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SOME COMMON FIXED POINT THEOREMS SATISFYING (CLR)-PROPERTY AND Φ-TYPE CONTRACTION IN $\mathcal{S}$-METRIC SPACES

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Abstract. The purpose of this paper is to prove some common fixed point theorems for two pairs of self-mappings satisfying (CLR)-property and some Φ-type contractive condition in the framework of $\mathcal{S}$-metric spaces. We also give some examples to validate the results. The results presented in this paper generalize, extend and unify several previous results in the existing literature.

1. Introduction

Banach's contraction principle in metric spaces is one of the most important results in the theory of fixed points and non-linear analysis. From 1922, when Stefan Banach ([**3**]) formulated the concept of contraction and proved the famous theorem, scientist and mathematicians around the world are publishing new results that are related either to establish a generalization of metric space or to get a improvement of contractive conditions.

In the literature, there are many extensions of the famous Banach contraction principle, which states that every self mapping $\mathcal{R}$ defined on a complete metric space (X, ρ) satisfying

$$\rho(\mathcal{R}(x), \mathcal{R}(y)) \leqslant \beta\, \rho(x, y), \tag{1.1}$$

for all $x, y \in X$, where $\beta \in (0, 1)$, has a unique fixed point and for every $r_0 \in X$ a sequence $\{\mathcal{R}^n r_0\}_{n \geqslant 1}$ is convergent to the fixed point. Inequality (1.1) also implies the continuity of $\mathcal{R}$.

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In 2006, Mustafa and Sims [**13**] introduced G-metric spaces as a generalizations of metric spaces and proved the existence of fixed points under different contractive conditions. In 2012, Sedghi *et al.* [**15**] introduced a new notion called S-metric space and studied its some properties and they also stated that S-metric space is a generalization of G-metric space. But Dung *et al.* [**4**] in 2014 showed by an example that S-metric space is not a generalization of G-metric space and conversely. Consequently, the class of S-metric spaces and the class of G-metric spaces are different.

On the other hand, Jungck and Rhoades [**10**] introduced the concept of weak compatibility in the year 1998. In 2002, Aamri and Moutawakil [**1**] introduced the new concept called $(E.A)$-property. In 2012, Imdad *et al.* [**7**] introduced the new concept called (CLR)-property for two pairs of self mappings and proved some common fixed point theorems using this new concept.

In 2016, Sedghi *et al.* [**17**] proved some existence of the unique common fixed point for the pair of weakly compatible self-mappings satisfying some Φ-type contractive conditions in the framework of $\mathcal{S}$-metric spaces and gave example to validate the results. The results presented in this paper extend and improve several results in the literature.

Recently, Sedghi *et al.* [**18**] proved some common fixed point theorems for four mappings satisfying generalized contractive condition in the set up of $\mathcal{S}$-metric spaces and gave examples to validate the results. The results presented in this paper extend and improve several results in the literature.

In this work, we prove some unique common fixed point theorems using (CLR)-property and satisfying some Φ-type contractive condition in the setting of $\mathcal{S}$-metric spaces and give some corollaries of the main results. We also illustrate some examples to support the results. Our results generalize, extend and enrich several existing results in the literature.

In the following we provide some basic definitions and preliminaries which we shall use in this paper.

2. Preliminaries

Following is the definition of $\mathcal{S}$-metric spaces (see, [**15**]).

Definition 2.1. ([**15**]) Let X be a nonempty set and let $\mathcal{S}\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $u, v, z, t \in X$:

$(S1)$ $\mathcal{S}(u,v,z) = 0$ if and only if $u = v = z$;

$(S2)$ $\mathcal{S}(u,v,z) \leqslant \mathcal{S}(u,u,t) + \mathcal{S}(v,v,t) + \mathcal{S}(z,z,t)$.

Then the function $\mathcal{S}$ is called an $\mathcal{S}$-metric on X and the pair $(X,\mathcal{S})$ is called an $\mathcal{S}$-metric space.

Example 2.1. ([**15**])

(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $\mathcal{S}(u,v,z) = \|v+z-2u\| + \|v-z\|$ is an $\mathcal{S}$-metric on X.

(2)Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $\mathcal{S}(u,v,z) = \|u-z\| + \|v-z\|$ is an $\mathcal{S}$-metric on X.

EXAMPLE 2.2. ([**16**]) Let $X = \mathbb{R}$ be the real line. Then $\mathcal{S}(u, v, z) = |u - z| + |v - z|$ for all $u, v, z \in \mathbb{R}$ is an $\mathcal{S}$-metric on X. This $\mathcal{S}$-metric on X is called the usual $\mathcal{S}$-metric on X.

EXAMPLE 2.3. ([**11**]) Let X be a non-empty set and d be an ordinary metric on X. Then $\mathcal{S}(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in \mathbb{R}$ is an $\mathcal{S}$-metric on X.

EXAMPLE 2.4. ([**18**]) Let X be a non-empty set and d_1, d_2 be two ordinary metrics on X. Then $\mathcal{S}(u, v, z) = d_1(u, z) + d_2(v, z)$ for all $u, v, z \in X$ is an $\mathcal{S}$-metric on X.

DEFINITION 2.2. Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. For $r > 0$ and $x \in X$ we define the open ball $\mathcal{B}_{\mathcal{S}}(u, r)$ and closed ball $\mathcal{B}_{\mathcal{S}}[u, r]$ with center u and radius r as follows, respectively:

$$\mathcal{B}_{\mathcal{S}}(u, r) = \{v \in X : \mathcal{S}(v, v, u) < r\},$$

$$\mathcal{B}_{\mathcal{S}}[u, r] = \{v \in X : \mathcal{S}(v, v, u) \leqslant r\}.$$

EXAMPLE 2.5. ([**16**]) Let $X = \mathbb{R}$. Denote $\mathcal{S}(u, v, z) = |v + z - 2u| + |v - z|$ for all $u, v, z \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{B}_{\mathcal{S}}(1, 2) &= \{v \in \mathbb{R} : \mathcal{S}(v, v, 1) < 2\} = \{v \in \mathbb{R} : |v - 1| < 1\} \\ &= \{v \in \mathbb{R} : 0 < v < 2\} = (0, 2), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_{\mathcal{S}}[2, 4] &= \{v \in \mathbb{R} : \mathcal{S}(v, v, 2) \leqslant 4\} = \{v \in \mathbb{R} : |v - 2| \leqslant 2\} \\ &= \{v \in \mathbb{R} : 0 \leqslant v \leqslant 4\} = [0, 4]. \end{aligned}$$

DEFINITION 2.3. ([**15**], [**16**]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and $A \subset X$.

(1) The subset A is said to be an open subset of X, if for every $u \in A$ there exists $r > 0$ such that $\mathcal{B}_{\mathcal{S}}(u, r) \subset A$.

(2) A sequence $\{u_n\}$ in X converges to $u \in X$ if $S(u_n, u_n, u) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $\mathcal{S}(u_n, u_n, u) < \varepsilon$. We denote this by $\lim_{n\to\infty} u_n = u$ or $u_n \to u$ as $n \to \infty$.

