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SHEFFER STROKE INK-ALGEBRAS

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Abstract. In this paper, a Sheffer stroke INK-algebra is introduced, and their properties are investigated. The independence of the axiom system of a Sheffer stroke INK-algebra has been demonstrated. After describing a G-part and a p-semisimple of a Sheffer stroke INK-algebra, the properties of this structures are examined. An ideal of Sheffer stroke INK-algebra is determined. The relations of a subalgebra and a closed ideal is given. A medial of Sheffer stroke INK-algebra is identified and its features are shown. Finally, a homomorphism between Sheffer stroke INK-algebras is introduced and it is presented that ideal is preserved under this homomorphism. Moreover, a kernel of a homomorphism is built and it is proved that the kernel is an ideal.

1. Introduction

The study of BCK/BCI-algebras was initiated by Imai and Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus [**15**]. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebra [**5, 8**]. Then Hu and Li (1983) are expanding algebra is called BCH-algebra, which is a generalization of BCK-and BCI-algebras. J. Neggers, S. S. Ahn, and H. S. Kim introduced a notion called Q-algebra, which is a generalization of BCH/ BCI/BCK-algebras, and generalized some theorems discussed in BCI-algebra [**9**].M. Kaviyarasu, K. Indhira, and V. M. Chandrasekaran introduced a new notion, called INK-algebras which are related to several classes of algebras of interest such as BCK/BCI-algebras [**6**]. Cho and Kim discussed further relations between INK- algebras and other topics especially quasigroups [**4**]. A. Borumand Saeid introduced the notion of fuzzy topological B-algebras [**1**]. Using the notion

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of interval-valued fuzzy set, A. Borumand Saeid introduced the concept of interval-valued fuzzy subalgebras of a B-algebra, and studied some of their properties [**2**]. For the general development of INK-algebras, the ideal theory and subalgebras play important role.

The Sheffer stroke operation, which was first introduced by H. M. Sheffer [**14**], engages many scientists' attention, because any Boolean function or axiom can be expressed by means of this operation [**7**]. It reducts axiom systems of many algebraic structures. So, many researchers want to use this operation on their studies. For example, relation between Sheffer stroke operation and Hilbert algebras [**10**], filters of strong Sheffer stroke non-associative MV-algebras [**11**], (Fuzzy) filters of Sheffer stroke BL-algebras [**12**] and the Sheffer stroke operation reducts of basic algebras [**13**] are given as some research on Sheffer stroke operation in recent years.

After giving basic definitions and notions about a Sheffer stroke and a INK-algebra, it is defined a Sheffer stroke INK-algebra. By presenting fundamental notions about this algebraic structure, it is proved that the axiom system of a Sheffer stroke INK-algebra is independent. It is demonstrated the relationships between a Sheffer stroke INK-algebra and a INK-algebra. A G-part of a Sheffer stroke INK-algebra and a p-semi simple Sheffer stroke INK-algebra are defined and some features related to these structures are given. After determining an ideal, a closed ideal, a subalgebra and a p-ideal of Sheffer stroke INK-algebra, the relationship between this structures are shown. Then a medial of Sheffer stroke INK-algebra is described and it is proved that $G(A)$ is a subalgebra of Sheffer stroke INK-algebra if $(A, |, 0)$ is a medial of Sheffer stroke INK-algebra. Finally defining a homomorphism between Sheffer stroke INK-algebras, it is demonstrated that the ideal is preserved under this homomorphism. Moreover, a kernel of a homomorphism is built and it is proved that the kernel is an ideal.

2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke and a INK-algebra.

Definition 2.1. [**3**] Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be *Sheffer stroke* if it satisfies the following conditions:
$(S1)$ $a_1|a_2 = a_2|a_1$,
$(S2)$ $(a_1|a_1)|(a_1|a_2) = a_1$,
$(S3)$ $a_1|((a_2|a_3)|(a_2|a_3)) = ((a_1|a_2)|(a_1|a_2))|a_3$,
$(S4)$ $(a_1|((a_1|a_1)|(a_2|a_2)))|(a_1|((a_1|a_1)|(a_2|a_2))) = a_1$,
for all $a_1, a_2, a_3 \in A$.

Definition 2.2. [**6**] A INK-algebra is a nonempty set A with a constant 0 and a binary operation $*$ satisfying the following axioms:
$(INK.1)$ $a_1 * a_1 = 0$,
$(INK.2)$ $a_1 * 0 = a_1$,
$(INK.3)$ $0 * a_1 = a_1$,
$(INK.4)$ $(a_2 * a_1) * (a_2 * a_3) = a_1 * a_3$,
for all $a_1, a_2, a_3 \in A$.

Definition 2.3. [**6**] Let $(A, *, 0)$ be a INK-algebra. For any nonempty subset S of A, we define

$$G(S) = \{a_1 \in S | 0 * a_1 = a_1\}.$$

In particular, if $S = A$, then we say that $G(A)$ is the G-part of A.

Definition 2.4. [**6**] For any INK-algebra, $(A, *, 0)$, the set

$$B(A) = \{a_1 \in A | 0 * a_1 = 0\}$$

is called the p-radical of A. If $B(A) = \{0\}$, then we say that A is a p-semisimple INK-algebra.

