let M be an idempotent ordered semiring with $a + ba = a$, for all $a, b \in M$, and d be a derivation of M.*

(i) *If $d(d(x)) = d(x)$ then $d(xd(x)) = d(x)$*
(ii) *If $d(d(x)) = x$ then $d(d(x)d(y)) = d(xy)$, for all $x \in M$.*

Proof. Let M be an idempotent ordered semiring with $a + ba = a$, for all $a, b \in M, d$ be a derivation of M and $x \in M$.

(i).

$$\begin{aligned} d(xd(x)) &= d(x)d(x) + xd(d(x)) \\ &= d(x)d(x) + xd(x) \\ &= d(x) + xd(x) \\ &= d(x). \end{aligned}$$

(ii).

$$\begin{aligned} d(d(x)d(y)) &= d(d(x))d(y) + d(x)d(d(y)) \\ &= xd(y) + d(x)y \\ &= d(xy). \end{aligned}$$

□

Theorem 3.4. *Let M be an idempotent ordered semiring in which $(M, +)$ is a positively ordered satisfying $a + ab = a$ and $a + ba = a$, for all $a, b \in M$. Then $d(x) \leqslant x$, for all $x \in M$.*

Proof. Let $a \in M$. Then

$$\begin{aligned} aa &= a, \\ \Rightarrow d(a) &= d(aa) \\ \Rightarrow d(a) &= d(a)a + ad(a) \\ \Rightarrow d(a) + a &= d(a)a + ad(a) + a \\ \Rightarrow d(a) + a &= d(a)a + a \\ \Rightarrow d(a) + a &= a. \\ \text{Therefore } d(a) &\leqslant a, \text{ for all } a \in M. \end{aligned}$$

□

THEOREM 3.5. *Let M be an idempotent ordered semiring in which $(M,+)$ is a positively ordered with unity satisfying $a+ab=a$ and $a+ba=a$, for all $a,b\in M$, d be a derivation of M. Then*

(i) $d(x)=d(1)x+1d(x)$,
(ii) *If* $x\geqslant d(1)$ *then* $d(x)\geqslant d(1)$
(iii) *If* $x\leqslant d(1)$ *then* $d(x)=x$
(iv) $d(xy)\leqslant y$, *for all* x,y *inM*.

PROOF. Let M be an ordered semiring in which $(M,+)$ is a positively ordered with unity satisfying $a+ab=a$ and $a+ba=a$, for all $a,b\in M$, d be a derivation of M and $x\in M$. Then $x1=x=1x$.

(i).

$$\begin{aligned} d(x) &= d(1x) \\ &= d(1)x+1d(x). \end{aligned}$$

(ii).

$$\begin{aligned} &\text{If } x\geqslant d(1) \text{ then} \\ d(x) &= d(1)x+1d(x) \\ &\geqslant d(1)x \\ &\geqslant d(1)d(1) \\ &= d(1). \end{aligned}$$

(iii).

$$\begin{aligned} &\text{If } x\leqslant d(1) \text{ then} \\ d(x) &= d(1x)=d(1)x+d(x)1 \\ &\geqslant d(1)x \\ &\geqslant xx \\ &= x \\ d(x) &\geqslant x. \end{aligned}$$

By Theorem 3.4, we have $d(x)\leqslant x$.
Therefore $d(x)=x$.

(iv). Let d be a derivation of an idempotent ordered semiring M.
Suppose $x\leqslant y$ Then $xy\leqslant yy=y$.
Therefore $d(xy)\leqslant d(y)\leqslant y$.

□

THEOREM 3.6. *Let d be a derivation of ordered semiring M in which $(M,+)$ is positively ordered. Then 1 and 0 are the maximal element and the minimal element of M respectively.*

PROOF. Let d be a derivation of ordered semiring M in which $(M,+)$ is positively ordered and $x\in M$. Then, such that $x1=x\Rightarrow x1\leqslant 1$, since $(M,+)$ is positively ordered then semigroup M is a negatively ordered .

