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EXISTENCE OF g-BEST PROXIMITY POINTS OF PROXIMAL $\mathscr{F}^*-$WEAK CONTRACTION MAPS

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Abstract. Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty subsets of a metric space X. Let $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{A} \to \mathscr{A}$. In this paper, we prove the existence of best approximate solution x^* in the space such that $\mathscr{S}x^*$ is as close to gx^* as possible. That is, we find the global minimizer of the map $x \mapsto \rho(gx, \mathscr{S}x)$ where $\mathscr{S}$ is either proximal $\mathscr{F}^*-$weak contraction of the first kind or second kind or both, and g is an isometry, in complete metric spaces. Examples are provided to illustrate the validity of our results.

1. Introduction

Let (X, ρ) be a metric space. Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty subsets of X. Let $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{A} \to \mathscr{A}$. Then the equation $\mathscr{S}x = gx$ may not have a solution. In this situation, we consider our attention in computing an approximate solution x^* in the space such that the distance between $\mathscr{S}x^*$ and gx^* is as small as possible. In fact, it is the global minimization of the mapping $x \mapsto \rho(gx, \mathscr{S}x)$ and if this map attains global minimum at x^* then $\rho(gx^*, \mathscr{S}x^*)$ indicates the global proximity between gx^* and $\mathscr{S}x^*$.

Since $\rho(gx, \mathscr{S}x) \geqslant \rho(\mathscr{A}, \mathscr{B})$ for all $x \in \mathscr{A}$, a best proximity point theorem finds the global minimization of $x \mapsto \rho(gx, \mathscr{S}x)$ by computing an approximate solution

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x^* which fulfill the condition that $\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$. Such x^* is known as $g-$best proximity point of the mapping S and g.

In 2011, Sadiq Basha [**5**] proved the existence of $g-$best proximity points of non-self proximal contractions in complete metric spaces. For extensions and more related works, we refer Sadiq Basha and Veeramani [**4**], Sadiq Basha [**6**], Sadiq Basha, Shahzad and Vetro [**8**].

In 2012, Wardowski [**10**], introduced a new concept of contraction, namely $\mathscr{F}-$contraction and proved the existence and uniqueness of fixed points of such mappings in complete metric spaces which generalizes Banach contraction principle in a different way. In 2014, Wardowski and Van Dung [**11**], introduced $\mathscr{F}-$weak contraction as a generalization of $\mathscr{F}-$contraction and proved a fixed point theorem for $\mathscr{F}-$weak contractions in complete metric spaces.

In 2022 Salamatbakhsh, Haghi and Fallahi [**9**], extended $\mathscr{F}-$weak contractions that are defined for selfmaps to non-selfmaps and introduced the notion of proximal $\mathscr{F}^*-$weak contraction of the first kind and strong proximal $\mathscr{F}^*-$weak contraction of the second kind and proved the existence of best proximity points in complete metric spaces.

In this paper, in Section 2, we recall some definitions and preliminaries that we use to prove our main results. Motivated by the works of Basha [**7**] and Salamatbakhsh, Haghi and Fallahi [**9**], in Section 3, we extend these results to find the existence of $g-$best proximity points of proximal $\mathscr{F}^*-$weak contractions of first kind or second kind or both. In Section 4, we draw some corollaries to our main results and provide examples in support of our results.

2. Preliminaries

In this section, we give some definitions of proximal contractions.

Let (X, ρ) be a metric space and $\mathscr{A}$ and $\mathscr{B}$ nonempty subsets of X.

$$\rho(\mathscr{A}, \mathscr{B}) = \inf\{\rho(x, y) : x \in \mathscr{A} \text{ and } y \in \mathscr{B}\}$$

$$\mathscr{A}_0 = \{x \in \mathscr{A} : \rho(x, y) = \rho(\mathscr{A}, \mathscr{B}) \text{ for some } \ y \in \mathscr{B}\}$$

$$\mathscr{B}_0 = \{y \in \mathscr{B} : \rho(x, y) = \rho(\mathscr{A}, \mathscr{B}) \text{ for some } x \in \mathscr{A}\}.$$

In 2000, Basha and Veeramani [**4**], stated that in the setting of normed linear spaces, if $\mathscr{A}$ and $\mathscr{B}$ are closed subsets such that $\rho(\mathscr{A}, \mathscr{B}) > 0$, then $\mathscr{A}_0$ and $\mathscr{B}_0$ are contained in the boundaries of $\mathscr{A}$ and $\mathscr{B}$, respectively. In 2003, Kirk, Reich, Veeramani [**2**] provided sufficient conditions that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonemtpies.

Throughout this paper, we assume that $\rho(\mathscr{A}, \mathscr{B}) > 0$.

DEFINITION 2.1. (Basha [**5**]) The set $\mathscr{B}$ is said to be *approximatively compact* with respect to $\mathscr{A}$ if every sequence $\{y_n\}$ of $\mathscr{B}$ satisfying the condition that $\rho(x, y_n) \to \rho(x, \mathscr{B})$ for some $x \in \mathscr{A}$ has a convergent subsequence.

It is trivial to see that every set is approximatively compact with respect to itself. Also, every compact set is approximatively compact. Moreover, $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty if $\mathscr{A}$ is compact and $\mathscr{B}$ is approximatively compact with respect to $\mathscr{A}$.

DEFINITION 2.2. (Basha [**5**]) A mapping $\mathscr{S}:\mathscr{A}\to\mathscr{B}$ is said to be *proximal contraction* if there exists a nonnegative real number $\alpha<1$ such that, for all u_1,u_2,x_1,x_2 in $\mathscr{A}$,

$$\left.\begin{array}{c}\rho(u_1,\mathscr{S}x_1)=\rho(\mathscr{A},\mathscr{B})\\ \rho(u_2,\mathscr{S}x_2)=\rho(\mathscr{A},\mathscr{B})\end{array}\right\}\Rightarrow\rho(u_1,u_2)\leqslant\alpha(\rho(x_1,x_2)).$$

DEFINITION 2.3. (Basha [**7**]) Let $\mathscr{S}:\mathscr{A}\to\mathscr{B}$ and $g:\mathscr{A}\to\mathscr{A}$. A point $x^*\in\mathscr{A}$ is said to be a $g-$*best proximity point* of the mapping $\mathscr{S}$ if $\rho(gx^*,\mathscr{S}x^*)=\rho(\mathscr{A},\mathscr{B})$.

If g is the identity map then x^* is a *best proximal point of* $\mathscr{S}$.

DEFINITION 2.4. (Basha [**7**]) A set $\mathscr{A}$ is said to have *uniform approximation* in $\mathscr{B}$ if, for given $\epsilon>0$, there exists $\delta>0$ such that

$$\left.\begin{array}{c}\rho(x_1,y_1)=\rho(\mathscr{A},\mathscr{B})\\ \rho(x_2,y_2)=\rho(\mathscr{A},\mathscr{B})\\ \rho(x_1,x_2)<\delta\end{array}\right\}\Rightarrow\rho(y_1,y_2)<\epsilon$$

for all $x_1,x_2\in\mathscr{A}$ and $y_1,y_2\in\mathscr{B}$.

DEFINITION 2.5. (Basha [**7**]) Let $\mathscr{S}:\mathscr{A}\to\mathscr{B}$ be a map. Then, $\mathscr{A}$ is said to have *uniform* $\mathscr{S}-$*approximation* in $\mathscr{B}$ if, for given $\epsilon>0$, there exists $\delta>0$ such that

$$\left.\begin{array}{c}\rho(x_1,\mathscr{S}u_1)=\rho(\mathscr{A},\mathscr{B})\\ \rho(x_2,\mathscr{S}u_2)=\rho(\mathscr{A},\mathscr{B})\\ \rho(x_1,x_2)<\delta\end{array}\right\}\Rightarrow\rho(\mathscr{S}u_1,\mathscr{S}u_2)<\epsilon$$

for all $x_1,x_2,u_1,u_2\in\mathscr{A}$.

DEFINITION 2.6. (Basha [**7**]) Given nonempty subsets $\mathscr{A}$ and $\mathscr{B}$ of a metric space, a map $\mathscr{S}:\mathscr{A}\to\mathscr{B}$ is said to be a *proximally quasi* $-$ *continuous* if

$$\left.\begin{array}{c}\rho(x_n,\mathscr{S}u_n)=\rho(\mathscr{A},\mathscr{B})\\ \rho(x,\mathscr{S}u)=\rho(\mathscr{A},\mathscr{B})\\ u_n\to u\end{array}\right\}\Rightarrow x_{n_k}\to x \text{ for some subsequence } x_{n_k} \text{ of } \{x_n\} \text{ for}$$

$u,x\in\mathscr{A}$ and for all sequences $\{x_n\}$ and $\{u_n\}\in\mathscr{A}$.

Here we note that a proximally quasi-continuous mapping is not necessarily continuous. ([**7**], Example 3.3)

DEFINITION 2.7. (Wardowski [**10**]) Let $\mathscr{F}:(0,\infty)\to\mathbb{R}$ be a mapping satisfying:

$\mathscr{F}_1$: $\mathscr{F}$ is strictly increasing, that is, $\alpha<\beta$ implies $\mathscr{F}(\alpha)<\mathscr{F}(\beta)$ for all $\alpha,\beta\in(0,+\infty)$,

$\mathscr{F}_2$: For every sequence $\{\alpha_n\}$ in $(0,+\infty)$ we have $\lim\limits_{n\to+\infty}\alpha_n=0$ if and only if $\lim\limits_{n\to+\infty}\mathscr{F}(\alpha_n)=-\infty$.