Definition 2.5. [**6**] Let $(A, *, 0)$ be a INK-algebra. A nonempty subset I of A is called an ideal of A if it satisfies:
(i) $0 \in I$,
(ii) $a_1 * a_2 \in I$ and $a_2 \in I$ imply $a_1 \in I$,
for all $a_1, a_2 \in A$.

Definition 2.6. [**6**] An ideal I of a INK-algebra $(A, *, 0)$ is called closed if $0 * a_1 \in I$, for all $a_1 \in I$.

Definition 2.7. [**6**] A nonempty subset S of a INK-algebra $(A, *, 0)$ is called a subalgebra of A if $a_1 * a_2 \in S$, for all $a_1, a_2 \in S$.

Definition 2.8. [**6**] A INK-algebra A is called a medial of INK-algebra if

$$(a_1 * a_2) * (a_3 * a_4) = (a_1 * a_3) * (a_2 * a_4)$$

holds for all $a_1, a_2, a_3, a_4 \in A$.

3. Sheffer stroke INK-algebras

In this part, we define a Sheffer stroke INK-algebra and give some properties.

Definition 3.1. A Sheffer stroke INK-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A and the following axioms are satisfied:
$(sINK.1)$ $(0|(a_1|a_1))|(0|(a_1|a_1)) = a_1$,
$(sINK.2)$ $((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)) = a_1|(a_3|a_3)$,
for all $a_1, a_2, a_3 \in A$.

A partial order $\leqslant$ on A is defined by

$$a_1 \leqslant a_2 \quad iff \quad (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0.$$

Let A be a Sheffer stroke INK-algebra, unless otherwise is indicated.

Lemma 3.1. *The axioms* $(sINK.1)$ *and* $(sINK.2)$ *are independent.*

Proof. Independence of (sINK.1):
We construct an example for this axiom which is false while (sINK.2) is true. Let $(\{0, 1\}, |_1)$ be the groupoid defined as follows:

$\vert_1$	0	1
0	0	1
1	0	0

Then $|_1$ satisfies (sINK.2) but not (sINK.1) when $a_1 = 1$.

Independence of (sINK.2):

We construct an example for this axiom which is false while (sINK.1) is true. Let $(\{0,1\}, |_2)$ be the groupoid defined as follows:

$\vert_2$	0	1
0	0	1
1	1	1

Then $|_2$ satisfies (sINK.1) but not (sINK.2) when $a_1 = 0$, $a_2 = 1$ and $a_3 = 0$. □

Lemma 3.2. *Let A be a Sheffer stroke INK-algebra. Then the following features hold for all $a_1, a_2, a_3 \in A$:*
(1) $(a_1|(0|0))|(a_1|(0|0)) = a_1$,
(2) $(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0$,
(3) $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$,
(4) $(0|0)|(a_1|a_1) = a_1$,
(5) $((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|a_2) = a_1|a_1$,
(6) $a_2|(a_2|(a_1|a_1)) = a_1|a_1$,
(7) $a_1|((a_2|(a_3|a_3))|(a_2|(a_3|a_3))) = a_2|((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))$,
(8) $a_1|(((a_1|(a_2|a_2))|(a_2|a_2))|((a_1|(a_2|a_2))|(a_2|a_2))) = 0|0$,
(9) $((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|a_2) = 0|0$.

Proof. (1) By (sINK.1), (S1) and (S3), we obtain

$$
\begin{aligned}
(a_1|(0|0))|(a_1|(0|0)) &= (((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|0))| \\
&\quad (((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|0)) \\
&= ((0|0)|((0|(a_1|a_1))|(0|(a_1|a_1))))| \\
&\quad ((0|0)|((0|(a_1|a_1))|(0|(a_1|a_1)))) \\
&= ((((0|0)|0)|((0|0)|0))|(a_1|a_1))|((((0|0)|0)|((0|0)|0))|(a_1|a_1)) \\
&= (((0|(0|0))|(0|(0|0)))|(a_1|a_1))|(((0|(0|0))|(0|(0|0)))|(a_1|a_1)) \\
&= (0|(a_1|a_1))|(0|(a_1|a_1)) \\
&= a_1.
\end{aligned}
$$

(2) By (S2), (sINK.2) and (1), we have

$$
\begin{aligned}
(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) &= (((a_1|(0|0))|(a_1|(0|0)))|(a_1|(0|0)))| \\
&\quad (((a_1|(0|0))|(a_1|(0|0)))|(a_1|(0|0))) \\
&= (0|(0|0))|(0|(0|0)) \\
&= 0.
\end{aligned}
$$

(3) Substituting $[a_2 := (a_1|a_1)]$ in (S2), we obtain $(a_1|a_1)|(a_1|(a_1|a_1)) = a_1$. Then $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$ from (S1).