(3) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $\mathcal{S}(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$ we have $\mathcal{S}(u_n, u_n, u_m) < \varepsilon$.

(4) The $\mathcal{S}$-metric space $(X, \mathcal{S})$ is called complete if every Cauchy sequence in X is convergent in X.

(5) Let τ be the set of all $A \subset X$ with $u \in A$ and there exists $r > 0$ such that $\mathcal{B}_{\mathcal{S}}(u, r) \subset A$. Then τ is a topology on X (induced by the $\mathcal{S}$-metric space).

(6) A nonempty subset A of X is $\mathcal{S}$-closed if closure of A coincides with A.

DEFINITION 2.4. ([**15**]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. A mapping $\mathcal{G} \colon X \to X$ is said to be a contraction if there exists a constant $0 \leqslant k < 1$ such that

$$\mathcal{S}(\mathcal{G}u, \mathcal{G}v, \mathcal{G}z) \leqslant k\, \mathcal{S}(u, v, z) \tag{2.1}$$

for all $u, v, z \in X$.

Note: If the $\mathcal{S}$-metric space $(X, \mathcal{S})$ is complete then the mapping defined as above has a unique fixed point (see, [**15**]).

DEFINITION 2.5. ([**15**]) Let $(X, \mathcal{S})$ and $(Y, \mathcal{S}')$ be two $\mathcal{S}$-metric spaces. A function $g\colon X \to Y$ is said to be continuous at a point $u_0 \in X$ if for every sequence $\{u_n\}$ in X with $\mathcal{S}(u_n, u_n, u_0) \to 0$, $\mathcal{S}'(g(u_n), g(u_n), g(u_0)) \to 0$ as $n \to \infty$. We say that g is continuous on X if g is continuous at every point $u_0 \in X$.

DEFINITION 2.6. Let X be a non-empty set and let $\mathcal{P}, \mathcal{Q}\colon X \to X$ be two self mappings of X. Then a point $z \in X$ is called a
(i) fixed point of operator $\mathcal{P}$ if $\mathcal{P}(z) = z$;
(ii) common fixed point of $\mathcal{P}$ and $\mathcal{Q}$ if $\mathcal{P}(z) = \mathcal{Q}(z) = z$.

DEFINITION 2.7. ([**2**]) Let $\mathcal{P}$ and $\mathcal{Q}$ be single valued self-mappings on a set X. If $u = \mathcal{P}v = \mathcal{Q}v$ for some $v \in X$, then v is called a coincidence point point of $\mathcal{P}$ and $\mathcal{Q}$, and u is called a point of coincidence of $\mathcal{P}$ and $\mathcal{Q}$.

DEFINITION 2.8. ([**8**]) Let $\mathcal{P}$ and $\mathcal{Q}$ be single valued self-mappings on a set X. Mappings $\mathcal{P}$ and $\mathcal{Q}$ are said to be commuting if $\mathcal{P}\mathcal{Q}v = \mathcal{Q}\mathcal{P}v$ for all $v \in X$.

DEFINITION 2.9. ([**9**]) Let $\mathcal{P}$ and $\mathcal{Q}$ be single valued self-mappings on a set X. Mappings $\mathcal{P}$ and $\mathcal{Q}$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $\mathcal{P}v = \mathcal{Q}v$ for some $v \in X$ implies $\mathcal{P}\mathcal{Q}v = \mathcal{Q}\mathcal{P}v$.

DEFINITION 2.10. ([**7**]) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and $\mathcal{A}, \mathcal{B}, \mathcal{R}, \mathcal{T}\colon X \to X$ be four self mappings of X. We say that two pairs $(\mathcal{A}, \mathcal{R})$ and $(\mathcal{B}, \mathcal{T})$ of self maps of $\mathcal{S}$-metric space $(X, \mathcal{S})$ are said to satisfy common limit range property with respect to $\mathcal{R}$ and $\mathcal{T}$ if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{R}u_n = \lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{B}v_n = \lim_{n\to\infty} \mathcal{T}v_n = z,$$

for some $z \in \mathcal{R}(X) \cap \mathcal{T}(X)$ and it is denoted by $(CLR_{\mathcal{R}\mathcal{T}})$.

LEMMA 2.1. *(*[**15**]*, Lemma 2.5) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. Then, we have $\mathcal{S}(u, u, v) = \mathcal{S}(v, v, u)$ for all $u, v \in X$.*

LEMMA 2.2. *(*[**15**]*, Lemma 2.12) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. If $u_n \to u$ and $v_n \to v$ as $n \to \infty$ then $S(u_n, u_n, v_n) \to S(u, u, v)$ as $n \to \infty$.*

LEMMA 2.3. *(*[**5**]*, Lemma 8) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space and A is a non-empty subset of X. Then A is said to be $\mathcal{S}$-closed if and only if for any sequence $\{u_n\}$ in A such that $u_n \to u$ as $n \to \infty$, then $u \in A$.*

LEMMA 2.4. *(*[**15**]*) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. If $r > 0$ and $u \in X$, then the ball $\mathcal{B}_{\mathcal{S}}(u, r)$ is a subset of X.*

LEMMA 2.5. *(*[**16**]*) The limit of $\{u_n\}$ in $\mathcal{S}$-metric space $(X, \mathcal{S})$ is unique.*

LEMMA 2.6. *(*[**15**]*) Let $(X, \mathcal{S})$ be an $\mathcal{S}$-metric space. Then the convergent sequence $\{u_n\}$ in X is Cauchy.*

In the following lemma we see the relationship between a metric and S-metric.

LEMMA 2.7. *([**6**]) Let (X, d) be a metric space. Then the following properties are satisfied:*

(1) *$\mathcal{S}_d(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in X$ is an $\mathcal{S}$-metric on X.*
(2) *$u_n \to u$ in (X, d) if and only if $u_n \to u$ in $(X, \mathcal{S}_d)$.*
(3) *$\{u_n\}$ is Cauchy in (X, d) if and only if $\{u_n\}$ is Cauchy in $(X, \mathcal{S}_d)$.*
(4) *(X, d) is complete if and only if $(X, \mathcal{S}_d)$ is complete.*

We call the function $\mathcal{S}_d$ defined in Lemma 2.7 (1) as the $\mathcal{S}$-metric generated by the metric d. It can be found an example of an $\mathcal{S}$-metric which is not generated by any metric in [**6, 14**].

PROPOSITION 2.1. *([**2**]) Let $\mathcal{P}$ and $\mathcal{Q}$ be weakly compatible self mappings on a set X. If $\mathcal{P}$ and $\mathcal{Q}$ have a unique point of coincidence $u = \mathcal{P}v = \mathcal{Q}v$, then u is the unique common fixed point of $\mathcal{P}$ and $\mathcal{Q}$.*

In 1977, *Matkowski* [**12**] introduced the Φ-maps as the following: let Φ be the set of all functions ϕ such that $\phi\colon [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, \infty)$. If $\phi \in \Phi$, then ϕ is called a Φ-map. Furthermore, if ϕ is a Φ-map, then

($\mathbf{\Phi_1}$) $\phi(t) < t$ for all $t \in (0, \infty)$;
($\mathbf{\Phi_2}$) $\phi(0) = 0$.