$\mathscr{F}_3$: There exists a number $k\in(0,1)$ such that $\lim\limits_{\alpha\to 0}\alpha^k\mathscr{F}(\alpha)=0$.

We denote $\Psi=$ The family of all functions $\mathscr{F}$ which satisfy the conditions $\mathscr{F}_1-\mathscr{F}_3$.

A mapping $T : X \to X$ is said to be a *Wardowski* $\mathscr{F}-$ *contraction* [**10**] if there exists $\tau > 0$ such that

$$\rho(Tx, Ty) > 0 \Rightarrow \tau + \mathscr{F}(\rho(Tx, Ty)) \leqslant \mathscr{F}(\rho(x, y))$$

for all $x, y \in X$, where $\mathscr{F} \in \Psi$.

In 2014, Wardowski and Van Dung [**11**], introduced $\mathscr{F}-$weak contraction as a generalization of Wardowski contraction as follows:

Definition 2.8. (Wardowski and Van Dung [**11**]) A mapping $T : X \to X$ is said to be a $\mathscr{F}-$ *weak contraction* if there exists $\tau > 0$ such that

$$\rho(Tx, Ty)>0 \Rightarrow \tau+\mathscr{F}(\rho(Tx, Ty))\leqslant\mathscr{F}(\max\{\rho(x, y),\rho(x, Tx),\rho(y, Ty),\frac{\rho(x, Ty) + \rho(y, Tx)}{2}\})$$

for all $x, y \in X$ and where $\mathscr{F} \in \Psi$.

Every $\mathscr{F}-$contraction is an $\mathscr{F}-$weak contraction. But its converse is not true. ([**11**], Example 2.3).

We denote Ψ^* = The family of all functions $\mathscr{F}$ which satisfy the conditions $\mathscr{F}_1$ and $\mathscr{F}_2$.

In 2022, Salamatbakhsh, Haghi and Fallahi [**9**], introduced the notion of proximal $\mathscr{F}^*-$weak contraction mappings by using $\mathscr{F}$ in Ψ^*.

Definition 2.9. (Salamatbakhsh, Haghi and Fallahi [**9**]) A mapping $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ is called a *proximal* $\mathscr{F}^*-$ *weak contraction of the first kind* if there exist $\mathscr{F} \in \Psi^*$ and $\tau > 0$ such that

$$\left.\begin{array}{l}\rho(u_1, \mathscr{S}a_1) = \rho(\mathscr{A}, \mathscr{B})\\ \rho(u_2, \mathscr{S}a_2) = \rho(\mathscr{A}, \mathscr{B})\end{array}\right\} \Rightarrow \tau + \mathscr{F}(\rho(u_1, u_2)) \leqslant \mathscr{F}(\rho(a_1, a_2)),$$

where $a_1, a_2, u_1, u_2 \in \mathscr{A}$ and $a_1 \neq a_2, u_1 \neq u_2$.

Example 2.1. Let $X = [0, 1] \times [0, 1]$ endowed with metric $\rho((u, v), (a, b)) = |u - a| + |v - b|$. Let $\mathscr{A} = \{(0, u); 0 \leqslant u \leqslant 1\}, \mathscr{B} = \{(1, v); 0 \leqslant v \leqslant 1\}$. It is clear that $\mathscr{A}_0 = \mathscr{A}$ and $\mathscr{B}_0 = \mathscr{B}$ and $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$. Define $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ by $\mathscr{S}((0, u)) = (1, \frac{u^2}{4}), 0 \leqslant u \leqslant 1$. Let $a_1 = (0, u)$ and $a_2 = (0, v)$ with $u \neq v$ so that $a_1 \neq a_2$. We choose $u_1, u_2 \in \mathscr{A}$ such that $\rho(u_1, \mathscr{S}a_1) = \rho(u_2, \mathscr{S}a_2) = \rho(\mathscr{A}, \mathscr{B}) = 1$. Also, $\mathscr{S}a_1 = \mathscr{S}((0, u)) = (1, \frac{u^2}{4})$ which implies that $u_1 = (0, \frac{u^2}{4})$ and $\mathscr{S}a_2 = \mathscr{S}((0, v)) = (1, \frac{v^2}{4})$ which implies that $u_2 = (0, \frac{v^2}{4})$, and $u_1 \neq u_2$. Also
$\rho(u_1, \mathscr{S}a_1) = \rho((0, \frac{u^2}{4}), (1, \frac{u^2}{4})) = |1| + |\frac{u^2}{4} - \frac{u^2}{4}| = 1$
$\rho(u_2, \mathscr{S}a_2) = \rho((0, \frac{v^2}{4}), (1, \frac{v^2}{4})) = |1| + |\frac{v^2}{4} - \frac{v^2}{4}| = 1.$
Therefore $\rho(u_1, \mathscr{S}a_1) = \rho(u_2, \mathscr{S}a_2) = \rho(\mathscr{A}, \mathscr{B})$. We choose $\tau = \log 2$ and $\mathscr{F}(\alpha) = \frac{-1}{\sqrt{\alpha}} + \log \alpha$, $\alpha > 0$. We consider

$$\begin{aligned}\tau + \mathscr{F}(\rho(u_1, u_2)) &= \log 2 + \mathscr{F}(\rho((0, \tfrac{u^2}{4}), (0, \tfrac{v^2}{4})))\\ &= \log 2 + \mathscr{F}(\tfrac{1}{4}|u^2 - v^2|)\\ &= \log 2 - \tfrac{1}{\sqrt{\frac{1}{4}|u^2-v^2|}} + \log(\tfrac{1}{4}|u^2 - v^2|)\\ &= \log 2 - \tfrac{2}{\sqrt{|u^2-v^2|}} + \log|u^2 - v^2| - \log 4\\ &\leqslant \log 2 - \tfrac{2}{\sqrt{|u^2-v^2|}} + \log 2 + \log|u - v| - 2\log 2\end{aligned}$$

$$= \frac{-2}{\sqrt{|u^2-v^2|}} + \log|u-v|$$
$$\leqslant \frac{-2}{2\sqrt{|u-v|}} + \log|u-v|$$
$$= \frac{-1}{\sqrt{|u-v|}} + \log|u-v|$$
$$= \mathscr{F}(|u-v|)$$
$$= \mathscr{F}(\rho((0,u),(0,v)))$$
$$= \mathscr{F}(\rho(a_1,a_2)).$$

Hence $\mathscr{S}$ is a proximal $\mathscr{F}^*-$weak contraction of the first kind.

DEFINITION 2.10. (Salamatbakhsh, Haghi and Fallahi [**9**]) Let $\mathscr{A},\mathscr{B}$ be nonempty subsets of a metric space X. $\mathscr{S}:\mathscr{A}\to\mathscr{B}$ is said to be a *proximal* $\mathscr{F}^*-$ *weak contraction of the second kind* if there exists $\mathscr{F}\in\Psi^*$ and $\tau>0$ such that

$$\left.\begin{array}{l}\rho(u_1,\mathscr{S}a_1)=\rho(\mathscr{A},\mathscr{B})\\ \rho(u_2,\mathscr{S}a_2)=\rho(\mathscr{A},\mathscr{B})\end{array}\right\}\Rightarrow \tau+\mathscr{F}(\rho(\mathscr{S}u_1,\mathscr{S}u_2))\leqslant \mathscr{F}(\rho(\mathscr{S}a_1,\mathscr{S}a_2)),$$

where $a_1,a_2,u_1,u_2\in\mathscr{A}$ and $\mathscr{S}a_1\neq\mathscr{S}a_2,\mathscr{S}u_1\neq\mathscr{S}u_2$.

DEFINITION 2.11. (Salamatbakhsh, Haghi and Fallahi [**9**]) A mapping $\mathscr{S}:\mathscr{A}\to\mathscr{B}$ is said to be a *strong proximal* $\mathscr{F}^*-$*weak contraction of the second kind* if the following conditions are satisfied:

(a) $\mathscr{S}$ is a proximally quasi-continuous,

(b) $\mathscr{S}$ is a proximal $\mathscr{F}^*-$weak contraction of the second kind.