Therefore $x \leqslant 1$. Hence 1 is the maximal element of M.
we have $x + 0 = x \Rightarrow\ 0 \leqslant x$, since $(M, +)$ is a positively ordered.
$\Rightarrow$ 0 is the minimal element of ordered semiring M. □

THEOREM 3.7. *Let M be an ordered semiring M in which $(M, +)$ is positively ordered and d be an onto derivation of M. Then $d(1) = 1$.*

PROOF. Let M be a ordered semiring M in which $(M, +)$ is positively ordered, d be an onto derivation of M and $x \in M$. Then there exists $a \in M$ such that $d(a) = 1$.

$$\begin{aligned} \text{We have } a &\leqslant 1 \\ &\Rightarrow d(a) \leqslant d(1) \\ &\Rightarrow d(a) \leqslant d(1) \leqslant 1 \\ &\Rightarrow 1 \leqslant d(1) \leqslant 1 \\ &\Rightarrow d(1) = 1. \end{aligned}$$

□

THEOREM 3.8. *Let d be a derivation of an idempotent ordered semiring with unity M in which $(M, +)$ is positively ordered satisfying $a + ab = a$ and $a + ba = a$, for all $a, b \in M$. Then $d(1) = 1$ if and only if $d(a) = a$*

PROOF. Let d be a derivation of an idempotent ordered semiring with unity M in which $(M, +)$ is positively ordered satisfying $a + ab = a$ and $a + ba = a$, for all $a, b \in M$. Suppose $a \in M$ and $d(1) = 1$. such that $a1 = a$ and $aa = a$. Then

$$\begin{aligned} d(a) &= d(a + a1) \\ &= d(a) + d(a1) \\ &= d(a) + d(a)1 + ad(1) \\ &= d(a) + ad(1) \\ \Rightarrow ad(1) &\leqslant d(a) \\ \Rightarrow a1 &\leqslant d(a) \\ \Rightarrow a &\leqslant d(a). \\ \text{Now } d(a) &= d(aa) \\ &= d(a)a + ad(a) \\ \Rightarrow d(a) + a &= d(a)a + ad(a) + a \\ \Rightarrow d(a) + a &= d(a)a + a \\ \Rightarrow d(a) + a &= a \\ \text{Therefore } d(a) &\leqslant a. \end{aligned}$$

Hence $d(a) = a$. Converse is obvious. □

THEOREM 3.9. *Let d be a derivation of ordered semiring M with identities $a + ab = a$ and $b + ab = b$, for all $a, b \in M$. Then x is an idempotent element M if and only if $d(x)$ is an idempotent of $(M, +)$.*

PROOF. Let d be a derivation of ordered semiring M with identities $a+ab = a$ and $b + ab = b$, for all $a, b \in M$. Suppose $x \in M$ is an idempotent. Then

$$\begin{aligned} xx &= x \\ \Leftrightarrow d(xx) &= d(x) \\ \Leftrightarrow d(x)x + xd(x) &= d(x) \\ \Leftrightarrow d(x) + d(x)x + xd(x) &= d(x) + d(x) \\ \Leftrightarrow d(x) + xd(x) &= d(x) + d(x) \\ \Leftrightarrow d(x) &= d(x) + d(x). \end{aligned}$$

□

THEOREM 3.10. *Let M be an idempotent ordered semiring in which $(M, +)$ is a positively ordered with $a + ab = a$, $a + ba = a$, for all $a, b \in M$, and d be a derivation of M. Then*

(i) *$d(xy) = d(x)d(y)$ for all $x, y \in M$.*
(ii) *$d(xy) \leqslant d(x + y)$.*

PROOF. Let M be an idempotent ordered semiring in which $(M, +)$ is a positively ordered with $a + ab = a$, $a + ba = a$, for all $a, b \in M$, and d be a derivation of M. Suppose $x, y \in M$.

(i).

$$\begin{aligned} \text{We have } d(x) &= d(x + xy) \\ &= d(x) + d(xy). \\ \text{Now } d(xy) &\leqslant d(x). \\ \text{Similarly } d(xy) &\leqslant d(y) \\ d(xy)d(xy) &\leqslant d(x)d(y). \\ \text{Therefore } d(xy) &\leqslant d(x)d(y) \\ d(xy) &= d(x)y + xd(y) \\ &\geqslant d(x)y \\ &\geqslant d(x)d(y), \text{ since by Theorem 3.4, we have } d(y) \leqslant y. \\ \text{Hence } d(xy) &= d(x)d(y). \end{aligned}$$

(ii). We have $d(xy) \leqslant d(x)$ and $d(xy) \leqslant d(y)$

$$\begin{aligned} &\Rightarrow d(xy) + d(xy) \leqslant d(x) + d(y) \\ &\Rightarrow d(xy) \leqslant d(x) + d(y). \\ &\text{Therefore } d(xy) \leqslant d(x + y). \end{aligned}$$

□

DEFINITION 3.7. Let d be a derivation of an ordered semiring M. Then the set of all elements $x \in M$ such that $d(x) = 0$ is called a kernel d and kernel d is denoted by Ker$d$.