3. Common fixed point theorems

In this section, we prove some unique common fixed point theorems for two pairs of self-mappings satisfying (CLR)-property and some Φ-type contractive conditions in the setting of $\mathcal{S}$-metric spaces.

THEOREM 3.1. *Let $(X, \mathcal{S})$ be a $\mathcal{S}$-metric space and let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{T} \colon X \to X$ be four self-mappings of X satisfying the following conditions:* (i)

$$\begin{aligned} \mathcal{S}(\mathcal{P}u, \mathcal{P}u, \mathcal{Q}v) \quad \leqslant \quad \max\Big\{ & \phi(\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{T}v)), \phi(\mathcal{S}(\mathcal{Q}v, \mathcal{Q}v, \mathcal{P}u)), \\ & \phi(\mathcal{S}(\mathcal{Q}v, \mathcal{Q}v, \mathcal{T}v)), \\ & \phi\Big(\mathcal{S}(\mathcal{Q}v, \mathcal{Q}v, \mathcal{P}u)\frac{[1 + \mathcal{S}(\mathcal{Q}v, \mathcal{Q}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{P}u, \mathcal{P}u, \mathcal{Q}v)]}\Big), \\ & \phi\Big(\mathcal{S}(\mathcal{P}u, \mathcal{P}u, \mathcal{T}v)\frac{[1 + \mathcal{S}(\mathcal{Q}v, \mathcal{Q}v, \mathcal{T}v)]}{[1 + \mathcal{S}(\mathcal{P}u, \mathcal{P}u, \mathcal{Q}v)]}\Big)\Big\}, \end{aligned} \tag{3.1}$$

for all $u, v \in X$, where $\phi \in \Phi$;

(ii) *the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ are weakly compatible.*

If the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy ($CLR_{\mathcal{R}\mathcal{T}}$)-property, then the mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point in X.

PROOF. Since by hypothesis the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy ($CLR_{\mathcal{R}\mathcal{T}}$)-property, we can find two sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{R}(u_n) = \lim_{n\to\infty} \mathcal{P}(u_n) = \lim_{n\to\infty} \mathcal{Q}(v_n) = \lim_{n\to\infty} \mathcal{T}(v_n) = \mu$$

for some $\mu \in \mathcal{R}(X) \cap \mathcal{T}(X)$. Then $\mu = \mathcal{T}\beta_1 = \mathcal{R}\beta_2$ for some $\beta_1, \beta_2 \in X$. Now, we show that $\mathcal{Q}\beta_1 = \mathcal{T}\beta_1$. For each $n \in \mathbb{N}$, from equation (3.1), we have

$$\begin{aligned}
\mathcal{S}(\mathcal{P}u_n, \mathcal{P}u_n, \mathcal{Q}\beta_1) \leqslant \max\Big\{ & \phi(\mathcal{S}(\mathcal{R}u_n, \mathcal{R}u_n, \mathcal{T}\beta_1)), \phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}u_n)), \\
& \phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)), \\
& \phi\Big(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}u_n)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{P}u_n, \mathcal{P}u_n, \mathcal{Q}\beta_1)]}\Big), \\
& \phi\Big(\mathcal{S}(\mathcal{P}u_n, \mathcal{P}u_n, \mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{P}u_n, \mathcal{P}u_n, \mathcal{Q}\beta_1)]}\Big)\Big\}.
\end{aligned}$$

Now, letting $n \to \infty$ in the above inequality and using $(S1)$, property of ϕ and Lemma 2.1, we get

$$\begin{aligned}
\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1) &\leqslant \max\Big\{ \phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{T}\beta_1)), \phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)), \\
&\quad \phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)), \\
&\quad \phi\Big(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}\Big), \\
&\quad \phi\Big(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}\Big)\Big\} \\
&= \max\Big\{ \phi(0), \phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)), \phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)), \\
&\quad \phi\Big(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)\frac{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}\Big), \\
&\quad \phi\Big(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}\Big)\Big\} \\
&= \max\Big\{ 0, \phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)), \phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)), \\
&\quad \phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)), 0\Big\} \\
&= \phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)) < \mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1),
\end{aligned}$$

which is a contradiction. Hence we conclude that $\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1) = 0$. It follows that $\mathcal{Q}\beta_1 = \mathcal{T}\beta_1$. Now, we show that $\mathcal{P}\beta_2 = \mathcal{R}\beta_2$. For each $n \in \mathbb{N}$, from equation (3.1), we have

$$\begin{aligned}
\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}v_n) \leqslant \max\Big\{ & \phi(\mathcal{S}(\mathcal{R}\beta_2, \mathcal{R}\beta_2, \mathcal{T}v_n)), \phi(\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{P}\beta_2)), \\
& \phi(\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{T}v_n)), \\
& \phi\Big(\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{P}\beta_2)\frac{[1+\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}v_n)]}\Big), \\
& \phi\Big(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{T}v_n)\frac{[1+\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}v_n)]}\Big)\Big\}.
\end{aligned}$$

Now, letting $n \to \infty$ in the above inequality and using Lemma 2.1, ($S1$) and the property of ϕ, we get

$$\begin{aligned}
\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu) &\leqslant \max\Big\{\phi(\mathcal{S}(\mu,\mu,\mu)), \phi(\mathcal{S}(\mu,\mu,\mathcal{P}\beta_2)), \phi(\mathcal{S}(\mu,\mu,\mu)),\\
&\qquad \phi\Big(\mathcal{S}(\mu,\mu,\mathcal{P}\beta_2)\frac{[1+\mathcal{S}(\mu,\mu,\mu)]}{[1+\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu)]}\Big),\\
&\qquad \phi\Big(\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu)\frac{[1+\mathcal{S}(\mu,\mu,\mu)]}{[1+\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu)]}\Big)\Big\}\\
&\leqslant \max\Big\{\phi(0), \phi(\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu)), \phi(0),\\
&\qquad \phi(\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu)), \phi(\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu))\Big\}\\
&= \max\Big\{0, \phi(\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu)), 0, \phi(\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu)),\\
&\qquad \phi(\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu))\Big\}\\
&= \phi(\mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu)) < \mathcal{S}(\mathcal{P}\beta_2,\mathcal{P}\beta_2,\mu),
\end{aligned}$$

which is a contradiction. Hence we conclude that $\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu) = 0$ and hence it follows that $\mathcal{P}\beta_2 = \mu$ and hence $\mathcal{P}\beta_2 = \mathcal{R}\beta_2 = \mathcal{Q}\beta_1 = \mathcal{T}\beta_1 = \mu$. Since the pair $(\mathcal{P}, \mathcal{R})$ is weakly compatible and $\mathcal{P}\beta_2 = \mathcal{R}\beta_2$ implies that $\mathcal{P}\mathcal{R}\beta_2 = \mathcal{R}\mathcal{P}\beta_2$ and hence $\mathcal{P}\mu = \mathcal{R}\mu$. Now since the pair $(\mathcal{Q}, \mathcal{T})$ is weakly compatible and $\mathcal{Q}\beta_1 = \mathcal{T}\beta_1$ implies that $\mathcal{T}\mathcal{Q}\beta_1 = \mathcal{Q}\mathcal{T}\beta_1$ and hence $\mathcal{Q}\mu = \mathcal{T}\mu$.