EXAMPLE 2.2. Let $X=[0,1]\times[0,1]$ endowed with metric $\rho((u,v),(a,b))=|u-a|+|v-b|$. Let $\mathscr{A}=\{(0,u);0\leqslant u\leqslant 1\},\mathscr{B}=\{(1,v);0\leqslant v\leqslant 1\}$. It is clear that $\mathscr{A}_0=\mathscr{A}$ and $\mathscr{B}_0=\mathscr{B}$ and $\mathscr{S}(\mathscr{A}_0)\subseteq\mathscr{B}_0$. Define $\mathscr{S}:\mathscr{A}\to\mathscr{B}$ by $\mathscr{S}((0,u))=(1,1-\frac{u}{2}),0\leqslant u\leqslant 1$. It is easy to see that $\mathscr{S}$ is proximally quasi-continuous. Let $a_1=(0,u)$ and $a_2=(0,v)$ with $u\neq v$ so that $a_1\neq a_2$ and $\mathscr{S}a_1\neq\mathscr{S}a_2$. We choose $u_1,u_2\in\mathscr{A}$ such that $\rho(u_1,\mathscr{S}a_1)=\rho(u_2,\mathscr{S}a_2)=\rho(\mathscr{A},\mathscr{B})=1$. Then $\mathscr{S}a_1=\mathscr{S}((0,u))=(1,1-\frac{u}{2})$ which implies that $u_1=(0,1-\frac{u}{2})$, and $\mathscr{S}a_2=\mathscr{S}((0,v))=(1,1-\frac{v}{2})$ which implies that $u_2=(0,1-\frac{v}{2})$. Now,

$\rho(u_1,\mathscr{S}a_1)=\rho((0,1-\frac{u}{2}),(1,1-\frac{u}{2}))=|1|+|1-\frac{u}{2}-1+\frac{u}{2}|=1$

$\rho(u_2,\mathscr{S}a_2)=\rho((0,1-\frac{v}{2}),(1,1-\frac{v}{2}))=|1|+|1-\frac{v}{2}-1+\frac{v}{2}|=1.$

Therefore $\rho(u_1,\mathscr{S}a_1)=\rho(u_2,\mathscr{S}a_2)=\rho(\mathscr{A},\mathscr{B})=1$. We choose $\tau=\log 2$ and $\mathscr{F}(\alpha)=\frac{-1}{\alpha}+\log\alpha+\alpha$, $\alpha>0$. Also, for such $u_1=(0,1-\frac{u}{2})$ and $u_2=(0,1-\frac{v}{2})$, we have $\mathscr{S}u_1\neq\mathscr{S}u_2$. Now

$$\begin{aligned}\tau+\mathscr{F}(\rho(\mathscr{S}u_1,\mathscr{S}u_2)) &= \log 2+\mathscr{F}(\rho(\mathscr{S}(0,1-\tfrac{u}{2}),\mathscr{S}(0,1-\tfrac{v}{2})))\\ &= \log 2+\mathscr{F}(\rho((1,1-\tfrac{1-\frac{u}{2}}{2}),(1,1-\tfrac{1-\frac{v}{2}}{2})))\\ &= \log 2+\mathscr{F}(\rho((1,\tfrac{u+2}{4}),(1,\tfrac{v+2}{4})))\\ &= \log 2+\mathscr{F}(\tfrac{1}{4}|(u-v|)\\ &= \log 2-\tfrac{1}{\frac{1}{4}|u-v|}+\log(\tfrac{1}{4}|u-v|)+\tfrac{1}{4}|u-v|\\ &= \log 2-\tfrac{4}{|u-v|}+\log(\tfrac{|u-v|}{4})+\tfrac{|u-v|}{4}\\ &= \log 2-\tfrac{4}{|u-v|}+\log(|u-v|)-\log 4+\tfrac{|u-v|}{4}\\ &= \log 2-\tfrac{4}{|u-v|}+\log(|u-v|)-2\log 2+\tfrac{|u-v|}{4}\end{aligned}$$

$$
\begin{aligned}
&= -\tfrac{4}{|u-v|} + \log(|u-v|) - \log 2 + \tfrac{|u-v|}{4}\\
&= -\tfrac{4}{|u-v|}\log(\tfrac{|u-v|}{2}) + \tfrac{|u-v|}{4}\\
&\leqslant -\tfrac{2}{|u-v|} + \log(\tfrac{|u-v|}{2}) + \tfrac{|u-v|}{2}\\
&= \mathscr{F}(\rho((1, 1-\tfrac{u}{2}), (1, 1-\tfrac{v}{2})))\\
&= \mathscr{F}(\rho(\mathscr{S}a_1, \mathscr{S}a_2)).
\end{aligned}
$$

Therefore $\mathscr{S}$ is a strong proximal $\mathscr{F}^*-$weak contraction of the second kind.

DEFINITION 2.12. (Mongkolkeha, Cho, Kumam [**3**]) Let $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{A} \to \mathscr{A}$ be an isometry. The mapping $\mathscr{S}$ is said to *preserve the isometric distance with respect to* g if $\rho(\mathscr{S}gx, \mathscr{S}gy) = \rho(\mathscr{S}x, \mathscr{S}y)$ for all $x, y \in \mathscr{A}$.

THEOREM 2.1. (*Salamatbakhsh, Haghi and Fallahi* [**9**]) *Let* $(\mathscr{M}, \rho)$ *be a complete metric space. Suppose that the following conditions are satisfied:*
(i) $\mathscr{A}, \mathscr{B}$ *are nonempty subsets of* $\mathscr{M}$ *and* $\mathscr{A}$ *is closed;*
(ii) $\mathscr{B}$ *is approximatively compact with respect to* $\mathscr{A}$*;*
(iii) $\mathscr{A}_0$ *is nonempty;*
(iv) $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ *is a proximal* $\mathscr{F}^*$*-weak contraction of the first kind;*
(v) $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$*;*
(vi) $\mathscr{F}$ *is continuous.*
Then there exists a unique element $a \in \mathscr{A}$ *such that* $\rho(a, \mathscr{S}a) = \rho(\mathscr{A}, \mathscr{B})$*. Further, for any fixed* $a_0 \in \mathscr{A}_0$*, the sequence* $\{a_n\}$ *defined by* $\rho(a_{n+1}, \mathscr{S}a_n) = \rho(\mathscr{A}, \mathscr{B})$ *is convergent to* a*.*

THEOREM 2.2. (*Salamatbakhsh, Haghi and Fallahi* [**9**]) *Let* $(\mathscr{M}, \rho)$ *be a complete metric space. Suppose that the following conditions are satisfied:*
(i) $\mathscr{A}, \mathscr{B}$ *are nonempty closed subsets of* $\mathscr{M}$*;*
(ii) $\mathscr{A}$ *is approximatively compact with respect to* $\mathscr{B}$*;*
(iii) $\mathscr{A}_0$ *is nonempty;*
(iv) $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ *is a strong proximal* $\mathscr{F}^*$*-weak contraction of the second kind;*
(v) $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$*;*
(vi) $\mathscr{F}$ *is continuous.*
Then there exists a unique element $a \in \mathscr{A}$ *such that* $\rho(a, \mathscr{S}a) = \rho(\mathscr{A}, \mathscr{B})$*. Further, for any fixed* $a_0 \in \mathscr{A}_0$*, the sequence* $\{a_n\}$ *defined by* $\rho(a_{n+1}, \mathscr{S}a_n) = \rho(\mathscr{A}, \mathscr{B})$ *is convergent to* a*.*

The following lemma is useful in proving our results.

LEMMA 2.1. (*Babu and Sailaja* [**1**]) *Suppose* (X, d) *is a metric space. Let* $\{x_n\}$ *be a sequence in* X *such that* $d(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$*. If* $\{x_n\}$ *is not a Cauchy sequence then there exist* $\epsilon > 0$ *and sequences of positive integers* $\{m_k\}$ *and* $\{n_k\}$ *with* $m_k > n_k > k$ *such that* $d(x_{m_k}, x_{n_k}) \geqslant \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ *and*
i) $\lim\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$ *ii)* $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$
iii) $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$ *iv)* $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon$.

3. Main results

LEMMA 3.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty subsets of a metric space X. Assume that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty. Let $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ be a map and $g : \mathscr{A} \to \mathscr{A}$ be an isometry. Also assume that $\mathscr{S}(x_0) \in \mathscr{B}_0$ for any $x_0 \in \mathscr{A}_0$. Then there exists a sequence $\{x_n\}$ in $\mathscr{A}_0$ such that $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, \dots$.*

PROOF. Let $x_0 \in \mathscr{A}_0$ be arbitrary. Since $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$, we have $\mathscr{S}x_0 \in \mathscr{B}_0$. So $\rho(x, \mathscr{S}x_0) = \rho(\mathscr{A}, \mathscr{B})$ for some $x \in \mathscr{A}$. This implies that $x \in \mathscr{A}_0$.
Since $\mathscr{A}_0 \subseteq g(\mathscr{A}_0)$, we have $x = gx_1$ for some $x_1 \in \mathscr{A}_0$. Therefore $\rho(gx_1, \mathscr{S}x_0) = \rho(\mathscr{A}, \mathscr{B})$ for some $x_1 \in \mathscr{A}_0$. Again, by repeating the same process, we obtain that $\rho(gx_2, \mathscr{S}x_1) = \rho(\mathscr{A}, \mathscr{B})$ for some $x_2 \in \mathscr{A}_0$.
In general, we obtain that there is a sequence $\{x_n\} \subseteq \mathscr{A}_0$ such that $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, \dots$. □

THEOREM 3.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty, closed subsets of a complete metric space X such that $\mathscr{B}$ is approximatively compact with respect to $\mathscr{A}$. Moreover, assume that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty. Let $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{A} \to \mathscr{A}$ satisfy the following conditions:*
a) $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ is a proximal $\mathscr{F}^*-$ weak contraction of the first kind
b) $\mathscr{B}_0$ contains $\mathscr{S}(\mathscr{A}_0)$
c) g is an isometry
d) $g(\mathscr{A}_0)$ contains $\mathscr{A}_0$
e) $\mathscr{F}$ is continuous.
Then for any fixed element $x_0 \in \mathscr{A}_0$, the sequence $\{x_n\}$ defined by $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ is Cauchy, and converges to an element x^* (say) in $\mathscr{A}_0$ that satisfies $\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$. Such x^* is unique.