THEOREM 3.11. *Let d be a derivation of an ordered semiring M in which $(M, +)$ is a positively ordered. Then Kerd is a $k-$ideal of M.*

PROOF. Let d be a derivation of an ordered semiring M in which $(M, +)$ is a positively ordered. By Theorem 3.2, we have $d(0) = 0$. Therefore $Kerd \neq \phi$. Let $x, y \in Kerd$. Then $d(x) = 0, d(y) = 0$
$\Rightarrow d(x + y) = d(x) + d(y) = 0 + 0 = 0.$
$\Rightarrow x + y \in Kerd.$
Suppose $x \in Kerd$ and $y \in M$ such that $y \leqslant x$.
$\Rightarrow d(y) \leqslant d(x) = 0.$
$\Rightarrow d(y) = 0.$
Hence $y \in Kerd$.
Suppose $x \in Kerd$ and $x + y \in Kerd$.
Then $d(x + y) = 0, d(x) = 0$
$\Rightarrow d(x) + d(y) = 0$
$\Rightarrow d(y) = 0$
$y \in Kerd$.
Hence Ker$d$ is a $k-$ideal of M. □

DEFINITION 3.8. An ordered semiring M is said to be ordered prime semiring if $xMy = 0$ then $x = 0$ or $y = 0$, for all $x, y \in M$.

THEOREM 3.12. *Let M be an ordered prime semiring and I be a proper ideal of M. If d is a derivation of M such that $d(u) = 0$, for all $u \in I$ then $d(m) = 0$ for all $m \in M$.*

PROOF. Let M be an ordered prime semiring, I be a proper ideal of M and d be a derivation of M such that $d(u) = 0$, for all $u \in I$. Suppose $0 \neq u \in I$ and $x \in M$. Then $xu \in I$ and $d(xu) = 0$
$\Rightarrow d(x)u + xd(u) = 0$
$\Rightarrow d(x)u = 0$ for all $x \in M$, since $d(u) = 0$ for all $u \in I$.
Replacing x by ms, where $s, m \in M$. We get

$$\begin{aligned} &\Rightarrow d(ms)u = 0 \\ &\Rightarrow [d(m)s + md(s)]u = 0u \\ &\Rightarrow d(m)su + md(s)u = 0 \\ &\Rightarrow d(m)su = 0, \text{ since } d(s)u = 0 \\ \text{Therefore } &d(m) = 0, \text{ for all } m \in M. \end{aligned}$$

□

THEOREM 3.13. *Let I be a proper ideal of an ordered integral semiring M. If d is a nonzero derivation of M then d is a nonzero on I.*

Proof. Let I be a proper ideal of an ordered integral semiring M and d be a nonzero derivation of M. Suppose $d(x) = 0$, for all $x \in I$, $y \in M, x \in I$ and Then $xy \in I$, since I is an ideal.
Therefore $d(xy) = 0$
$d(xy) = d(x)y + xd(y) = 0 + xd(y) = xd(y) = 0$.
Therefore $d(y) = 0$, for all $y \in M$.
Which is a contradiction to d is a nonzero derivation.
Hence d is a nonzero on I. □

Theorem 3.14. *Let d be a derivation of an ordered integral semiring M with unity and a be a nonzero element of M. If $ad(x) = 0$, for all $x \in M$ then d is a zero derivation on M.*

Proof. Let d be a derivation of an ordered integral semiring M and $ad(x) = 0$, for all $x \in M$. Suppose $x, y \in M$. Then

$$\begin{aligned}
&ad(xy) = 0\\
&\Rightarrow a[d(x)y + xd(y)] = 0\\
&\Rightarrow ad(x)y + axd(y) = 0\\
&\Rightarrow axd(y) = 0,\\
&\Rightarrow a1d(y) = 0,\\
&\Rightarrow ad(y) = 0,
\end{aligned}$$

Therefore $d(y) = 0$, since M is an integral semiring M. □

Theorem 3.15. *Let M be an additively idempotent ordered semiring and d be an identity function from M into M. Then d is a derivation of M if and only if d is a homomorphism from M into M.*

Proof. Let M be an additively idempotent ordered semiring and d be an identity function from M into M. Suppose d is a derivation of M $a, b \in M$ and.Then $d(ab) = d(a)b + ad(b) = ab + ab = ab = d(a)d(b)$.
Therefore d is a homomorphism.