Now to show that μ is a common fixed point of $\mathcal{P}$ and $\mathcal{R}$. For this, we consider

$$\begin{aligned}
\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1) &\leqslant \max\Big\{\phi(\mathcal{S}(\mathcal{R}\mu,\mathcal{R}\mu,\mathcal{T}\beta_1)), \phi(\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mathcal{P}\mu)),\\
&\qquad \phi(\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mathcal{T}\beta_1)),\\
&\qquad \phi\Big(\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mathcal{P}\mu)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{Q}\beta_1)]}\Big),\\
&\qquad \phi\Big(\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{Q}\beta_1)]}\Big)\Big\}\\
&= \max\Big\{\phi(\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mu)), \phi(\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mathcal{P}\mu)),\\
&\qquad \phi(\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mu)),\\
&\qquad \phi\Big(\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mathcal{P}\mu)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mu)]}{[1+\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{Q}\beta_1)]}\Big),\\
&\qquad \phi\Big(\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mu)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1,\mathcal{Q}\beta_1,\mu)]}{[1+\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{Q}\beta_1)]}\Big)\Big\}.
\end{aligned}$$

Using the condition ($S1$), Lemma 2.1, property of ϕ and $\mu = \mathcal{Q}\beta_1$ in the above inequality, we obtain

$$\begin{aligned}
\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1) &\leqslant \max\Big\{\phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)), \phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}\mu)), \\
&\qquad \phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{Q}\beta_1)), \\
&\qquad \phi\Big(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}\mu)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{Q}\beta_1)]}{[1+\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)]}\Big), \\
&\qquad \phi\Big(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{Q}\beta_1)]}{[1+\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)]}\Big)\Big\} \\
&= \max\Big\{\phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)), \phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}\mu)), \phi(0), \\
&\qquad \phi\Big(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}\mu)\frac{[1+0]}{[1+\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)]}\Big), \\
&\qquad \phi\Big(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)\frac{[1+0]}{[1+\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)]}\Big)\Big\} \\
&= \max\Big\{\phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)), \phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)), \phi(0), \\
&\qquad \phi\Big(\frac{\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}\mu)}{[1+\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)]}\Big), \phi\Big(\frac{\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)}{[1+\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)]}\Big)\Big\} \\
&\leqslant \max\Big\{\phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)), \phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)), 0, \\
&\qquad \phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)), \phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1))\Big\} \\
&= \phi(\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1)) < \mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1),
\end{aligned}$$

which is a contradiction. Hence we conclude that $\mathcal{S}(\mathcal{P}\mu, \mathcal{P}\mu, \mathcal{Q}\beta_1) = 0$. This will imply that $\mathcal{Q}\beta_1 = \mathcal{P}\mu$ and hence $\mathcal{P}\mu = \mathcal{R}\mu = \mu$. This shows that μ is a common fixed point of $\mathcal{P}$ and $\mathcal{R}$.

Now we show that μ is a common fixed point of $\mathcal{Q}$ and $\mathcal{T}$. For this, we consider the inequality (3.1) and using Lemma 2.1, ($S1$) and the property of ϕ, we have

$$\begin{aligned}
\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}\mu)) &\leqslant \max\Big\{\phi(\mathcal{S}(\mathcal{R}\beta_2, \mathcal{R}\beta_2, \mathcal{T}\mu)), \phi(\mathcal{S}(\mathcal{Q}\mu, \mathcal{Q}\mu, \mathcal{P}\beta_2)), \\
&\qquad \phi(\mathcal{S}(\mathcal{Q}\mu, \mathcal{Q}\mu, \mathcal{T}\mu)), \\
&\qquad \phi\Big(\mathcal{S}(\mathcal{Q}\mu, \mathcal{Q}\mu, \mathcal{P}\beta_2)\frac{[1+\mathcal{S}(\mathcal{Q}\mu, \mathcal{Q}\mu, \mathcal{T}\mu)]}{[1+\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}\mu)]}\Big), \\
&\qquad \phi\Big(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{T}\mu)\frac{[1+\mathcal{S}(\mathcal{Q}\mu, \mathcal{Q}\mu, \mathcal{T}\mu)]}{[1+\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}\mu)]}\Big)\Big\} \\
&= \max\Big\{\phi(\mathcal{S}(\mu, \mu, \mathcal{T}\mu)), \phi(\mathcal{S}(\mathcal{T}\mu, \mathcal{T}\mu, \mu)), \phi(\mathcal{S}(\mathcal{T}\mu, \mathcal{T}\mu, \mathcal{T}\mu)), \\
&\qquad \phi\Big(\mathcal{S}(\mathcal{T}\mu, \mathcal{T}\mu, \mu)\frac{[1+\mathcal{S}(\mathcal{T}\mu, \mathcal{T}\mu, \mathcal{T}\mu)]}{[1+\mathcal{S}(\mu, \mu, \mathcal{T}\mu)]}\Big), \\
&\qquad \phi\Big(\mathcal{S}(\mu, \mu, \mathcal{T}\mu)\frac{[1+\mathcal{S}(\mathcal{T}\mu, \mathcal{T}\mu, \mathcal{T}\mu)]}{[1+\mathcal{S}(\mu, \mu, , \mathcal{T}\mu)]}\Big)\Big\}.
\end{aligned}$$

Using the condition $(S1)$, Lemma 2.1, the property of ϕ, $\mathcal{Q}\mu = \mathcal{T}\mu$ and $\mu = \mathcal{P}\beta_2$ in the above inequality, we obtain

$$
\begin{aligned}
\mathcal{S}(\mu,\mu,\mathcal{T}\mu)) &\leqslant \max\Big\{\phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)), \phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)), \phi(0),\\
&\qquad \phi\Big(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)\frac{[1+0]}{[1+\mathcal{S}(\mu,\mu,\mathcal{T}\mu)]}\Big),\\
&\qquad \phi\Big(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)\frac{[1+0]}{[1+\mathcal{S}(\mu,\mu,,\mathcal{T}\mu)]}\Big)\Big\}\\
&= \max\Big\{\phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)), \phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)), 0,\\
&\qquad \phi\Big(\frac{\mathcal{S}(\mu,\mu,\mathcal{T}\mu)}{[1+\mathcal{S}(\mu,\mu,\mathcal{T}\mu)]}\Big), \phi\Big(\frac{\mathcal{S}(\mu,\mu,\mathcal{T}\mu)}{[1+\mathcal{S}(\mu,\mu,,\mathcal{T}\mu)]}\Big)\Big\}\\
&\leqslant \max\Big\{\phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)), \phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)), 0, \phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)),\\
&\qquad \phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu))\Big\}\\
&= \phi(\mathcal{S}(\mu,\mu,\mathcal{T}\mu)) < \mathcal{S}(\mu,\mu,\mathcal{T}\mu),
\end{aligned}
$$

which is a contradiction. Hence we conclude that $\mathcal{S}(\mu,\mu,\mathcal{T}\mu) = 0$. This will imply that $\mathcal{T}\mu = \mu$ and hence $\mathcal{Q}\mu = \mathcal{T}\mu = \mu$. This shows that μ is a common fixed point of $\mathcal{Q}$ and $\mathcal{T}$. Hence μ is a common fixed point of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$.