PROOF. Let $x_0 \in \mathscr{A}_0$ be arbitrary. By Lemma 3.1, we have a sequence $\{x_n\} \subset \mathscr{A}_0$ such that $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, \dots$. Suppose $x_n = x_{n+1}$ for some $n \in \mathbb{N}$. Since $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$, in this case, we have $\rho(gx_n, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$. Hence the conclusion of the theorem holds with x_n inplace of x^*. Therefore, without loss of generality, we assume that $x_{n+1} \neq x_n$ for $n = 0, 1, 2, \dots$. Since $\mathscr{S}$ is a proximal $\mathscr{F}^*-$weak contraction of the first kind, we have $\tau + \mathscr{F}(\rho(gx_{n+2}, gx_{n+1})) \leqslant \mathscr{F}(\rho(x_{n+1}, x_n))$. Since g is an isometry, it follows that $\tau + \mathscr{F}(\rho(x_{n+2}, x_{n+1})) \leqslant \mathscr{F}(\rho(x_{n+1}, x_n))$. This implies that $\mathscr{F}(\rho(x_{n+2}, x_{n+1})) \leqslant \mathscr{F}(\rho(x_{n+1}, x_n)) - \tau$. By repeating this process, we get $\mathscr{F}(\rho(x_{n+1}, x_n)) \leqslant \mathscr{F}(\rho(x_1, x_0)) - n\tau$. On letting $n \to \infty$, we have $\lim_{n\to\infty} \mathscr{F}(\rho(x_{n+1}, x_n)) = -\infty$. By $\mathscr{F}_2$, we have $\lim_{n\to\infty} \rho(x_{n+1}, x_n) = 0$. We now show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not Cauchy, by Lemma 2.1, there exist $\epsilon > 0$ and two subsequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that

$$\rho(x_{n_k}, x_{m_k}) \geqslant \epsilon \text{ and } \rho(x_{n_k-1}, x_{m_k}) < \epsilon \tag{3.1}$$

and $\lim_{k\to\infty} \rho(x_{n_k}, x_{m_k}) = \epsilon$ and $\lim_{k\to\infty} \rho(x_{n_k-1}, x_{m_k-1}) = \epsilon$. Since $\rho(gx_{n_k}, \mathscr{S}x_{n_k-1}) = \rho(gx_{m_k}, \mathscr{S}x_{m_k-1}) = \rho(\mathscr{A}, \mathscr{B})$, by proximal $\mathscr{F}^*-$weak contraction of the first kind, we have $\tau + \mathscr{F}(\rho(gx_{n_k}, gx_{m_k})) \leqslant \mathscr{F}(\rho(x_{n_k-1}, x_{m_k-1}))$. Since g is an isometry, we have $\tau + \mathscr{F}(\rho(x_{n_k}, x_{m_k})) \leqslant \mathscr{F}(\rho(x_{n_k-1}, x_{m_k-1}))$. On letting $k \to \infty$, we have $\tau + \mathscr{F}(\epsilon) \leqslant \mathscr{F}(\epsilon)$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space and $\mathscr{A}$ is closed in X, it follows that there exists $x^* \in \mathscr{A}$ such that $x_n \to x^*$ as $n \to \infty$. Since g is continuous and $\{x_n\}$ converges to x^*, we have $\{gx_n\}$ converges to gx^*. Now, we have

$$\begin{aligned}\rho(gx^*, \mathscr{B}) &\leqslant \rho(gx^*, \mathscr{S}x_n)\\ &\leqslant \rho(gx^*, gx_{n+1}) + \rho(gx_{n+1}, \mathscr{S}x_n)\\ &= \rho(gx^*, gx_{n+1}) + \rho(\mathscr{A}, \mathscr{B})\\ &\leqslant \rho(gx^*, gx_{n+1}) + \rho(gx^*, \mathscr{B}).\end{aligned}$$

Therefore $\rho(gx^*, \mathscr{S}x_n) \to \rho(gx^*, \mathscr{B})$.

Since $\mathscr{B}$ is approximatively compact with respect to the set $\mathscr{A}$, it follows that the sequence $\{\mathscr{S}x_n\}$ has a subsequence $\{\mathscr{S}x_{n_k}\}$ converging to some element y in $\mathscr{B}$. Now, for the subsequence $\{x_{n_k}\}$ of $\{x_n\}$, we have $\rho(gx_{n_k+1}, \mathscr{S}x_{n_k}) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, \ldots$. On taking limits as $k \to \infty$, it follows that $\rho(gx^*, y) = \rho(\mathscr{A}, \mathscr{B})$ and hence gx^* is a member of $\mathscr{A}_0$. Since $\mathscr{A}_0 \subseteq g(\mathscr{A}_0)$, $gx^* = gu$ for some element $u \in \mathscr{A}_0$. Since g is an isometry, we have $\rho(x^*, u) = \rho(gx^*, gu) = 0$. Therefore, x^* and u are identical and hence x^* is a member of $\mathscr{A}_0$. Since $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$, we have $\rho(z, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$ for some $z \in \mathscr{A}$. Also, since $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 0, 1, 2, \ldots$. Since $\mathscr{S}$ is a proximal $\mathscr{F}^*-$weak contraction of the first kind, we have $\tau + \mathscr{F}(\rho(gx_{n+1}, z)) \leqslant \mathscr{F}(\rho(x_n, x^*))$ which implies that $\mathscr{F}(\rho(gx_{n+1}, z)) < \mathscr{F}(\rho(x_n, x^*))$. Since $\mathscr{F}$ is strictly increasing we have $\rho(gx_{n+1}, z) \leqslant \rho(x_n, x^*) \to 0$ as $n \to \infty$. Hence $\{gx_n\}$ converges to z. Since the sequence $\{gx_n\}$ converges to gx^*, we have Thus z and gx^* must be identical. Hence, it can be concluded that $\rho(gx^*, \mathscr{S}x^*) = \rho(z, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$.

Suppose that there is another element $b^* \in \mathscr{A}_0$ such that $\rho(gb^*, \mathscr{S}b^*) = \rho(\mathscr{A}, \mathscr{B})$. Also, we have $\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B}) \Rightarrow \tau + \mathscr{F}(\rho(gx^*, gb^*)) \leqslant \mathscr{F}(\rho(x^*, b^*))$.

Since $\mathscr{S}$ is a proximal $\mathscr{F}^*-$weak contraction of the first kind, and g is an isometry, we have $\mathscr{F}(\rho(x^*, b^*)) = \mathscr{F}(\rho(gx^*, gb^*)) < \tau + \mathscr{F}(\rho(gx^*, gb^*)) \leqslant \mathscr{F}(\rho(x^*, b^*))$, a contradiction. Hence $\rho(x^*, b^*) = 0$. Therefore $x^* = b^*$. Hence the theorem follows. □

THEOREM 3.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty, closed subsets of a complete metric space X such that $\mathscr{A}$ is approximatively compact with respect to $\mathscr{B}$. Moreover, assume that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty. Let $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{A} \to \mathscr{A}$ satisfy the following conditions:*

a) $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ is a strong proximal $\mathscr{F}^*-$ weak contraction of the second kind
b) $\mathscr{B}_0$ contains $\mathscr{S}(\mathscr{A}_0)$
c) g is an isometry
d) $g(\mathscr{A}_0)$ contains $\mathscr{A}_0$
e) $\mathscr{F}$ is continuous
f) $\mathscr{S}$ preserves isometric distance with respect to g.

Then for any fixed element $x_0 \in \mathscr{A}_0$, the sequence $\{x_n\}$ defined by $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ is Cauchy, and converges to an element x^* (say) in $\mathscr{A}_0$ that satisfies $\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$.

PROOF. Let $x_0 \in \mathscr{A}_0$ be arbitrary. By Lemma 3.1, we have a sequence $\{x_n\} \subset \mathscr{A}_0$ such that $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, \ldots$. Without loss of generality, we assume that $x_{n+1} \neq x_n$ for $n = 0, 1, 2, \ldots$. Since $\mathscr{S}$ is a proximal $\mathscr{F}^*-$weak contraction of the second kind, we have $\tau + \mathscr{F}(\rho(\mathscr{S}gx_{n+2}, \mathscr{S}gx_{n+1})) \leqslant \mathscr{F}(\rho(\mathscr{S}x_{n+1}, \mathscr{S}x_n))$. Since $\mathscr{S}$ preserves isometric distance with respect to g $\tau + \mathscr{F}(\rho(\mathscr{S}x_{n+2}, \mathscr{S}x_{n+1})) \leqslant \mathscr{F}(\rho(\mathscr{S}x_{n+1}, \mathscr{S}x_n))$. This implies that
$\mathscr{F}(\rho(\mathscr{S}x_{n+2}, \mathscr{S}x_{n+1})) \leqslant \mathscr{F}(\rho(\mathscr{S}x_{n+1}, \mathscr{S}x_n)) - \tau$. By repeating this process, we get $\mathscr{F}(\rho(\mathscr{S}x_{n+1}, \mathscr{S}x_n)) \leqslant \mathscr{F}(\rho(\mathscr{S}x_1, \mathscr{S}x_0)) - n\tau$. On letting $n \to \infty$, we have $\lim\limits_{n\to\infty} \mathscr{F}(\rho(\mathscr{S}x_{n+1}, \mathscr{S}x_n)) = -\infty$. By $\mathscr{F}_2$, we have $\lim\limits_{n\to\infty} \rho(\mathscr{S}x_{n+1}, \mathscr{S}x_n) = 0$. We now show that $\{\mathscr{S}x_n\}$ is a Cauchy sequence. If $\{\mathscr{S}x_n\}$ is not Cauchy by Lemma 2.1, there exist $\epsilon > 0$ and two subsequences of positive integers $\{\mathscr{S}x_{m_k}\}$ and $\{\mathscr{S}x_{n_k}\}$ with $n_k > m_k > k$ such that