Conversely suppose that d is a homomorphism from M into M.
Let $a, b \in M$. Then
$d(ab) = d(a)d(b) = ab = ab + ab = d(a)b + ad(b)$.
Hence d is a derivation from M into M. □

Theorem 3.16. *Let M be an idempotent ordered semiring in which $(M, +)$ is a positively ordered with $d(1) = 1$ and d be a derivation of M. Then d is a homomorphism from M into M.*

Proof. By Theorem 3.8, we have $d(x) = x$. Let $x, y \in M$.
Then $d(xy) = xy = d(x)d(y)$.
Hence d is a homomorphism from M into M. □

THEOREM 3.17. *Let d be a derivation of an additively idempotent ordered semiring M, $d(d(a)) = d(a)$, for all $a \in M$ and $b + d(b) = d(b)$, for all $b \in M$. Then d is a homomorphism from M into M.*

PROOF. Let d be a derivation of an additively idempotent ordered semiring M and $d(d(a)) = d(a)$, for all $a \in M$ and $b + d(b) = d(b)$, for all $b \in M$.
We have $d(d(ab)) = d(ab)$, for all $a, b \in M$.

$$\begin{aligned}
&\Rightarrow d[d(a)b + ad(b)] = d(ab)\\
&\Rightarrow d(a)b + d(a)d(b) + d(a)d(b) + ad(b) = d(ab)\\
&\Rightarrow d(a)[b + d(b)] + (d(a) + a)d(b) = d(ab)\\
&\Rightarrow d(a)d(b) + d(a)d(b) = d(ab)\\
&\Rightarrow d(a)d(b) = d(ab), \text{ for all } a, b \in M, .
\end{aligned}$$

Hence d is a homomorphism from M into M. □

4. Derivations of fuzzy ideals in ordered semirings

In this section, we introduce the notion of characteristic ideal and fuzzy characteristic ideal with respect to derivation, derivation of image and pre-image of fuzzy ideals in ordered semirings. We prove that if μ is a fuzzy prime ideal and d is a derivation of ordered idempotent semiring M with $a+ab = a$, for all $a, b \in M$, then $d(\mu)$ and $d^{-1}(\mu)$ are fuzzy prime ideals of M.

DEFINITION 4.1. Let M be an ordered semiring. A fuzzy subset μ of M is called a fuzzy subsemiring of M if

(i) $\mu(x + y) \geqslant min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geqslant min\{\mu(x), \mu(y)\}$
(iii) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$, for all $x, y \in M$.

DEFINITION 4.2. Let μ be a non-empty fuzzy subset of an ordered semiring M. Then μ is called a fuzzy left (right) ideal of M if

(i) $\mu(x + y) \geqslant min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geqslant \mu(y)\big(\mu(x)\big)$
(iii) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$, for all $x, y \in M$.

DEFINITION 4.3. Let μ be a non-empty fuzzy subset of an ordered semiring M. Then μ is called a fuzzy prime ideal of M if

(i) $\mu(x + y) \geqslant min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) = max\{\mu(x), \mu(y)\}$
(iii) $x \leqslant y \Rightarrow \mu(x) \geqslant \mu(y)$, for all $x, y \in M$.

DEFINITION 4.4. An ideal I of an ordered semiring M is said to be characteristic if $d(I) = I$, for all derivations d of M.

DEFINITION 4.5. A fuzzy ideal μ of an ordered semiring M is said to be fuzzy ideal characteristic if $\mu(d(x)) = \mu(x)$, for all derivations d of M and $x \in M$.

THEOREM 4.1. *Let μ be a fuzzy ideal and d be a derivation of an idempotent ordered semiring M with $a + ab = a$ and $a + ba = a$, for all $a, b \in M$. Then the mapping $\mu^d : M \to [0, 1]$, defined by $\mu^d(x) = \mu(d(x))$, for all $x \in M$, is a fuzzy ideal of M.*

PROOF. Let μ be a fuzzy ideal, d be a derivation of an idempotent ordered semiring M with $a + ab = a$ and $a + ba = a$, for all $a, b \in M$. Suppose $x, y \in M$. Then

$$\begin{aligned}\mu^d(x+y) &= \mu(d(x+y)) \\ &= \mu(d(x)+d(y)) \\ &\geqslant \min\{\mu(d(x)), \mu(d(y))\} \\ &= \min\{\mu^d(x), \mu^d(y)\} \\ \mu^d(xy) &= \mu(d(xy)) \\ &= \mu(d(x)d(y)), \text{ by Theorem 3.10} \\ &\geqslant \max\{\mu(d(x)), \mu(d(y))\} \\ &= \max\{\mu^d(x), \mu^d(y)\}.\end{aligned}$$