(4) By (S1), (S2) and (2), we get

$$\begin{aligned}(0|0)|(a_1|a_1) &= (((a_1|(a_1|a_1))|(a_1|(a_1|a_1)))|((a_1|(a_1|a_1))|(a_1|(a_1|a_1))))|(a_1|a_1)\\ &= (a_1|(a_1|a_1))|(a_1|a_1)\\ &= (a_1|a_1)|(a_1|(a_1|a_1))\\ &= a_1.\end{aligned}$$

(5) By (S2), (1) and (sINK.2), we obtain

$$\begin{aligned}((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|a_2) &= ((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(0|0))\\ &= (a_1|(0|0))\\ &= a_1|a_1.\end{aligned}$$

(6) By (sINK.1), (sINK.2), (S2) and (1), we get

$$\begin{aligned}a_2|(a_2|(a_1|a_1)) &= ((a_2|(0|0))|(a_2|(0|0)))|(a_2|(a_1|a_1))\\ &= 0|(a_1|a_1)\\ &= a_1|a_1.\end{aligned}$$

(7) By (S1) and (S3), we have

$$\begin{aligned}a_1|((a_2|(a_3|a_3))|(a_2|(a_3|a_3))) &= (((a_1|a_2)|(a_1|a_2))|(a_3|a_3))\\ &= (((a_2|a_1)|(a_2|a_1))|(a_3|a_3))\\ &= a_2|((a_1|(a_3|a_3))|(a_1|(a_3|a_3))).\end{aligned}$$

(8) In (S3), by substituting $[a_2 := a_1|(a_2|a_2)]$ and $[a_3 := a_2|a_2]$ and applying (S1), (S2), (S3) and (2), we get

$$\begin{aligned}&a_1|(((a_1|(a_2|a_2))|(a_2|a_2))|((a_1|(a_2|a_2))|(a_2|a_2)))\\ &= a_1|((((a_2|a_2))|(a_1|(a_2|a_2)))|(((a_2|a_2))|(a_1|(a_2|a_2))))\\ &= ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_2|a_2))\\ &= (a_1|(a_2|a_2))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\\ &= 0|0.\end{aligned}$$

(9) By (S1), (S2), (S3) and (2), we obtain

$$\begin{aligned}&((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|a_2)\\ &= ((a_2|a_2)|((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))\\ &= (((a_2|a_2)|a_1)|((a_2|a_2)|a_1))|(a_1|(a_2|a_2))\\ &= (a_1|(a_2|a_2))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))\\ &= 0|0.\end{aligned}$$

□

Proposition 3.1. *Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. Then*

$$((a_2|(a_3|a_3))|(a_2|(a_3|a_3)))|(a_1|a_1) = a_3|(a_2|(a_1|a_1)),$$

for all $a_1, a_2, a_3 \in A$.

Proof. Let $(A,|,0)$ be a Sheffer stroke INK-algebra. Put $[a_1 := (a_2|(a_3|a_3))|(a_2|(a_3|a_3))]$ and $[a_3 := a_1]$ in (sINK.2) and by using (S2), Lemma 3.2 (6), we get

$$\begin{aligned}((a_2|(a_3|a_3))|(a_2|(a_3|a_3)))|(a_1|a_1) &= ((a_2|(a_2|(a_3|a_3)))|(a_2|(a_2\\ &\quad |(a_3|a_3))))|(a_2|(a_1|a_1))\\ &= ((a_3|a_3)|(a_3|a_3))|(a_2|(a_1|a_1))\\ &= a_3|(a_2|(a_1|a_1)).\end{aligned}$$

□

Proposition 3.2. *Let $(A,|,0)$ be a Sheffer stroke INK-algebra. Then*
(1) *If $(a_1|(0|0))|(a_1|(0|0)) = 0$ then $a_1 = 0$,*
(2) $((((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)))|((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3))) \leqslant (a_1|(a_3|a_3))|(a_1|(a_3|a_3))$,
(3) *If $a_1 \leqslant a_2$, then $((a_2|(a_1|a_1))|(a_2|(a_1|a_1))) \leqslant ((a_2|(a_3|a_3))|(a_2|(a_3|a_3)))$ and $((a_3|(a_2|a_2))|(a_3|(a_2|a_2))) \leqslant ((a_3|(a_1|a_1))|(a_3|(a_1|a_1)))$,*
(4) $a_2|(a_2|(a_2|(a_1|a_1))) = a_2|(a_1|a_1)$,
(5) $0|(a_2|(a_1|a_1)) = ((0|(a_2|a_2))|(0|(a_2|a_2)))|(0|(a_1|a_1))$,
(6) $((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_1|a_1) = 0|0$,
(7) *If $a_1|(a_2|a_2) = 0|0$ and $a_2|(a_1|a_1) = 0|0$ then $a_1 = a_2$,*
for all $a_1, a_2, a_3 \in A$.

Proof. (1) $(a_1|(0|0))|(a_1|(0|0)) = a_1$ from Lemma 3.2 (1). If $(a_1|(0|0))|(a_1|(0|0)) = 0$, then $a_1 = 0$.

(2) By (S2) and (sINK.2), we get
$((((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)))|((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))$
$|(a_2|(a_3|a_3)))|((a_1|(a_3|a_3))|(a_1|(a_3|a_3))|(a_1|(a_3|a_3))|(a_1|(a_3|a_3)))$
$= ((((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)))|((a_2|$
$(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)))|(a_1|(a_3|a_3))$
$= ((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_1|(a_3|a_3))$
$= (a_1|(a_3|a_3))|((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))$
$= 0|0.$
Hence $((((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)))|((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3))) \leqslant (a_1|(a_3|a_3))|(a_1|(a_3|a_3))$ from (S2).