$$\rho(\mathscr{S}x_{n_k}, \mathscr{S}x_{m_k}) \geqslant \epsilon \text{ and } \rho(\mathscr{S}x_{n_k-1}, \mathscr{S}x_{m_k}) < \epsilon \tag{3.2}$$

and $\lim\limits_{k\to\infty} \rho(\mathscr{S}x_{n_k}, \mathscr{S}x_{m_k}) = \epsilon$ and $\lim\limits_{k\to\infty} \rho(\mathscr{S}x_{n_k-1}, \mathscr{S}x_{m_k-1}) = \epsilon$.
Since $\rho(gx_{n_k}, \mathscr{S}x_{n_k-1}) = \rho(gx_{m_k}, \mathscr{S}x_{m_k-1}) = \rho(\mathscr{A}, \mathscr{B})$, by proximal $\mathscr{F}^*-$weak contraction of second kind, we have
$\tau + \mathscr{F}(\rho(\mathscr{S}gx_{n_k}, \mathscr{S}gx_{m_k})) \leqslant \mathscr{F}(\rho(\mathscr{S}x_{n_k-1}, \mathscr{S}x_{m_k-1}))$. Since $\mathscr{S}$ preserves isometric distance with respect to g, we have
$\tau + \mathscr{F}(\rho(\mathscr{S}x_{n_k}, \mathscr{S}x_{m_k})) \leqslant \mathscr{F}(\rho(\mathscr{S}x_{n_k-1}, \mathscr{S}x_{m_k-1}))$. On letting $k \to \infty$, we have $\tau + \mathscr{F}(\epsilon) \leqslant \mathscr{F}(\epsilon)$, a contradiction. Hence $\{\mathscr{S}x_n\}$ is a Cauchy sequence. Since X is a complete metric space and $\mathscr{B}$ is closed there exists $y \in \mathscr{B}$ such that $\mathscr{S}x_n \to y$ as $n \to \infty$. Further,

$$\begin{aligned}\rho(y, \mathscr{A}) &\leqslant \rho(y, gx_{n+1})\\ &\leqslant \rho(y, \mathscr{S}x_n) + \rho(\mathscr{S}x_n, gx_{n+1})\\ &\leqslant \rho(y, \mathscr{S}x_n) + \rho(\mathscr{B}, \mathscr{A})\\ &\leqslant \rho(y, \mathscr{S}x_n) + \rho(y, \mathscr{A}).\end{aligned}$$

This implies that $\rho(y, gx_{n+1}) \to \rho(y, \mathscr{A})$. Since $\mathscr{A}$ is approximatively compact with respect to the set $\mathscr{B}$, then $\{gx_n\}$ has a convergent subsequence $\{gx_{n_k}\}$ such that $\lim\limits_{k\to\infty} gx_{n_k} = x \in \mathscr{A}$. Hence it follows that $\rho(x, y) = \rho(\mathscr{A}, y)$.
Since $\rho(\mathscr{B}, \mathscr{A}) \leqslant \rho(y, \mathscr{A}) \leqslant \rho(y, gx_{n+1}) \leqslant \rho(y, \mathscr{S}x_n) + \rho(\mathscr{S}x_n, gx_{n+1})$, we have $\rho(\mathscr{B}, \mathscr{A}) \leqslant \rho(y, gx_{n+1}) \leqslant \rho(y, \mathscr{S}x_n) + \rho(\mathscr{B}, \mathscr{A})$. On letting $n \to \infty$, we get $\rho(\mathscr{B}, \mathscr{A}) \leqslant \lim\limits_{n\to\infty} \rho(y, gx_{n+1}) \leqslant \lim\limits_{n\to\infty} \rho(y, \mathscr{S}x_n) + \rho(\mathscr{B}, \mathscr{A})$, i.e., $\rho(\mathscr{B}, \mathscr{A}) \leqslant \rho(y, x) \leqslant \rho(\mathscr{B}, \mathscr{A})$. Therefore $\rho(x, y) = \rho(\mathscr{A}, \mathscr{B})$ and hence x is a member of $\mathscr{A}_0$. Since $\mathscr{A}_0 \subseteq g(\mathscr{A}_0), x = gx^*$ for some element $x^* \in \mathscr{A}_0$.
Since g is an isometry, we have $gx_n \to x = gx^*$ implies that $x_n \to x^*$. Since $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$, it follows that $\rho(u^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$ for some $u^* \in \mathscr{A}$. Also, since $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, \ldots$, and $x_n \to x^*$, since $\mathscr{S}$ is a proximally quasi-continuous, we have $u_{n_k} \to u^*$ for some subsequence $\{u_{n_k}\}$ of $\{u_n\}$. i.e., $\{gx_{n_k+1}\} \to u^*$ as $k \to \infty$. Therefore $gx^* = u^*$ so that $\rho(gx^*, \mathscr{S}x^*) = \rho(u^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$.

□

THEOREM 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty, closed subsets of a complete metric space X. Assume that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty. Let $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{A} \to \mathscr{A}$ satisfy the following conditions:*
a) $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ is a proximal $\mathscr{F}^*-$ weak contraction of the first kind
b) $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ is a strong proximal $\mathscr{F}^*-$ weak contraction of the second kind
c) $\mathscr{B}_0$ contains $\mathscr{S}(\mathscr{A}_0)$
d) g is an isometry
e) $g(\mathscr{A}_0)$ contains $\mathscr{A}_0$
f) $\mathscr{S}$ preserves isometric distance with respect to g
g) $\mathscr{F}$ is continuous.
Then for any fixed element $x_0 \in \mathscr{A}_0$, the sequence $\{x_n\}$ defined by $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ is Cauchy, and converges to an element x^* (say) in $\mathscr{A}_0$ that satisfies $\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$. Such x^* is unique.

PROOF. Let $x_0 \in \mathscr{A}_0$ be arbitrary. By Lemma 3.1, we consider a sequence $\{x_n\} \subset \mathscr{A}_0$ such that $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, ...$. As in the proof of Theorem 3.1, it follows that $\{x_n\} \subseteq \mathscr{A}_0$ is a Cauchy sequence. Since X is a complete and $\mathscr{A}$ is closed, then there exists $x^* \in \mathscr{A}$ such that $x_n \to x^*$ as $n \to \infty$. Since g is an isometry we have $gx_n \to gx^*$ as $n \to \infty$. Also as in the proof of Theorem 3.2, it follows that $\{\mathscr{S}x_n\}$ is a Cauchy sequence. So there exists $b \in \mathscr{B}$ such that $\mathscr{S}x_n \to b$ as $n \to \infty$. Now, $\rho(gx_{n+1}, \mathscr{S}x_n) \to \rho(x^*, b) = \rho(\mathscr{A}, \mathscr{B})$ as $n \to \infty$, so that $x^* \in \mathscr{A}_0$. Since $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$, it follows that $\rho(z, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$ for some $z \in \mathscr{A}$. Again proceeding as in the proof of Theorem 3.1, it follows that $\rho(z, \mathscr{S}x^*) = \rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$, and x^* is the unique best proximity point of $\mathscr{S}$. Hence the theorem follows. □

In the following, we extend the existence of best proximity points in the setting of fairly complete space (Theorem 2.5 and Theorem 2.6, [**9**]) to the existence of $g-$best proximity points in complete metric spaces.

THEOREM 3.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty, closed subsets of a complete metric space X such that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty. Let the mapping $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ be such that $\mathscr{A}$ has uniform $\mathscr{S}-$approximation in $\mathscr{B}$. Moreover, assume that*
a) $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ is a proximal $\mathscr{F}^*-$ weak contraction of the first kind
b) $\mathscr{B}_0$ contains $\mathscr{S}(\mathscr{A}_0)$
c) $g : \mathscr{A} \to \mathscr{A}$ is an isometry
d) $g(\mathscr{A}_0)$ contains $\mathscr{A}_0$
e) $\mathscr{F}$ is continuous.
Then for any fixed element $x_0 \in \mathscr{A}_0$, the sequence $\{x_n\}$ defined by $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ is Cauchy, and converges to an element x^* (say) in $\mathscr{A}_0$ that satisfies $\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$. Such x^* is unique.