Now, we show uniqueness of the common fixed point. Let us assume that μ' be another common fixed point of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$ such that $\mathcal{P}\mu' = \mathcal{Q}\mu' = \mathcal{R}\mu' = \mathcal{T}\mu' = \mu$ with $\mu' \neq \mu$. Again we consider the given inequality (3.1) and using the condition $(S1)$, Lemma 2.1 and the properties of ϕ, we have

$$
\begin{aligned}
\mathcal{S}(\mu,\mu,\mu') &= \mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{Q}\mu')\\
&\leqslant \max\Big\{\phi(\mathcal{S}(\mathcal{R}\mu,\mathcal{R}\mu,\mathcal{T}\mu')), \phi(\mathcal{S}(\mathcal{Q}\mu',\mathcal{Q}\mu',\mathcal{P}\mu)),\\
&\qquad \phi(\mathcal{S}(\mathcal{Q}\mu',\mathcal{Q}\mu',\mathcal{T}\mu')),\\
&\qquad \phi\Big(\mathcal{S}(\mathcal{Q}\mu',\mathcal{Q}\mu',\mathcal{P}\mu)\frac{[1+\mathcal{S}(\mathcal{Q}\mu',\mathcal{Q}\mu',\mathcal{T}\mu')]}{[1+\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{Q}\mu')]}\Big),\\
&\qquad \phi\Big(\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{T}\mu')\frac{[1+\mathcal{S}(\mathcal{Q}\mu',\mathcal{Q}\mu',\mathcal{T}\mu')]}{[1+\mathcal{S}(\mathcal{P}\mu,\mathcal{P}\mu,\mathcal{Q}\mu')]}\Big)\Big\}\\
&= \max\Big\{\phi(\mathcal{S}(\mu,\mu,\mu')), \phi(\mathcal{S}(\mu',\mu',\mu)), \phi(\mathcal{S}(\mu',\mu',\mu')),\\
&\qquad \phi\Big(\mathcal{S}(\mu',\mu',\mu)\frac{[1+\mathcal{S}(\mu',\mu',\mu')]}{[1+\mathcal{S}(\mu,\mu,\mu')]}\Big),\\
&\qquad \phi\Big(\mathcal{S}(\mu,\mu,\mu')\frac{[1+\mathcal{S}(\mu',\mu',\mu')]}{[1+\mathcal{S}(\mu,\mu,\mu')]}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \max\Big\{\phi(\mathcal{S}(\mu,\mu,\mu')),\phi(\mathcal{S}(\mu,\mu,\mu')),\phi(0),\\
&\qquad \phi\Big(\mathcal{S}(\mu,\mu,\mu')\frac{[1+0]}{[1+\mathcal{S}(\mu,\mu,\mu')]}\Big),\\
&\qquad \phi\Big(\mathcal{S}(\mu,\mu,\mu')\frac{[1+0]}{[1+\mathcal{S}(\mu,\mu,\mu')]}\Big)\Big\}\\
&= \max\Big\{\phi(\mathcal{S}(\mu,\mu,\mu')),\phi(\mathcal{S}(\mu,\mu,\mu')),0,\\
&\qquad \phi\Big(\frac{\mathcal{S}(\mu,\mu,\mu')}{[1+\mathcal{S}(\mu,\mu,\mu')]}\Big),\phi\Big(\frac{\mathcal{S}(\mu,\mu,\mu')}{[1+\mathcal{S}(\mu,\mu,\mu')]}\Big)\Big\}\\
&\leqslant \max\Big\{\phi(\mathcal{S}(\mu,\mu,\mu')),\phi(\mathcal{S}(\mu,\mu,\mu')),0,\phi(\mathcal{S}(\mu,\mu,\mu')),\\
&\qquad \phi(\mathcal{S}(\mu,\mu,\mu'))\Big\}\\
&= \phi(\mathcal{S}(\mu,\mu,\mu')) < \mathcal{S}(\mu,\mu,\mu'),
\end{aligned}
$$

which is a contradiction. Hence we conclude that $\mathcal{S}(\mu,\mu,\mu')=0$, tat is, $\mu=\mu'$. This shows that the common fixed point of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$ is unique. This completes the proof. □

THEOREM 3.2. *Let $(X,\mathcal{S})$ be a $\mathcal{S}$-metric space and let $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{T}\colon X\to X$ be four self-mappings of X satisfying the following conditions: (i)*

$$
\begin{aligned}
\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v) \;&\leqslant\; d_1\,\phi(\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{T}v)) + d_2\,\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u))\\
&\quad + d_3\,\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v))\\
&\quad + d_4\,\phi\Big(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)\frac{[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big)\\
&\quad + d_5\,\phi\Big(\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v)\frac{[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big),
\end{aligned} \tag{3.2}
$$

for all $u,v\in X$, where $\phi\in\Phi$ and $d_1,d_2,d_3,d_4,d_5>0$ are nonnegative reals such that $d_1+d_2+d_3+d_4+d_5<1$;

(ii) the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ are weakly compatible.

If the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ satisfy $(CLR_{\mathcal{RT}})$-property, then the mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point in X.