(3) Let $a_1 \leqslant a_2$. Then $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$. Since $(a_1|(a_3|a_3))|(a_1|(a_3|a_3)) = 0$, by $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$, by (2), we get $((((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)))|((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3))) = 0$. Hence, $((a_2|(a_1|a_1))|(a_2|(a_1|a_1))) \leqslant ((a_2|(a_3|a_3))|(a_2|(a_3|a_3)))$. Similiarly, $((a_3|(a_2|a_2))|(a_3|(a_2|a_2))) \leqslant ((a_3|(a_1|a_1))|(a_3|(a_1|a_1)))$.

(4) By (S2), Lemma 3.2 (1), (sINK.1) and (sINK.2), we obtain

$$\begin{aligned}a_2|(a_2|(a_2|(a_1|a_1))) &= ((a_2|(0|0))|(a_2|(0|0)))|(a_2|(a_2|(a_1|a_1)))\\ &= 0|(a_2|(a_1|a_1))\\ &= 0|((a_2|(a_1|a_1))|(a_2|(a_1|a_1))|(a_2|(a_1|a_1))|(a_2|(a_1|a_1)))\\ &= a_2|(a_1|a_1).\end{aligned}$$

(5) By Lemma 3.2 (2), (S2), (sINK.1) and (sINK.2), we have

$$\begin{aligned}0|(a_2|(a_1|a_1)) &= ((a_2|(a_2|a_2))|(a_2|(a_2|a_2)))|(a_2|(a_1|a_1))\\ &= (a_2|(a_1|a_1))\\ &= ((0|(a_2|a_2))|(0|(a_2|a_2)))|(0|(a_1|a_1)).\end{aligned}$$

(6) By Lemma 3.2 (2), (6) and (S2), we get

$$\begin{aligned}((a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1))))|(a_1|a_1) &= ((a_1|a_1)|(a_1|a_1))|(a_1|a_1)\\ &= a_1|(a_1|a_1)\\ &= 0|0.\end{aligned}$$

(7) If $a_1|(a_2|a_2) = 0|0$ and $a_2|(a_1|a_1) = 0|0$ then

$$\begin{aligned}a_1 &= (a_1|(0|0))|(a_1|(0|0))\\ &= (a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2)))\\ &= (a_2|a_2)|(a_2|a_2)\\ &= a_2\end{aligned}$$

and

$$\begin{aligned}a_2 &= (a_2|(0|0))|(a_2|(0|0))\\ &= (a_2|(a_2|(a_1|a_1)))|(a_2|(a_2|(a_1|a_1)))\\ &= (a_1|a_1)|(a_1|a_1)\\ &= a_1\end{aligned}$$

from (S2), Lemma 3.2 (1) and (6). □

THEOREM 3.1. *Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. If we define*

$$a_1 * a_2 := (a_1|(a_2|a_2))|(a_1|(a_2|a_2)),$$

then $(A, *, 0)$ is a INK-algebra.

PROOF. By using (S2), (sINK.1), (sINK.2), Lemma 3.2 (1) and (2), we have
$(INK.1): a_1 * a_1 = (a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0.$
$(INK.2)$: $a_1 * 0 = (a_1|(0|0))|(a_1|(0|0)) = a_1.$
$(INK.3)$: $0 * a_1 = (0|(a_1|a_1))|(0|(a_1|a_1)) = a_1.$
$(INK.4)$:

$$\begin{aligned}(a_2 * a_1) * (a_2 * a_3) &= (((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)))|\\ &\quad\ (((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)))\\ &= (a_1|(a_3|a_3))|(a_1|(a_3|a_3))\\ &= a_1 * a_3.\end{aligned}$$

Then $(A, *, 0)$ is a INK-algebra. □

THEOREM 3.2. *Let $(A, *, 0, 1)$ be a bounded INK-algebra. If we define $a_1|a_2 := (a_1 * a_2^0)^0$ and $a_1^0 = 1 * a_1$, where $a_1 * (1 * a_1) = a_1$ and $1 * (1 * a_1) = a_1$, then $(A, |, 0)$ is a Sheffer stroke INK-algebra.*

PROOF. $(sINK.1)$: By using (INK.3), we have

$$\begin{aligned}(0|(a_1|a_1))|(0|(a_1|a_1)) &= (0|a_1^0)|(0|a_1^0)\\ &= (0 * a_1)^0|(0 * a_1)^0\\ &= ((0 * a_1)^0)^0\\ &= 0 * a_1\\ &= a_1.\end{aligned}$$

$(sINK.2)$: By using (INK.4), we obtain

$$\begin{aligned}((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_3|a_3)) &= ((a_2 * a_1)^0|(a_2 * a_1)^0)|(a_2 * a_3)^0\\ &= ((a_2 * a_1)^0)^0|(a_2 * a_3)^0\\ &= (a_2 * a_1)|(a_2 * a_3)^0\\ &= ((a_2 * a_1) * (a_2 * a_3))^0\\ &= (a_1 * a_3)^0\\ &= (a_1|a_3^0)\\ &= (a_1|(a_3|a_3)).\end{aligned}$$

Then $(A, |, 0)$ is a Sheffer stroke INK-algebra. □

4. The G-part of Sheffer stroke INK-algebras

In this section, we investigate the some properties of the G-part in Sheffer stroke INK-algebras.

DEFINITION 4.1. Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. For any non-empty subset S, of A, we define

$$G(S) = \{a_1 \in S : (0|(a_1|a_1)) = a_1|a_1\}.$$

In particular, if $S = A$, then we say that $G(A)$ is the G-part of A.