PROOF. Let $x_0 \in \mathscr{A}_0$ be arbitrary. By Lemma 3.1, $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for some sequence $\{x_n\} \subset \mathscr{A}_0$. Now, proceeding as in the proof of Theorem 3.1, it follows that $\{x_n\}$ is a Cauchy sequence in $\mathscr{A}$. Since g is an isometry, we have $\{gx_n\}$ is a Cauchy sequence. Since $\mathscr{A}$ has uniform $\mathscr{S}-$approximation in $\mathscr{B}$ and since $\{gx_n\}$ is Cauchy, given $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $\rho(gx_n, gx_m) < \epsilon$ for all $n > m \geqslant N$. We choose $\delta = \epsilon$. Then for all $n > m \geqslant N$, we have

$$\left.\begin{array}{r} \rho(gx_{m+1}, \mathscr{S}x_m) = \rho(\mathscr{A}, \mathscr{B}) \\ \rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B}) \\ \rho(gx_{m+1}, gx_{n+1}) < \delta \text{ for all } n > m \geqslant N \end{array}\right\} \Rightarrow \rho(\mathscr{S}x_n, \mathscr{S}x_m) < \epsilon.$$

Therefore $\{\mathscr{S}x_n\}$ is a Cauchy sequence. Since X is a complete and $\mathscr{A}$ and $\mathscr{B}$ are closed subsets of X, we have $\mathscr{A}$ and $\mathscr{B}$ are complete and so there exist $w^* \in \mathscr{A}$ and $y^* \in \mathscr{B}$ such that $\lim_{n\to\infty} gx_n = w^*$ and $\lim_{n\to\infty} \mathscr{S}x_n = y^*$. Therefore $\rho(w^*, y^*) = \lim_{n\to\infty} \rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$. So w^* is an element of $\mathscr{A}_0$. Since $\mathscr{A}_0 \subseteq g(\mathscr{A}_0), w^* = gx^*$ for some x^* in $\mathscr{A}_0$. Since g is an isometry, we have

$$gx_n \to w^* = gx^* \text{ implies that } x_n \to x^*. \tag{3.3}$$

Since $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$, it follows that $\rho(u^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$ for some $u^* \in \mathscr{A}$. Also, since $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, \dots$. Since $\mathscr{S}$ is a proximal $\mathscr{F}^*-$weak contraction of the first kind, we have $\mathscr{F}(\rho(gx_{n+1}, u^*)) \leqslant \tau + \mathscr{F}(\rho(gx_{n+1}, u^*)) \leqslant \mathscr{F}(\rho(x_n, x^*))$ which implies that $\mathscr{F}(\rho(gx_{n+1}, u^*)) \leqslant \mathscr{F}(\rho(x_n, x^*))$. Since $\mathscr{F}$ is strictly increasing we have $\rho(gx_{n+1}, u^*) \leqslant \rho(x_n, x^*) \to 0$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} gx_{n+1} = u^* \tag{3.4}$$

Therefore from (3.3) and (3.4), we have $gx^* = u^*$, and $\rho(gx^*, \mathscr{S}x^*) = \rho(u^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$. Uniqueness of x^* follows as in the proof of Theorem 3.1. □

THEOREM 3.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty, closed subsets of a complete metric space X such that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty. Moreover, assume that $\mathscr{B}$ has uniform approximation in $\mathscr{A}$. Let $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ and $g : \mathscr{A} \to \mathscr{A}$ satisfy the following conditions:*

a) $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ is a strong proximal $\mathscr{F}^*-$ weak contraction of the second kind
b) $\mathscr{B}_0$ contains $\mathscr{S}(\mathscr{A}_0)$
c) $g : \mathscr{A} \to \mathscr{A}$ is an isometry
d) $g(\mathscr{A}_0)$ contains $\mathscr{A}_0$
e) $\mathscr{S}$ preserves isometric distance with respect to g
f) $\mathscr{F}$ is continuous.
Then for any fixed element $x_0 \in \mathscr{A}_0$, the sequence $\{x_n\}$ defined by $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ is Cauchy, and converges to an element x^* (say) in $\mathscr{A}_0$ that satifies $\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$. Such x^* is unique.

PROOF. Let $x_0 \in \mathscr{A}_0$ be arbitrary. By Lemma 3.1, we have a sequence $\{x_n\} \subset \mathscr{A}_0$ such that $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for $n = 1, 2, \dots$. Now, proceeding as in the proof of Theorem 3.2, we have $\{\mathscr{S}x_n\}$ is a Cauchy sequence. Since $\mathscr{B}$ has uniform approximation in $\mathscr{A}$ and since $\{\mathscr{S}x_n\}$ is Cauchy sequence, it follows that $\{gx_n\}$ is a Cauchy sequence. Since X is complete, $\mathscr{A}$ and $\mathscr{B}$ are closed subsets of X, we have $\mathscr{A}$ and $\mathscr{B}$ are complete and so there exist $w^* \in \mathscr{A}$ and $y^* \in \mathscr{B}$ such that $\lim_{n\to\infty} gx_n = w^*$ and $\lim_{n\to\infty} \mathscr{S}x_n = y^*$. Therefore $\rho(w^*, y^*) = \lim_{n\to\infty} \rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ and hence w^* is an element of $\mathscr{A}_0$. Since $\mathscr{A}_0 \subseteq g(\mathscr{A}_0), w^* = gx^*$ for some element x^* in $\mathscr{A}_0$. Since g is an isometry, we have

$$gx_n \to w^* = gx^* \text{ implies that } x_n \to x^*. \tag{3.5}$$

Since $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$, it follows that $\rho(u^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$ for some $x^* \in \mathscr{A}$. Also, since $\rho(gx_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ for all n, and $x_n \to x^*$. Since $\mathscr{S}$ is a proximally quasi-continuous, we have $u_{n_k} \to u^*$ for some subsequence $\{u_{n_k}\}$ of $\{u_n\}$. That is

$$gx_{n_k+1} \to u^* \text{ as } k \to \infty. \tag{3.6}$$

Therefore from (3.5) and (3.6), we have $gx^* = u^*$. Hence,
$\rho(gx^*, \mathscr{S}x^*) = \rho(u^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$.
Suppose that there is another element $b^* \in \mathscr{A}_0$ such that $\rho(gb^*, \mathscr{S}b^*) = \rho(\mathscr{A}, \mathscr{B})$ also we have $\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$ that implies $\tau + \mathscr{F}(\rho(gx^*, gb^*)) \leqslant \mathscr{F}(\rho(x^*, b^*))$. Since $\mathscr{S}$ is a proximal $\mathscr{F}^*-$weak contraction of the second kind and $\mathscr{S}$ preserves isometric distance with respect to g, we have
$\mathscr{F}(\rho(\mathscr{S}x^*, \mathscr{S}b^*)) = \mathscr{F}(\rho(\mathscr{S}gx^*, \mathscr{S}gb^*)) < \tau + \mathscr{F}(\rho(\mathscr{S}gx^*, \mathscr{S}gb^*)) \leqslant \mathscr{F}(\rho(\mathscr{S}x^*, \mathscr{S}b^*))$, a contradiction. Hence $\rho(\mathscr{S}x^*, \mathscr{S}b^*) = 0$. Therefore $\mathscr{S}x^* = \mathscr{S}b^*$. Since $\mathscr{B}$ has uniform approximation in $\mathscr{A}$, it follows that

$$\left.\begin{array}{r}\rho(gx^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})\\ \rho(gb^*, \mathscr{S}b^*) = \rho(\mathscr{A}, \mathscr{B})\\ 0 = \rho(\mathscr{S}x^*, \mathscr{S}b^*) < \delta\end{array}\right\} \Rightarrow \rho(gx^*, gb^*) < \epsilon.$$

Since g is an isometry, we have $\rho(x^*, b^*) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $x^* = b^*$. □

4. Corollaries and examples

REMARK 4.1. If g is the identity mapping in Theorem 3.1 and Theorem 3.2 then Theorem 2.1 and Theorem 2.2 follow as corollaries, respectively. Further, Theorem 2.4 of [**9**] follow as corollary to Theorem 3.3 when g is the identity mapping.

When g is the identity in Theorem 3.4, we have the following corollary.

COROLLARY 4.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty, closed subsets of a complete metric space X such that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty. Let the mapping $\mathscr{S}: \mathscr{A} \to \mathscr{B}$ be such that $\mathscr{A}$ has uniform $\mathscr{S}-$approximation in $\mathscr{B}$. Moreover, assume that*
a) $\mathscr{S}: \mathscr{A} \to \mathscr{B}$ is a proximal $\mathscr{F}^*-$ weak contraction of the first kind
b) $\mathscr{B}_0$ contains $\mathscr{S}(\mathscr{A}_0)$
c) $\mathscr{F}$ is continuous.
Then for any fixed element $x_0 \in \mathscr{A}_0$, the sequence $\{x_n\}$ defined by $\rho(x_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ is Cauchy, and converges to an element x^* (say) in $\mathscr{A}_0$ that satisfies $\rho(x^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$. Such x^* is unique.

When g is the identity in Theorem 3.5, we have the following corollary.