PROOF. Since by hypothesis the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ satisfy $(CLR_{\mathcal{RT}})$-property, we can find two sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty}\mathcal{R}(u_n)=\lim_{n\to\infty}\mathcal{P}(u_n)=\lim_{n\to\infty}\mathcal{Q}(v_n)=\lim_{n\to\infty}\mathcal{T}(v_n)=\mu$$

for some $\mu \in \mathcal{R}(X) \cap \mathcal{T}(X)$. Then $\mu = \mathcal{T}\beta_1 = \mathcal{R}\beta_2$ for some $\beta_1, \beta_2 \in X$. Now, we show that $\mathcal{Q}\beta_1 = \mathcal{T}\beta_1$. For each $n \in \mathbb{N}$, from equation (3.2), we have

$$\begin{aligned}\mathcal{S}(\mathcal{P}u_n, \mathcal{P}u_n, \mathcal{Q}\beta_1) \leqslant\ & d_1\,\phi(\mathcal{S}(\mathcal{R}u_n, \mathcal{R}u_n, \mathcal{T}\beta_1)) + d_2\,\phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}u_n)) \\ & + d_3\,\phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)) \\ & + d_4\,\phi\Big(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{P}u_n)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{P}u_n, \mathcal{P}u_n, \mathcal{Q}\beta_1)]}\Big) \\ & + d_5\,\phi\Big(\mathcal{S}(\mathcal{P}u_n, \mathcal{P}u_n, \mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{P}u_n, \mathcal{P}u_n, \mathcal{Q}\beta_1)]}\Big).\end{aligned}$$

Now, letting $n \to \infty$ in the above inequality and using $(S1)$, property of ϕ and Lemma 2.1, we get

$$\begin{aligned}\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1) \leqslant\ & d_1\,\phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{T}\beta_1)) + d_2\,\phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)) \\ & + d_3\,\phi(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)) \\ & + d_4\,\phi\Big(\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}\Big) \\ & + d_5\,\phi\Big(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{Q}\beta_1, \mathcal{Q}\beta_1, \mathcal{T}\beta_1)]}{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}\Big) \\ =\ & d_1\,\phi(0) + d_2\,\phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)) + d_3\,\phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)) \\ & + d_4\,\phi\Big(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)\frac{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}\Big) \\ & + d_5\,\phi\Big(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{T}\beta_1)\frac{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}{[1+\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)]}\Big) \\ =\ & d_1\,.(0) + d_2\,\phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)) + d_3\,\phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)) \\ & + d_4\,\phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)) + d_5\,.(0) \\ =\ & (d_2 + d_3 + d_4)\,\phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)) \\ \leqslant\ & (d_1 + d_2 + d_3 + d_4 + d_5)\,\phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)) \\ \leqslant\ & \phi(\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1)), \text{ since } d_1 + d_2 + d_3 + d_4 + d_5 < 1 \\ <\ & \mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1),\end{aligned}$$

which is a contradiction. Hence we conclude that $\mathcal{S}(\mathcal{T}\beta_1, \mathcal{T}\beta_1, \mathcal{Q}\beta_1) = 0$. It follows that $\mathcal{Q}\beta_1 = \mathcal{T}\beta_1$. Now, we show that $\mathcal{P}\beta_2 = \mathcal{R}\beta_2$. For each $n \in \mathbb{N}$, from equation (3.2), we have

$$\begin{aligned}\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}v_n) \leqslant\ & d_1\,\phi(\mathcal{S}(\mathcal{R}\beta_2, \mathcal{R}\beta_2, \mathcal{T}v_n)) + d_2\,\phi(\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{P}\beta_2)) \\ & + d_3\,\phi(\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{T}v_n)) \\ & + d_4\,\phi\Big(\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{P}\beta_2)\frac{[1+\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}v_n)]}\Big) \\ & + d_5\,\phi\Big(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{T}v_n)\frac{[1+\mathcal{S}(\mathcal{Q}v_n, \mathcal{Q}v_n, \mathcal{T}v_n)]}{[1+\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mathcal{Q}v_n)]}\Big).\end{aligned}$$

Now, letting $n \to \infty$ in the above inequality and using Lemma 2.1, $(S1)$ and the property of ϕ, we get

$$\begin{aligned}
\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu) &\leqslant d_1\, \phi(\mathcal{S}(\mu, \mu, \mu)) + d_2\, \phi(\mathcal{S}(\mu, \mu, \mathcal{P}\beta_2)) + d_3\, \phi(\mathcal{S}(\mu, \mu, \mu)) \\
&\quad + d_4\, \phi\Big(\mu, \mu, \mathcal{P}\beta_2)\frac{[1 + \mathcal{S}(\mu, \mu, \mu)]}{[1 + \mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)]}\Big) \\
&\quad + d_5\, \phi\Big(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)\frac{[1 + \mathcal{S}(\mu, \mu, \mu)]}{[1 + \mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)]}\Big) \\
&\leqslant d_1\, .\phi(0) + d_2\, \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)) + d_3\, .\phi(0) \\
&\quad + d_4\, \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)) + d_5\, \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)) \\
&= d_1\, .(0) + d_2\, \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)) + d_3\, .(0) \\
&\quad + d_4\, \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)) + d_5\, \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)) \\
&= (d_2 + d_4 + d_5)\, \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)) \\
&\leqslant (d_1 + d_2 + d_3 + d_4 + d_5)\, \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)) \\
&\leqslant \phi(\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu)), \text{ since } d_1 + d_2 + d_3 + d_4 + d_5 < 1 \\
&< \mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu),
\end{aligned}$$

which is a contradiction. Hence we conclude that $\mathcal{S}(\mathcal{P}\beta_2, \mathcal{P}\beta_2, \mu) = 0$ and hence it follows that $\mathcal{P}\beta_2 = \mu$ and hence $\mathcal{P}\beta_2 = \mathcal{R}\beta_2 = \mathcal{Q}\beta_1 = \mathcal{T}\beta_1 = \mu$. Since the pair $(\mathcal{P}, \mathcal{R})$ is weakly compatible and $\mathcal{P}\beta_2 = \mathcal{R}\beta_2$ implies that $\mathcal{P}\mathcal{R}\beta_2 = \mathcal{R}\mathcal{P}\beta_2$ and hence $\mathcal{P}\mu = \mathcal{R}\mu$. Now since the pair $(\mathcal{Q}, \mathcal{T})$ is weakly compatible and $\mathcal{Q}\beta_1 = \mathcal{T}\beta_1$ implies that $\mathcal{T}\mathcal{Q}\beta_1 = \mathcal{Q}\mathcal{T}\beta_1$ and hence $\mathcal{Q}\mu = \mathcal{T}\mu$. Now to show that μ is a common fixed point of $\mathcal{P}$ and $\mathcal{R}$. Rest of the proof follows from Theorem 3.1. This completes the proof. □

REMARK 3.1. (i) Completeness of the space X is relaxed in Theorem 3.1 and Theorem 3.2.

(ii) Continuity of the mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$ are relaxed in Theorem 3.1 and Theorem 3.2.