DEFINITION 4.2. For any Sheffer stroke INK-algebra $(A, |, 0)$, the set

$$B(A) = \{a_1 \in A : (0|(a_1|a_1)) = 0|0\}$$

is called the p-radical of A. If $B(A) = \{0\}$, then we say that A is a p-semisimple Sheffer stroke INK-algebra.

PROPOSITION 4.1. *Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. Then $a_1 \in G(A)$ if and only if $(0|(a_1|a_1))|(0|(a_1|a_1)) \in G(A)$.*

Proof. Let $(A, |, 0)$ be a Sheffer stroke INK-algebra and $a_1 \in G(A)$. Then $0|(0|(a_1|a_1)) = 0|(a_1|a_1)$. Hence $(0|(a_1|a_1))|(0|(a_1|a_1)) \in G(A)$ from (S2). Conversely, if $(0|(a_1|a_1))|(0|(a_1|a_1)) \in G(A)$, then $0|(0|(a_1|a_1)) = 0|(a_1|a_1)$. By Lemma 3.2 (6), we obtain $a_1|a_1 = 0|(0|(a_1|a_1)) = (0|(a_1|a_1))$. We get $(0|(a_1|a_1)) = a_1|a_1$. Therefore $a_1 \in G(A)$. □

Proposition 4.2. *Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. Then A is a p-semisimple if and only if $0|(0|(a_1|a_1)) = a_1|a_1$, for all $a_1 \in A$.*

Proof. Let A be a p-semisimple. Then $B(A) = \{0\}$ that is $(0|(0|0)) = 0|0$ and $0|(a_1|a_1) \neq 0|0$ for all $a_1 \neq 0$. By using (sINK.1), (sINK.2) and (S2), we get

$$\begin{aligned} 0|(0|(a_1|a_1)) &= ((0|(0|0))|(0|(0|0)))|(0|(a_1|a_1)) \\ &= 0|(a_1|a_1) \\ &= a_1|a_1. \end{aligned}$$

Conversely, let $0|(0|(a_1|a_1)) = a_1|a_1$ implies $(0|(a_1|a_1)) \neq 0|0$. If $(0|(a_1|a_1)) = 0|0$ for $a_1 \neq 0$, then $((0|(0|0)) = a_1|a_1$ which implies $a_1 = 0$. Hence A is a p-semisimple. □

Proposition 4.3. *Let $(A, |, 0)$ is a Sheffer stroke INK-algebra and $a_1, a_2 \in A$. Then $a_2 \in B(A)$ if and only if $a_1|(a_1|(a_2|a_2)) = 0|0$.*

Proof. Let $a_2 \in B(A)$ then $(0|(a_2|a_2)) = 0|0$. By using Lemma 3.2 (1), (S2) and (sINK.2), we have

$$\begin{aligned} a_1|(a_1|(a_2|a_2)) &= ((a_1|(0|0))|(a_1|(0|0)))|(a_1|(a_2|a_2)) \\ &= 0|(a_2|a_2) \\ &= 0|0. \end{aligned}$$

Conversely, let $a_1|(a_1|(a_2|a_2)) = 0|0$. Then

$$\begin{aligned} 0|0 &= a_1|(a_1|(a_2|a_2)) \\ &= ((a_1|(0|0))|(a_1|(0|0)))|(a_1|(a_2|a_2)) \\ &= 0|(a_2|a_2) \end{aligned}$$

and so $a_2 \in B(A)$. □

Proposition 4.4. *Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. If $G(A) = A$, then A is a p-semisimple.*

Proof. Let $G(A) = A$. Also $G(A) \cap B(A) = \{0\}$. So $A \cap B(A) = \{0\}$. That is $B(A) = \{0\}$. Hence A is a p-semisimple. □

5. On ideals of Sheffer stroke INK-algebras

Definition 5.1. Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. A nonempty subset I of A is called an ideal of A if it satisfies:
(i) $0 \in I$,
(ii) $((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \in I$ and $a_2 \in I$ imply $a_1 \in I$,
for all $a_1, a_2 \in A$.

Definition 5.2. An ideal I of a Sheffer stroke INK-algebra $(A,|,0)$ is called closed if

$$(0|(a_1|a_1))|(0|(a_1|a_1)) \in I$$

for all $a_1 \in I$.

Definition 5.3. A non-empty subset S of a Sheffer stroke INK-algebra $(A,|,0)$ is said to be a subalgebra of A if

$$((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \in S$$

for all $a_1, a_2 \in S$.