COROLLARY 4.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonempty, closed subsets of a complete metric space X such that $\mathscr{A}_0$ and $\mathscr{B}_0$ are nonempty. Moreover, assume that $\mathscr{B}$ has uniform approximation in $\mathscr{A}$. Let $\mathscr{S}: \mathscr{A} \to \mathscr{B}$ satisfy the following conditions:*
a) $\mathscr{S}: \mathscr{A} \to \mathscr{B}$ is a strong proximal $\mathscr{F}^*-$ weak contraction of the second kind
b) $\mathscr{B}_0$ contains $\mathscr{S}(\mathscr{A}_0)$
c) $\mathscr{S}$ preserves isometric distance with respect to g

d) $\mathscr{F}$ is continuous.
Then for any fixed element $x_0 \in \mathscr{A}_0$, the sequence $\{x_n\}$ defined by $\rho(x_{n+1}, \mathscr{S}x_n) = \rho(\mathscr{A}, \mathscr{B})$ is Cauchy, and converges to an element x^* (say) in $\mathscr{A}_0$ that satisfies $\rho(x^*, \mathscr{S}x^*) = \rho(\mathscr{A}, \mathscr{B})$. Such x^* is unique.

REMARK 4.2. Every proximal contraction $\mathscr{S}$ with contraction constant $\alpha \in (0,1)$ is a proximal $\mathscr{F}^*-$weak contraction of the first kind with $\mathscr{F}(a) = \ln(a), a \in (0,\infty)$ and $\tau = -\ln\alpha$.

REMARK 4.3. By Remark 4.2, we get the Theorem 3.1 in [**5**] follows as a corollary to Theorem 3.1.

EXAMPLE 4.1. Let $X = [0,1] \times [0,1]$ endowed with metric $\rho((u,v),(a,b)) = |u-a| + |v-b|$. Let $\mathscr{A} = \{(0,u); 0 \leqslant u \leqslant 1\}, \mathscr{B} = \{(1,v); 0 \leqslant v \leqslant 1\}$. It is clear that $\mathscr{A}_0 = \mathscr{A}$ and $\mathscr{B}_0 = \mathscr{B}$ and $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$. Moreover, $\mathscr{B}$ is approximatively compact with respect to $\mathscr{A}$. Define $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ by $\mathscr{S}((0,u)) = (1, \frac{u^2}{4}), 0 \leqslant u \leqslant 1$. Define $g : \mathscr{A} \to \mathscr{A}$ by $g((0,u)) = (0, 1-u)$. Therefore

$$\begin{aligned}\rho(gx, gy) &= \rho(g(0,u), g(0,v))\\ &= \rho((0,1-u),(0,1-v)) = |0|+|u-v| = |u-v| = |(0,u)-(0,v)| = \rho(x,y).\end{aligned}$$

Therefore g is an isomtery. For each $a = (0,u) \in \mathscr{A}_0$, there exist $b = (0,v) \in \mathscr{A}_0$ such that $(0,u) = g(b) = g(0,v) = (0,1-v) \Leftrightarrow u = 1-v$ implies $v = 1-u$. Therefore $\rho((0,u),(0,1-u)) = 1 = \rho(\mathscr{A},\mathscr{B})$ so that $b = (0,1-u) \in g(\mathscr{A}_0)$. Therefore $g(\mathscr{A}_0)$ contains $\mathscr{A}_0$. Let $a_1 = (0,u)$ and $a_2 = (0,v)$ with $u \neq v$, then $a_1, a_2 \in \mathscr{A}$ and $a_1 \neq a_2$. We choose $gu_1, gu_2 \in \mathscr{A}$ such that $\rho(gu_1, \mathscr{S}a_1) = \rho(gu_2, \mathscr{S}a_2) = \rho(\mathscr{A},\mathscr{B}) = 1$. $\mathscr{S}a_1 = \mathscr{S}((0,u)) = (1, \frac{u^2}{4})$ which implies that $gu_1 = (0, \frac{u^2}{4})$ $\mathscr{S}a_2 = \mathscr{S}((0,v)) = (1, \frac{v^2}{4})$ which implies that $gu_2 = (0, \frac{v^2}{4})$. Here we observe that $gu_1 \neq gu_2$. Now,

$$\rho(gu_1, \mathscr{S}a_1) = \rho((0, \tfrac{u^2}{4}), (1, \tfrac{u^2}{4})) = |1| + |\tfrac{u^2}{4} - \tfrac{u^2}{4}| = 1$$
$$\rho(gu_2, \mathscr{S}a_2) = \rho((0, \tfrac{v^2}{4}), (1, \tfrac{v^2}{4})) = |1| + |\tfrac{v^2}{4} - \tfrac{v^2}{4}| = 1.$$

Therefore $\rho(gu_1, \mathscr{S}a_1) = \rho(gu_2, \mathscr{S}a_2) = \rho(\mathscr{A},\mathscr{B}) = 1$. We choose $\tau = \log 2$ and $\mathscr{F}(\alpha) = \frac{-1}{\alpha} + \log\alpha,\ \alpha > 0$. Then $\mathscr{F} \in \Psi^*$, and $\mathscr{F}$ is continuous on $(0,\infty)$. We consider $\tau + \mathscr{F}(\rho(u_1,u_2)) = \tau + \mathscr{F}(\rho(gu_1, gu_2))$, since g is an isometry

$$\begin{aligned}
&= \log 2 + \mathscr{F}(\rho((0, \tfrac{u^2}{4}), (0, \tfrac{v^2}{4})))\\
&= \log 2 + \mathscr{F}(\tfrac{1}{4}|u^2 - v^2|)\\
&= \log 2 - \frac{1}{\frac{1}{4}|u^2-v^2|} + \log(\tfrac{1}{4}|u^2 - v^2|)\\
&= \log 2 - \frac{4}{|u^2-v^2|} + \log|u^2 - v^2| - \log 4\\
&\leqslant \log 2 - \frac{4}{|u^2-v^2|} + \log 2 + \log|u-v| - 2\log 2\\
&= \frac{-4}{|u^2-v^2|} + \log|u-v|\\
&\leqslant \frac{-1}{|u-v|} + \log|u-v|\\
&= \mathscr{F}(|u-v|)\\
&= \mathscr{F}(\rho((0,u),(0,v)))\\
&= \mathscr{F}(\rho(a_1,a_2)).
\end{aligned}$$

Therefore $\mathscr{S}$ is proximal $\mathscr{F}^*-$weak contraction of the first kind.
Hence $\mathscr{S}$ satisfies all the hypotheses of Theorem 3.1 and $(-2+2\sqrt{2}, -2+2\sqrt{2})$ is a unique best proximity point of $\mathscr{S}$.

EXAMPLE 4.2. Let $X = [0,1] \times [0,1]$ endowed with metric $\rho((u,v),(a,b)) = |u-a| + |v-b|$. Let $\mathscr{A} = \{(0,u); 0 \leqslant u \leqslant 1\}, \mathscr{B} = \{(1,v); 0 \leqslant v \leqslant 1\}$. It is clear that $\mathscr{A}_0 = \mathscr{A}$ and $\mathscr{B}_0 = \mathscr{B}$ and $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$. Moreover, $\mathscr{A}$ has uniform $\mathscr{S}-$approximation in $\mathscr{B}$. Define $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ by $\mathscr{S}((0,u)) = (1, 1 - \frac{u^2}{4}), 0 \leqslant u \leqslant 1$. Define $g : \mathscr{A} \to \mathscr{A}$ by $g((0,u)) = (0, 1-u)$. Therefore

$$\begin{aligned}\rho(gx,gy) &= \rho(g(0,u), g(0,v))\\ &= \rho((0,1-u),(0,1-v)) = |0| + |u-v| = |u-v| = |(0,u)-(0,v)| = \rho(x,y).\end{aligned}$$

Therefore g is an isomtery. For each $a = (0,u) \in \mathscr{A}_0$, there exist $b = (0,v) \in \mathscr{A}_0$ such that $(0,u) = g(b) = g(0,v) = (0,1-v) \Leftrightarrow u = 1-v$ implies $v = 1-u$. Therefore $\rho((0,u),(0,1-u)) = 1 = \rho(\mathscr{A},\mathscr{B})$ so that $b = (0,1-u) \in g(\mathscr{A}_0)$.
Therefore $g(\mathscr{A}_0)$ contains $\mathscr{A}_0$. Let $a_1 = (0,u)$ and $a_2 = (0,v)$ with $u \neq v$, then $a_1, a_2 \in \mathscr{A}$ and $a_1 \neq a_2$. We choose $gu_1, gu_2 \in \mathscr{A}$ such that $\rho(gu_1, \mathscr{S}a_1) = \rho(gu_2, \mathscr{S}a_2) = \rho(\mathscr{A},\mathscr{B}) = 1$. Then $\mathscr{S}a_1 = \mathscr{S}((0,u)) = (1, 1-\frac{u^2}{4})$ which implies that $gu_1 = (0, 1-\frac{u^2}{4})$, and $\mathscr{S}a_2 = \mathscr{S}((0,v)) = (1, 1-\frac{v^2}{4})$ which implies that $gu_2 = (0, 1-\frac{v^2}{4})$. Here we observe that $gu_1 \neq gu_2$. Now,

$$\rho(gu_1, \mathscr{S}a_1) = \rho((0,1-\tfrac{u^2}{4}),(1,1-\tfrac{u^2}{4})) = |1| + |1 - \tfrac{u^2}{4} - 1 + \tfrac{u^2}{4}| = 1$$
$$\rho(gu_2, \mathscr{S}a_2) = \rho((0,1-\tfrac{v^2}{4}),(1,1-\tfrac{v^2}{4})) = |1| + |1 - \tfrac{v^2}{4} - 1 + \tfrac{v^2}{4}| = 1.$$