Let $x, y \in M$ and $x \leqslant y$. Then

$$\begin{gathered}d(x) \leqslant d(y) \\ \Rightarrow \mu(d(x)) \geqslant \mu(d(y)). \\ \text{Therefore } \mu^d(x) \geqslant \mu^d(y).\end{gathered}$$

Hence μ^d is a fuzzy ideal of M. □

THEOREM 4.2. *Let μ be a fuzzy ideal and d be an onto derivation of an idempotent ordered semiring M with identity $a + ab = a$, for all $a, b \in M$. Then μ is a fuzzy ideal characteristic if and only if each level ideal μ of M is a characteristic ideal.*

PROOF. Suppose μ is a fuzzy ideal characteristic and $x \in \mu_s$ where $s \in [0, 1]$ and d is a derivation of M. Then

$$\begin{gathered}\mu(x) \geqslant s \\ \Rightarrow \mu((d(x)) \geqslant s \\ \Rightarrow d(x) \in \mu_s \\ \text{Therefore } d(\mu_s) \subseteq \mu_s.\end{gathered}$$

Let $x \in \mu_s$. Then there exists $y \in M$ such that $d(y) = x$.

$$\begin{gathered}\Rightarrow \mu(d(y)) = \mu(x) \geqslant s \text{ and } \mu(y) \geqslant s \ \Rightarrow y \in \mu_s \\ \Rightarrow d(y) \in \mu_s \text{ and } x = d(y) \in d(\mu_s). \\ \text{Therefore } \mu_s \subseteq d(\mu_s).\end{gathered}$$

Hence $\mu_s = d(\mu_s)$.

Conversely suppose that each level ideal μ_s of M is a characteristic, $x \in M, d$ is an onto derivation of M and $\mu(x) = s$.
Then $x \in \mu_s$ and $x \notin \mu_t$ for $s < t$
$\Rightarrow d(x) \in d(\mu_s) = \mu_s$.
By Theorem 4.1, $\mu^d : M \to [0,1]$ defined by $\mu^d(x) = \mu(d(x))$, is an fuzzy ideal.
Let $w = \mu^d(x)$ and $w > s$. Then $d(x) \in \mu_w = d(\mu_w) \Rightarrow x \in \mu_w$.
Which is a contradiction, since $w > s$.
Therefore $\mu^d(x) = \mu(d(x)) = s = \mu(x)$.
Hence μ is a fuzzy ideal characteristic. □

Definition 4.6. Let d be a derivation of an ordered semiring M and μ be a fuzzy subset of M. We define a fuzzy subset $d(\mu)$ of M by

$$d(\mu)(x) = \begin{cases} \sup\limits_{y \in d^{-1}(x)} \mu(y), & \text{if } d^{-1}(x) \neq \phi \\ 0\ , & \text{otherwise .} \end{cases}$$

Definition 4.7. Let d be an onto derivation of an ordered semiring M. If μ is a fuzzy subset of M then pre image of μ under d is the fuzzy subset of M defined by $d^{-1}(\mu)(x) = \mu(d(x))$, for all $x \in M$ and it is denoted by $d^{-1}(\mu)$.

Theorem 4.3. *Let d be a derivation of an ordered semiring M and μ be a $d-$invariant fuzzy subset of M. If $x = d(a)$ then $d(\mu)(x) = \mu(a)$, for all $a \in M$.*

Proof. Let d be a derivation of an ordered semiring M and μ be a $d-$invariant fuzzy subset of M. Suppose $x = d(a)$. Then $d^{-1}(x) = a$.
Let $t \in d^{-1}(x)$. Then $x = d(t) \Rightarrow d(a) = x = d(t)$. Since μ is a $d-$invariant fuzzy subset of $M \ \Rightarrow \mu(a) = \mu(t)$. Therefore $d(\mu)(x) = \sup\limits_{t \in d^{-1}(x)} \mu(t) = \mu(a)$. □

Theorem 4.4. *Let d be a derivation of an ordered idempotent semiring M and μ be a fuzzy ideal of M. If $\mu\ o\ d = \eta$ then η is a fuzzy ideal of an ordered semiring M.*