Proposition 5.1. *Let $(A,|,0)$ be a Sheffer stroke INK-algebra. Then $B(A)$ is an ideal of A.*

Proof. Since $0|(0|0) = 0|0$, then $0 \in B(A)$. Let $((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \in B(A)$ and $a_2 \in B(A)$. Then $0|(a_1|(a_2|a_2)) = 0|0$ and $0|(a_2|a_2) = 0|0$. By using (S2), Proposition 3.2 (5), Lemma 3.2 (1), we obtain

$$\begin{aligned} 0|0 &= 0|(a_1|(a_2|a_2)) \\ &= ((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2)) \\ &= ((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|0) \\ &= 0|(a_1|a_1). \end{aligned}$$

Therefore, $a_1 \in B(A)$. Hence $B(A)$ is an ideal of A. □

Lemma 5.1. *Let $(A,|,0)$ be a Sheffer stroke INK-algebra. Then $B(A)$ is a subalgebra of A.*

Proof. Clearly, $0 \in B(A)$ and $B(A)$ is non-empty. Let $a_1, a_2 \in B(A)$. By (S2), (sINK.1) and Proposition 3.2 (5), we have

$$\begin{aligned} (0|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2)))) &= (0|(a_1|(a_2|a_2))) \\ &= ((0|(a_1|a_1))|(0|(a_1 \\ &\quad |a_1)))|(0|(a_2|a_2)) \\ &= ((0|0)|(0|0))|(0|0) \\ &= (0|(0|0)) \\ &= 0|0. \end{aligned}$$

Then, $B(A)$ is a subalgebra of A. □

Lemma 5.2. *Let $(A,|,0)$ be a Sheffer stroke INK-algebra and S is a subalgebra of A. Then S is an ideal of A.*

Proof. Let S be a subalgebra of A and $a_1, a_2 \in S$. Then
(i) $(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0 \in S$,
(ii) $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in S$ and $a_2 \in S$ imply $a_1 \in S$.
Therefore, S is an ideal of A. □

PROPOSITION 5.2. *Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. Every closed ideal of A is a Sheffer stroke INK-subalgebra.*

PROOF. Let I be a closed ideal of a Sheffer stroke $(A, |, 0)$ and $a_1, a_2 \in I$. Then $((0|(a_1|a_1))|(0|(a_1|a_1)))$, $((0|(a_2|a_2))|(0|(a_2|a_2))) \in I$. By Proposition 3.2 (5), we obtain $((0|(a_1|(a_2|a_2)))|(0|(a_1|(a_2|a_2))) = (((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2)))|(((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2)))$. Hence, $((0|(a_1|(a_2|a_2)))|(0|(a_1|(a_2|a_2))) \in I$. So, $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in I$. Thus, I is a Sheffer stroke INK-subalgebra. □

DEFINITION 5.4. Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. A nonempty subset I of A is called a p-ideal of A if it satisfies:
(i) $0 \in I$,
(ii) $((((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_3|a_3)))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_3|a_3)))) \in I$ and $a_2 \in I$ imply $a_1 \in I$,
for all $a_1, a_2, a_3 \in A$.

THEOREM 5.1. *Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. Then every p-ideal of A is an ideal of A.*

PROOF. Let I be a p-ideal of A. Putting $[a_3 := a_2]$ in Definition 5.4 (ii) and by Lemma 3.2 (1), (2), Definition 5.1, we have
$(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_2|(a_2|a_2)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_2|(a_2|a_2)))$
$= (((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|0))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|0))$
$= ((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \in I$
and $a_2 \in I$ imply $a_1 \in I$. Therefore, I is an ideal of A. □

LEMMA 5.3. *Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. Then*

$$((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) = ((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)$$

for all $a_1, a_2, a_3 \in A$.

PROOF. Let $(A, |, 0)$ be a Sheffer stroke INK-algebra. By (S2), (sINK.2), Proposition 3.1, Lemma 3.2 (2) and (5), we have

$$\begin{aligned}
&((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3)))\\
&|((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)\\
&= a_3|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2))\\
&= a_3|(((((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_1|(a_2|a_2))))|((((a_1|(a_3|a_3))|\\
&\quad (a_1|(a_3|a_3)))|(a_1|(a_2|a_2))))|(a_2|a_2))\\
&= a_3|(((a_3|(a_2|a_2))|(a_3|(a_2|a_2)))|(a_2|a_2))\\
&= (((a_3|(a_2|a_2))|(a_3|(a_2|a_2)))|(a_3|a_3))|(((a_3|(a_2|a_2))|(a_3|(a_2|a_2)))|\\
&\quad (a_3|a_3))|(a_2|a_2)\\
&= (a_2|(a_2|a_2))\\
&= 0|0.
\end{aligned}$$

Similarly, $((((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)))$ $|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) = 0|0$. So, we obtain

$$\begin{aligned}
&((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3)))\\
&= ((((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)))
\end{aligned}$$

from Proposition 3.2 (7). By (S2), we get $((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) = ((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)$. □

THEOREM 5.2. *If* $(A, |, 0)$ *is a Sheffer stroke INK-algebra for order 3. Then*
(i) $G(A) \neq A$,
(ii) $G(A)$ *is an ideal of* A *if* $|G(A)| = 1$.

PROOF. *(i)* Let $A = \{0, a, b\}$ be a Sheffer stroke INK-algebra. Assume $G(A) = A$. Then $0|(0|0) = 0|0$, $0|(a|a) = a|a$, $0|(b|b) = b|b$. Also we know $a_1|(a_1|a_1) = 0|0$ and $a_1|(0|0) = a_1|a_1$, for all $a_1 \in A$. Therefore, $a|(0|0) = a|a$, $a|(a|a) = 0|0$ and $b|(b|b) = 0|0$.

Let $a|(b|b) = 0|0$. It is argued for $b|(a|a) = 0|0, a|a, b|b$.