4. Consequences of Theorem 3.1

COROLLARY 4.1. *Let $(X, \mathcal{S})$ be a $\mathcal{S}$-metric space and let $\mathcal{P}, \mathcal{R}\colon X \to X$ be two self-mappings of X satisfying the following conditions:* (i)

$$\begin{aligned}
\mathcal{S}(\mathcal{P}u, \mathcal{P}u, \mathcal{P}v) \leqslant \max\Big\{ &\phi(\mathcal{S}(\mathcal{R}u, \mathcal{R}u, \mathcal{R}v)), \phi(\mathcal{S}(\mathcal{P}v, \mathcal{P}v, \mathcal{P}u)), \\
&\phi(\mathcal{S}(\mathcal{P}v, \mathcal{P}v, \mathcal{R}v)), \\
&\phi\Big(\mathcal{S}(\mathcal{P}v, \mathcal{P}v, \mathcal{P}u)\frac{[1 + \mathcal{S}(\mathcal{P}v, \mathcal{P}v, \mathcal{R}v)]}{[1 + \mathcal{S}(\mathcal{P}u, \mathcal{P}u, \mathcal{P}v)]}\Big), \\
&\phi\Big(\mathcal{S}(\mathcal{P}u, \mathcal{P}u, \mathcal{R}v)\frac{[1 + \mathcal{S}(\mathcal{P}v, \mathcal{P}v, \mathcal{R}v)]}{[1 + \mathcal{S}(\mathcal{P}u, \mathcal{P}u, \mathcal{P}v)]}\Big)\Big\},
\end{aligned}$$

for all $u, v \in X$, where $\phi \in \Phi$;

(ii) *the pair $(\mathcal{P}, \mathcal{R})$ is weakly compatible.*

If the pair $(\mathcal{P},\mathcal{R})$ satisfies $(CLR_{\mathcal{RT}})$-property, then the mappings $\mathcal{P}$ and $\mathcal{R}$ have a unique common fixed point in X.

Proof. Putting $\mathcal{P}=\mathcal{Q}$ and $\mathcal{R}=\mathcal{T}$ in inequality (3.1). Then all conditions of Theorem 3.1 are satisfied and hence the result follows. □

Corollary 4.2. *Let $(X,\mathcal{S})$ be a $\mathcal{S}$-metric space and let $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{T}\colon X\to X$ be four self-mappings of X satisfying the following conditions: (i)*

$$\begin{aligned}\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v) \quad \leqslant \quad \mathcal{K}\max\Big\{&\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{T}v),\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u),\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v),\\ &\frac{\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]},\\ &\frac{\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big\},\end{aligned}$$

for all $u,v\in X$, where $\mathcal{K}\in[0,1)$;

(ii) the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ are weakly compatible.

If the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ satisfy $(CLR_{\mathcal{RT}})$-property, then the mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof. Putting $\phi(t)=\mathcal{K}\,t$ for all $t\geqslant 0$ in inequality (3.1). Then all conditions of Theorem 3.1 are satisfied and hence the result follows. □

Corollary 4.3. *Let $(X,\mathcal{S})$ be a $\mathcal{S}$-metric space and let $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{T}\colon X\to X$ be four self-mappings of X satisfying the following conditions: (i)*

$$\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v) \quad \leqslant \quad \mathcal{A}_1\,\mathcal{N}_1^{\mathcal{S}}(u,u,v)+\mathcal{A}_2\,\mathcal{N}_2^{\mathcal{S}}(u,u,v),$$

for all $u,v\in X$, where $\phi\in\Phi$, $\mathcal{A}_1,\mathcal{A}_2$ are nonnegative reals with $\mathcal{A}_1+\mathcal{A}_2<1$,

$$\mathcal{N}_1^{\mathcal{S}}(u,u,v)=\max\Big\{\phi(\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{T}v)),\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)),\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v))\Big\},$$

and

$$\begin{aligned}\mathcal{N}_2^{\mathcal{S}}(u,u,v) \quad = \quad \max\Big\{&\phi\Big(\frac{\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big),\\ &\phi\Big(\frac{\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big)\Big\},\end{aligned}$$

(ii) the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ are weakly compatible.

If the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ satisfy $(CLR_{\mathcal{RT}})$-property, then the mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point in X.

Proof. Follows from Theorem 3.1 and Theorem 3.2. □

Example 4.1. Let $X=[0,2]$ and $\mathcal{S}(u,v,z)=\max\{|u-v|,|v-z|,|z-u|\}$ for all $u,v,z\in X$ and $\phi\in\Phi$. Denote $\mathcal{P},\mathcal{R}\colon X\to X$ by

$$\mathcal{P}(u)=1 \quad \text{and} \quad \mathcal{R}(u)=2-u.$$

We obtain that $\mathcal{P}$ and $\mathcal{R}$ satisfy the inequality of Corollary 4.1. Indeed, we have

$$\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{P}v)=0,$$

and

$$\max\Big\{\phi(\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{R}v)),\phi(\mathcal{S}(\mathcal{P}v,\mathcal{P}v,\mathcal{P}u)),\phi(\mathcal{S}(\mathcal{P}v,\mathcal{P}v,\mathcal{R}v)),$$

$$\phi\Big(\mathcal{P}v,\mathcal{P}v,\mathcal{P}u)\frac{[1+\mathcal{S}(\mathcal{P}v,\mathcal{P}v,\mathcal{R}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{P}v)]}\Big),$$

$$\phi\Big(\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{R}v)\frac{[1+\mathcal{S}(\mathcal{P}v,\mathcal{P}v,\mathcal{R}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{P}v)]}\Big)\Big\}$$

$$= \max\Big\{\phi(|u-v|),\phi(0),\phi(|1-v|),\phi(0),\phi([|1-v|(1+|1-v|)])\Big\}$$

$$= \max\Big\{\phi(|u-v|),0,\phi(|1-v|),0,\phi([|1-v|(1+|1-v|)])\Big\}.$$

That is,

$$\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{P}v)$$

$$\leqslant \max\Big\{\phi(\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{R}v)),\phi(\mathcal{S}(\mathcal{P}v,\mathcal{P}v,\mathcal{P}u)),$$
$$\phi(\mathcal{S}(\mathcal{P}v,\mathcal{P}v,\mathcal{R}v)),$$
$$\phi\Big(\mathcal{P}v,\mathcal{P}v,\mathcal{P}u)\frac{[1+\mathcal{S}(\mathcal{P}v,\mathcal{P}v,\mathcal{R}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{P}v)]}\Big),$$
$$\phi\Big(\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{R}v)\frac{[1+\mathcal{S}(\mathcal{P}v,\mathcal{P}v,\mathcal{R}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{P}v)]}\Big)\Big\}.$$

It is easy to show that $\mathcal{P}$ and $\mathcal{R}$ are weakly compatible maps, that is, the pair $(\mathcal{P},\mathcal{R})$ is weakly compatible. Now, we show that the pair $(\mathcal{P},\mathcal{R})$ satisfies (CLR) property. For this, consider the sequence $\{q_n\} = \{1+\frac{1}{2n+1}\}_{n\geqslant 1}$. Clearly the sequence $\{q_n\}$ is in X and note that $\mathcal{P}q_n = 1$ and $\mathcal{R}q_n = 2-q_n = 2-\{1+\frac{1}{2n+1}\}$ for all $n\in\mathbb{N}$. This will implies that

$$\mathcal{S}(\mathcal{P}q_n,\mathcal{P}q_n,1) = \mathcal{S}(1,1,1) = \max\{1,1,1\} = 1 \text{ as } n\to\infty.$$

This shows that $\mathcal{P}q_n \to 1$ as $n\to\infty$.