Therefore $\rho(gu_1, \mathscr{S}a_1) = \rho(gu_2, \mathscr{S}a_2) = \rho(\mathscr{A},\mathscr{B}) = 1$. We choose $\tau = \log 2$ and $\mathscr{F}(\alpha) = \frac{-1}{\sqrt{\alpha}} + \log\alpha$, $\alpha > 0$. Then $\mathscr{F} \in \Psi^*$. Here we note that $\mathscr{F}$ is continuous on $(0,\infty)$. Thus, we have

$$\begin{aligned}\tau + \mathscr{F}(\rho(u_1,u_2)) &= \tau + \mathscr{F}(\rho(gu_1,gu_2)), \text{ since } g \text{ is an isometry}\\
&= \log 2 + \mathscr{F}(\rho((0,1-\tfrac{u^2}{4}),(0,1-\tfrac{v^2}{4})))\\
&= \log 2 + \mathscr{F}(\tfrac{1}{4}|u^2-v^2|)\\
&= \log 2 - \frac{1}{\sqrt{\frac{1}{4}|u^2-v^2|}} + \log(\tfrac{1}{4}|u^2-v^2|)\\
&= \log 2 - \frac{2}{\sqrt{|u^2-v^2|}} + \log|u^2-v^2| - \log 4\\
&\leqslant \log 2 - \frac{2}{\sqrt{|u^2-v^2|}} + \log 2 + \log|u-v| - 2\log 2\\
&= \frac{-2}{\sqrt{|u^2-v^2|}} + \log|u-v|\\
&\leqslant \frac{-1}{\sqrt{|u-v|}} + \log|u-v|\\
&= \mathscr{F}(|u-v|)\\
&= \mathscr{F}(\rho((0,u),(0,v)))\\
&= \mathscr{F}(\rho(a_1,a_2)).\end{aligned}$$

Therefore $\mathscr{S}$ is proximal $\mathscr{F}^*-$weak contraction of the first kind.
Hence $\mathscr{S}$ satisfies all the hypotheses of Theorem 3.4 and $(0,0)$ is a unique best proximity point of $\mathscr{S}$.

EXAMPLE 4.3. Let $X = [0,1] \times [0,1]$ endowed with metric $\rho((u,v),(a,b)) = |u-a| + |v-b|$. Let $\mathscr{A} = \{(0,u); 0 \leqslant u \leqslant 1\}, \mathscr{B} = \{(1,v); 0 \leqslant v \leqslant 1\}$. It is clear that $\mathscr{A}_0 = \mathscr{A}$ and $\mathscr{B}_0 = \mathscr{B}$ and $\mathscr{S}(\mathscr{A}_0) \subseteq \mathscr{B}_0$. Moreover, $\mathscr{B}$ has uniform approximation in $\mathscr{A}$. Define $\mathscr{S} : \mathscr{A} \to \mathscr{B}$ by $\mathscr{S}((0,u)) = (1, 1-\frac{u}{2}), 0 \leqslant u \leqslant 1$. It is easy to see that $\mathscr{S}$ is proximally quasi-continuous. Define $g : \mathscr{A} \to \mathscr{A}$ by $g((0,u)) = (0, 1-u)$. Therefore

$$\rho(gx, gy) = \rho(g(0,u), g(0,v)) = \rho((0,1-u),(0,1-v)) = |0|+|u-v| = |u-v| = |(0,u)-(0,v)| = \rho(x,y).$$

Therefore g is an isomtery. For each $a = (0,u) \in \mathscr{A}_0$, there exist $b = (0,v) \in \mathscr{A}_0$ such that $(0,u) = g(b) = g(0,v) = (0,1-v) \Leftrightarrow u = 1-v$ implies $v = 1-u$. Therefore $\rho((0,u),(0,1-u)) = 1 = \rho(\mathscr{A},\mathscr{B})$ so that $b = (0,1-u) \in g(\mathscr{A}_0)$.

Therefore $g(\mathscr{A}_0)$ contains $\mathscr{A}_0$. Let $a_1 = (0,u)$ and $a_2 = (0,v)$ with $u \neq v$, then $a_1, a_2 \in \mathscr{A}$ and $a_1 \neq a_2$. We choose $gu_1, gu_2 \in \mathscr{A}$ such that $\rho(gu_1, \mathscr{S}a_1) = \rho(gu_2, \mathscr{S}a_2) = \rho(\mathscr{A},\mathscr{B}) = 1$. Then $\mathscr{S}a_1 = \mathscr{S}((0,u)) = (1, 1-\frac{u}{2})$ which implies that $gu_1 = (0, 1-\frac{u}{2})$, and $\mathscr{S}a_2 = \mathscr{S}((0,v)) = (1, 1-\frac{v}{2})$ which implies that $gu_2 = (0, 1-\frac{v}{2})$. Here we observe that $gu_1 \neq gu_2$. Now,

$$\rho(gu_1, \mathscr{S}a_1) = \rho((0,1-\tfrac{u}{2}),(1,1-\tfrac{u}{2})) = |1| + |1-\tfrac{u}{2}-1+\tfrac{u}{2}| = 1$$
$$\rho(gu_2, \mathscr{S}a_2) = \rho((0,1-\tfrac{v}{2}),(1,1-\tfrac{v}{2})) = |1| + |1-\tfrac{v}{2}-1+\tfrac{v}{2}| = 1.$$

Therefore $\rho(gu_1, \mathscr{S}a_1) = \rho(gu_2, \mathscr{S}a_2) = \rho(\mathscr{A},\mathscr{B}) = 1$. We choose $\tau = 2$ and $\mathscr{F}(\alpha) = \sqrt{\alpha} - \frac{1}{\alpha}$, $\alpha > 0$. Then $\mathscr{F} \in \Psi^*$ and $\mathscr{F}$ is continuous on $(0,\infty)$.

Let $x = (0,u) \in \mathscr{A}, y = (0,v) \in \mathscr{A}$.

$$\begin{aligned}\rho(\mathscr{S}gx, \mathscr{S}gy) &= \rho(\mathscr{S}(0,1-u), \mathscr{S}(0,1-v)) \\ &= \rho((1, 1-\tfrac{1-u}{2}),(1,1-\tfrac{1-v}{2})) \\ &= |1-1| + |(1-\tfrac{1-u}{2}) - (1-\tfrac{1-v}{2})| \\ &= \tfrac{|u-v|}{2}.\end{aligned}$$

$$\begin{aligned}\rho(\mathscr{S}x, \mathscr{S}y) &= \rho(\mathscr{S}(0,u), \mathscr{S}(0,v)) \\ &= \rho((1,1-\tfrac{u}{2}),(1,\tfrac{1-v}{2})) \\ &= |\tfrac{u-v}{2}|\end{aligned}$$

so that $\rho(\mathscr{S}gx, \mathscr{S}gy) = \rho(\mathscr{S}x, \mathscr{S}y)$ for any $x, y \in \mathscr{A}$. Therefore, $\mathscr{S}$ preserves isometric distance with respect to g. Hence, for any $u_1, u_2 \in \mathscr{A}$ with $u_1 \neq u_2$ we have $\mathscr{S}u_1 \neq \mathscr{S}u_2$ and so

$$\begin{aligned}\tau + \mathscr{F}(\rho(\mathscr{S}u_1, \mathscr{S}u_2)) &= \tau + \mathscr{F}(\rho(\mathscr{S}gu_1, \mathscr{S}gu_2)) \\ &= 2 + \mathscr{F}(\rho(\mathscr{S}(0,1-\tfrac{u}{2}), \mathscr{S}(0,1-\tfrac{v}{2}))) \\ &= 2 + \mathscr{F}(\rho((1,1-\tfrac{1-\frac{u}{2}}{2}),(1,1-\tfrac{1-\frac{v}{2}}{2}))) \\ &= 2 + \mathscr{F}(\rho((1,\tfrac{u+2}{4}),(1,\tfrac{v+2}{4}))) \\ &= 2 + \mathscr{F}(\tfrac{1}{4}|u-v|) \\ &= 2 + \sqrt{\tfrac{1}{4}|u-v|} - \tfrac{1}{\frac{1}{4}|u-v|} \\ &= 2 + \sqrt{\tfrac{1}{4}|u-v|} - \tfrac{2}{|u-v|} - \tfrac{2}{|u-v|} \\ &\leqslant 2 + \sqrt{\tfrac{1}{4}|u-v|} - 2 - \tfrac{2}{|u-v|} \\ &= \sqrt{\tfrac{1}{4}|u-v|} - \tfrac{2}{|u-v|} \\ &\leqslant \sqrt{\tfrac{1}{2}|u-v|} - \tfrac{2}{|u-v|} \\ &= \mathscr{F}(\tfrac{1}{2}|u-v|) \\ &= \mathscr{F}(\rho((1,1-\tfrac{u}{2}),(1,1-\tfrac{v}{2}))) \\ &= \mathscr{F}(\rho(\mathscr{S}a_1, \mathscr{S}a_2)).\end{aligned}$$

Therefore $\mathscr{S}$ is strong proximal $\mathscr{F}^*$−weak contraction of the second kind.

Hence $\mathscr{S}$ satisfies all the hypotheses of Theorem 3.5 and $(0,0)$ is a unique best proximity point of $\mathscr{S}$.

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