If $b|(a|a) = 0|0$ then $a|(b|b) = 0|0 = b|(a|a)$ and

$$\begin{aligned}((a|(b|b))|(a|(b|b)))|(a|a) &= ((a|(b|b))|(a|(b|b)))|(a|(0|0)) \\ &= b|(0|0) \\ &= b|b\end{aligned}$$

and

$$\begin{aligned}((b|(a|a))|(b|(a|a)))|(a|a) &= ((b|(a|a))|(b|(a|a)))|(a|(0|0)) \\ &= a|(0|0) \\ &= a|a\end{aligned}$$

from (sINK.2) and Lemma 3.2 (1). Since

$$((a|(b|b))|(a|(b|b)))|(a|a) = ((b|(a|a))|(b|(a|a)))|(a|a),$$

it follows that $a = b$, a contradiction. So, $b|(a|a) \neq 0|0$.

If $b|(a|a) = a|a$ and by using Lemma 5.3 and (sINK.1) then

$$\begin{aligned}a|a &= b|(a|a) \\ &= ((0|(b|b))|(0|(b|b)))|(a|a) \\ &= ((0|(a|a))|(0|(a|a)))|(b|b) \\ &= a|(b|b) \\ &= 0|0,\end{aligned}$$

which is a contradiction.

If $b|(a|a) = b|b$ then

$$\begin{aligned}b|b &= b|(a|a) \\ &= ((0|(b|b))|(0|(b|b)))|(a|a) \\ &= ((0|(a|a))|(0|(a|a)))|(b|b) \\ &= a|(b|b) \\ &= 0|0,\end{aligned}$$

which is a contradiction.

Let $a|(b|b) = a|a$ then by using Lemma 3.2 (2), (S2) and (sINK.1), we get

$$\begin{aligned} ((a|(a|(b|b)))|(a|(a|(b|b))))|(b|b) &= ((a|(a|a))|(a|(a|a)))|(b|b) \\ &= 0|(b|b) \\ &= b|b \\ &\neq 0|0, \end{aligned}$$

which is a contradiction with Proposition 3.2 (6).

Finally, let $a|(b|b) = b|b$. It is argued for $b|(a|a) = 0|0, a|a, b|b$.

If $b|(a|a) = 0|0$ then

$$\begin{aligned} b|b &= a|(b|b) \\ &= ((0|(a|a))|(0|(a|a)))|(b|b) \\ &= ((0|(b|b))|(0|(b|b)))|(a|a) \\ &= b|(a|a) \\ &= 0|0, \end{aligned}$$

which is a contradiction.

If $b|(a|a) = a|a$ then

$$\begin{aligned} b|b &= a|(b|b) \\ &= ((0|(a|a))|(0|(a|a)))|(b|b) \\ &= ((0|(b|b))|(0|(b|b)))|(a|a) \\ &= b|(a|a) \\ &= a|a, \end{aligned}$$

which is a contradiction.

If $b|(a|a) = b|b$ then

$$\begin{aligned} a|a &= 0|(a|a) \\ &= ((b|(b|b))|(b|(b|b)))|(a|a) \\ &= ((b|(a|a))|(b|(a|a)))|(b|b) \\ &= b|(b|b) \\ &= 0|0, \end{aligned}$$

which is a contradiction.

Thus, it is conclued that there exist some other elements in $G(A)$ which is not in A.

(ii) Let $A = \{0, a, b\}$ be a Sheffer stroke INK-algebra of order 3. If the order of $G(A)$ is 1, then $G(A) = \{0\}$ is the trivial ideal of A.

Conversely, assume $G(A)$ is an ideal of A. By (i), $|G(A)| = 1$ or $|G(A)| = 2$. Let $|G(A)| = 2$. Then $G(A) = \{0, a\}$ or $G(A) = \{0, b\}$. If $G(A) = \{0, a\}$, since

$G(A)$ is an ideal of A, $((b|(a|a))|(b|(a|a))) \notin G(A)$. So $((b|(a|a))|(b|(a|a))) = b$.

$$\begin{aligned} a|a &= (0|(a|a)) \\ &= ((b|(b|b))|(b|(b|b)))|(a|a) \\ &= ((b|(a|a))|(b|(a|a)))|(b|b) \\ &= b|(b|b) \\ &= 0|0, \end{aligned}$$

which is a contradiction. Hence $|G(A)| \neq 2$ and so $|G(A)| = 1$. □

Definition 5.5. A Sheffer stroke INK-algebra A is called a medial of Sheffer stroke INK-algebra if

$$((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|(a_4|a_4)) = ((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_4|a_4))$$

holds for all $a_1, a_2, a_3, a_4 \in A$.

Theorem 5.3. *Let $(A, |, 0)$ be a medial of Sheffer stroke INK-algebra. Then $G(A)$ is a subalgebra of A.*

Proof. Let $a_1, a_2 \in G(A)$. Then

$$((0|(a_1|a_1))|(0|(a_1|a_1))) = a_1 \text{ and } ((0|(a_2|a_2))|(0|(a_2|a_2))) = a_2.$$

Hence,

$$\begin{aligned} ((0|(a_1|(a_2|a_2)))|(0|(a_1|(a_2|a_2)))) &= (((0|(0|0))|(0|(0|0)))|(a_1|(a_2|a_2)))| \\ &\quad (((0|(0|0))|(0|(0|0)))|(a_1|(a_2|a_2))) \\ &= (((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2)))| \\ &\quad (((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2))) \\ &= ((a_1|(a_2|a_2))|(a_1|(a_2|a_2))). \end{aligned}$$

Therefore, $((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \in G(A)$. So, $G(A)$ is a subalgebra of A. □

Theorem 5.4. *Let $(A, |, 0)$ be a medial of Sheffer stroke INK-algebra. Then*
(i) $((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) = a_1|(a_2|(a_3|a_3))$,
(ii) $a_1|(a_2|(a_3|a_3)) = a_2|(a_1|(a_3|a_3))$,
(iii) $a_1|(a_2|a_2) = a_2|(a_1|a_1)$,
for all $a_1, a_2, a_3 \in G(A)$.