Proof. Let $x, y \in M$. Then

$$\begin{aligned} \eta(x+y) &= \mu(d(x+y)) \\ &= \mu(d(x) + d(y)) \\ &\geqslant \min\{\mu(d(x)), \mu(d(y))\} \\ &= \min\{\eta(x), \eta(y)\} \\ \eta(xy) &= \mu(d(xy)) \\ &= \mu[d(x)d(y)] \\ &\geqslant \max\{\mu(d(x)), \mu(d(y))\} \\ &= \max\{\eta(x), \eta(y)\}. \end{aligned}$$

Suppose $x, y \in M$ and $x \leqslant y$. Since d is a derivation of M, we have

$$\begin{aligned} d(x) &\leqslant d(y) \\ \Rightarrow \mu(d(x)) &\geqslant \mu(d(y)). \\ \text{Therefore } \eta(x) &\geqslant \eta(y). \end{aligned}$$

Hence η is a fuzzy ideal of M. □

THEOREM 4.5. *Let M be an idempotent ordered semiring with identity $a+ab = a$, for all $a, b \in M$, and $d : M \to M$ be an onto derivation of M. If μ is a derivation $d-$invariant fuzzy prime ideal of M then $d(\mu)$ is a fuzzy prime ideal of M.*

PROOF. Let M be an idempotent ordered semiring with $a + ab = a$, for all $a, b \in M$.
Suppose $d(a) = x$.By Theorem 4.3, we have $d(\mu)(x) = \mu(a)$.
Let $x, y \in M$. Then there exist $a, b \in M$ such that $d(a) = x, d(b) = y$.

$$\begin{aligned} \text{Then } d(a+b) &= d(a) + d(b) = x + y \\ \Rightarrow d(\mu)(x+y) &= \mu(a+b) \\ &\geqslant \min\{\mu(a), \mu(b)\} \\ &= \min\{d(\mu)(x), d(\mu)(y)\}. \\ \text{By Theorem 3.10, } d(ab) &= d(a)d(b) = xy. \\ \text{Then } d(\mu)(xy) &= \sup_{t \in d^{-1}(x)} \mu(t) \\ &= \mu(ab) \\ &= \max\{\mu(a), \mu(b)\} \\ &= \max\{d(\mu(x)), d(\mu(y))\}. \end{aligned}$$

Let $x, y \in M$ and $x \leqslant y$. Then there exist $a, b \in M$ such that $d(a) = x$ and $d(b) = y$. Therefore $x \leqslant y$. Then

$$\begin{aligned} d(a) &\leqslant d(b) \\ \Rightarrow \mu(a) &\geqslant \mu(b) \\ \Rightarrow d(\mu)(x) &\geqslant d(\mu)(y). \end{aligned}$$

Hence $d(\mu)$ is a fuzzy ideal of M. □

THEOREM 4.6. *If μ is a fuzzy prime ideal and d is a derivation of an idempotent ordered semiring M with $a + ab = a$, for all $a, b \in M$, then $d^{-1}(\mu)$ is a fuzzy prime ideal of M.*

Proof. Suppose d is a derivation of an idempotent ordered semiring M, μ is a fuzzy ideal of M and $x_1, x_2 \in M$.

$$\begin{aligned} d^{-1}(\mu)(x_1 + x_2) &= \mu(d(x_1 + x_2)) \\ &= \mu[d(x_1) + d(x_2)] \\ &\geqslant \min\{\mu(d(x_1) + \mu d(x_2)\} \\ &= \min\{d^{-1}(\mu(x_1)), d^{-1}(\mu(x_2))\}. \\ d^{-1}(\mu)(x_1 x_2) &= \mu(d(x_1 x_2)) \\ &= \mu(d(x_1)d(x_2)) \\ &= \max\{\mu(d(x_1)), \mu(d(x_2))\} \\ &= \max\{d^{-1}(\mu(x_1)), d^{-1}(\mu(x_2))\}. \end{aligned}$$

Let $x_1, x_2 \in M$ and $x_1 \leqslant x_2$. Then

$$\Rightarrow d(x_1) \leqslant d(x_2)$$
$$\Rightarrow \mu(d(x_1)) \geqslant \mu(d(x_2))$$
$$\Rightarrow d^{-1}(\mu(x_1)) \geqslant d^{-1}(\mu(x_2)).$$

Therefore $d^{-1}(\mu)$ is a fuzzy prime ideal of M. □

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Received by editors 22.12.2021; Revised version 14.5.2023; Available online 11.6.2023.

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