Proof. *(i)* By using Lemma 3.2 (1), (S2) and (sINK.1), we obtain

$$\begin{aligned} ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) &= ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(a_3|a_3)) \\ &= ((a_1|(0|0))|(a_1|(0|0)))|(a_2|(a_3|a_3)) \\ &= a_1|(a_2|(a_3|a_3)). \end{aligned}$$

(ii) By using (sINK.1), we get

$$\begin{aligned} a_1|(a_2|(a_3|a_3)) &= ((0|(a_1|a_1))|(0|(a_1|a_1)))|(a_2|(a_3|a_3)) \\ &= ((0|(a_2|a_2))|(0|(a_2|a_2)))|(a_1|(a_3|a_3)) \\ &= a_2|(a_1|(a_3|a_3)). \end{aligned}$$

(iii) By using Lemma 3.2 (1) and (2), (sINK.1) we get

$$\begin{aligned} a_1|(a_2|a_2) &= (0|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2)))) \\ &= (0|(a_1|(a_2|a_2))) \\ &= ((a_2|(a_2|a_2))|(a_2|(a_2|a_2)))|(a_1|(a_2|a_2)) \\ &= ((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(a_2|(a_2|a_2)) \\ &= ((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))|(0|0) \\ &= a_2|(a_1|a_1). \end{aligned}$$

□

6. Homomorphism on Sheffer stroke INK-algebras

In this section, we present some definitions and concepts about homomorphism between Sheffer stroke INK-algebras.

DEFINITION 6.1. Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke INK-algebras. A mapping $f : A \to B$ is called a homomorphism if

$$f(a_1|_A a_2) = f(a_1)|_B f(a_2)$$

for all $a_1, a_2 \in A$.

LEMMA 6.1. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke INK-algebras and $f : A \to B$ be a homomorphism. If I is an ideal of A, then $f(I)$ is an ideal of B.*

PROOF. Suppose that I is an ideal of A. Since $0_A \in I$, we obtain $0_B = f(0_A) \in f(I)$. Let $f(a_2), f((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2))) = (f(a_1)|_B(f(a_2)|_B f(a_2)))|_B (f(a_1)|_B(f(a_2)|_B f(a_2))) \in f(I)$. We get $a_2, (a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)) \in I$. Since I is an ideal of A, $a_1 \in I$, i.e., $f(a_1) \in f(I)$. □

LEMMA 6.2. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke INK-algebras and $f : A \to B$ be a homomorphism. If I is an ideal of B, then $f^{-1}(I)$ is an ideal of A.*

PROOF. Suppose that I is an ideal of B. Since $f(0_A) = 0_B \in I$, we have $0_A = f^{-1}f(0_A) = f^{-1}(0_B) \in f^{-1}(I)$. Let $a_2, (a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)) \in f^{-1}(I)$, i.e., $f(a_2), (f(a_1)|_B(f(a_2)|_B f(a_2)))|_B(f(a_1)|_B(f(a_2)|_B f(a_2))) = f(a_1|_A(a_2|_A a_2))|_A (a_1|_A(a_2|_A a_2)) \in ff^{-1}(I) \subseteq I$. Since I is an ideal of B, we obtain $f(a_1) \in I$ i.e., $a_1 \in f^{-1}(I)$. □

DEFINITION 6.2. Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke INK-algebras and $f : A \to B$ be a homomorphism. Then the set $\{a_1 \in A | f(a_1) = 0_B\}$ is called the kernel of f and it is denoted by $Kerf$.

LEMMA 6.3. *Let $f : A \to B$ be a homomorphism of Sheffer stroke INK-algebras. Then $Kerf$ is an ideal of A.*

PROOF. Obviously, $0 \in Kerf$, since $f(0_A) = 0_B$. Let $a_2 \in Kerf$ and $(a_1|_A(a_2 |_A a_2))|_A(a_1|_A(a_2|_A a_2)) \in Kerf$. Then $f(a_2) = 0_B$ and $f((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2

$|_A a_2))) = 0_B$.

$$\begin{aligned} f(a_1) &= (f(a_1)|_B(0_B|_B0_B))|_B(f(a_1)|_B(0_B|_B0_B)) \\ &= (f(a_1)|_B(f(a_2)|_Bf(a_2)))|_B(f(a_1)|_B(f((a_2)|_Bf(a_2)))) \\ &= f((a_1|_A(a_2|_Aa_2))|_A(a_1|_A(a_2|_Aa_2))) \\ &= 0_B, \end{aligned}$$

it follows $a_1 \in Kerf$. Thus, $Kerf$ is an ideal of A. □

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