Also note that

$$\mathcal{S}(\mathcal{R}q_n,\mathcal{R}q_n,1) = \mathcal{S}\Big(2-\{1+\frac{1}{2n+1}\},2-\{1+\frac{1}{2n+1}\},1\Big)$$

$$= \max\Big\{2-\{1+\frac{1}{2n+1}\},2-\{1+\frac{1}{2n+1}\},1\Big\}$$

$$= 2-\{1+\frac{1}{2n+1}\} \to 1 \text{ as } n\to\infty.$$

This shows that $\mathcal{R}q_n \to 1$ as $n\to\infty$.

Thus there exists a sequence $\{q_n\}$ in X such that $\mathcal{P}q_n \to 1$ and $\mathcal{R}q_n \to 1$ as $n\to\infty$. Hence the pair $(\mathcal{P},\mathcal{R})$ satisfies (CLR) property.

Hence all the assumptions in Corollary 4.1 are satisfied. Consequently $\mathcal{P}$ and $\mathcal{R}$ have a unique common fixed point, say, $u = 1$ in X.

Example 4.2. Let $X = [0,1]$. We define the function $\mathcal{S}\colon X^3 \to [0,\infty)$ by

$$\mathcal{S}(u,v,z) = \begin{cases} 0, & \text{if } u = v = z, \\ \max\{u,v,z\}, & \text{if otherwise,} \end{cases}$$

for all $u,v,z \in X$, then $\mathcal{S}$ is an $\mathcal{S}$-metric on X. Define four self-maps $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{T}$: $X \to X$ on X by $\mathcal{P}(u) = \frac{u}{4}$, $\mathcal{Q}(u) = \frac{u}{4}$, $\mathcal{T}(u) = u$ and $\mathcal{R}(u) = \frac{u}{2}$ for all $u \in X$. Let $u,v \in X$. We also define $\phi\colon [0,\infty) \to [0,\infty)$ by $\phi(\alpha) = \frac{\alpha}{2}$ for all $\alpha \in [0,\infty)$. Clearly ϕ is continuous on $[0,\infty)$ satisfying $\phi(0) = 0$ and $0 < \phi(\alpha) < \alpha$ for all $\alpha > 0$. Now consider the following cases:

Case I. (1) Let $u < v$. Then we have

$$\begin{aligned}
\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v) &= \mathcal{S}\Big(\frac{u}{4},\frac{u}{4},\frac{v}{4}\Big) = \max\Big\{\frac{u}{4},\frac{u}{4},\frac{v}{4}\Big\} = \frac{v}{4},\\
\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{T}v) &= \mathcal{S}\Big(\frac{u}{2},\frac{u}{2},v\Big) = \max\Big\{\frac{u}{2},\frac{u}{2},v\Big\} = v,\\
\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u) &= \mathcal{S}\Big(\frac{v}{4},\frac{v}{4},\frac{u}{4}\Big) = \max\Big\{\frac{v}{4},\frac{v}{4},\frac{u}{4}\Big\} = \frac{v}{4},\\
\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v) &= \mathcal{S}\Big(\frac{v}{4},\frac{v}{4},v\Big) = \max\Big\{\frac{v}{4},\frac{v}{4},v\Big\} = v,\\
\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v) &= \mathcal{S}\Big(\frac{u}{4},\frac{u}{4},v\Big) = \max\Big\{\frac{u}{4},\frac{u}{4},v\Big\} = v,\\
\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{R}u) &= \mathcal{S}\Big(\frac{v}{4},\frac{v}{4},\frac{u}{2}\Big) = \max\Big\{\frac{v}{4},\frac{v}{4},\frac{u}{2}\Big\} = \frac{v}{4}.
\end{aligned}$$

Now using inequality (3.1) and the property of ϕ, we have

$$\begin{aligned}
\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v) &= \frac{v}{4}\\
&\leqslant \max\Big\{\phi(\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{T}v)),\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)),\\
&\qquad \phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)),\\
&\qquad \phi\Big(\frac{\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big),\\
&\qquad \phi\Big(\frac{\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big)\Big\}\\
&= \max\Big\{\phi(v),\phi\Big(\frac{v}{4}\Big),\phi(v),\phi\Big(\frac{v(1+v)}{(v+4)}\Big),\phi\Big(\frac{4v(1+v)}{(v+4)}\Big)\Big\}\\
&= \max\Big\{\frac{v}{2},\frac{v}{8},\frac{v}{2},\frac{v(1+v)}{2(v+4)},\frac{4v(1+v)}{2(v+4)}\Big\}\\
&= \frac{4v(1+v)}{2(v+4)},
\end{aligned}$$

that is,

$$\frac{v}{4} \leqslant \frac{4v(1+v)}{2(v+4)}.$$

Taking $u = 0$ and $v = 1$, we obtain

$$\frac{1}{4} \leqslant \frac{4}{5},$$

which is true.

Hence we conclude that

$$\begin{aligned} \mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v) \;\; &\leqslant \;\; \max\Big\{\phi(\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{T}v)),\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)), \\ &\qquad \phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)), \\ &\qquad \phi\Big(\frac{\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big), \\ &\qquad \phi\Big(\frac{\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big)\Big\}. \end{aligned}$$

(2) Now using inequality (3.2) of Theorem 3.2, we have

$$\begin{aligned} \mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v) &= \frac{v}{4} \\ &\leqslant d_1\,\phi(\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{T}v)) + d_21\,\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)), \\ &\quad +d_3\,\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)) \\ &\quad +d_4\,\phi\Big(\frac{\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big) \\ &\quad +d_5\,\phi\Big(\frac{\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big) \\ &= d_1\,\phi(v) + d_2\,\phi(\frac{v}{4}) + d_3\,\phi(v) \\ &\quad +d_4\,\phi\Big(\frac{v(1+v)}{(v+4)}\Big) + d_5\,\phi\Big(\frac{4v(1+v)}{(v+4)}\Big), \end{aligned}$$

that is,

$$\begin{aligned} \frac{v}{4} &\leqslant d_1\,\phi(v) + d_2\,\phi(\frac{v}{4}) + d_3\,\phi(v) \\ &\quad +d_4\,\phi\Big(\frac{v(1+v)}{(v+4)}\Big) + d_5\,\phi\Big(\frac{4v(1+v)}{(v+4)}\Big) \\ &= d_1\,\Big(\frac{v}{2}\Big) + d_2\,\Big(\frac{v}{8}\Big) + d_3\,\Big(\frac{v}{2}\Big) \\ &\quad +d_4\,\Big(\frac{v(1+v)}{2(v+4)}\Big) + d_5\,\Big(\frac{4v(1+v)}{2(v+4)}\Big). \end{aligned}$$

Taking $u = 0$ and $v = 1$, we obtain

$$\begin{aligned} \frac{1}{4} &\leqslant (\frac{d_1}{2}) + (\frac{d_2}{8}) + (\frac{d_3}{2}) \\ &\quad +(\frac{d_4}{5}) + (\frac{4d_5}{5}). \end{aligned}$$

The above inequality is satisfied for $d_1 = \frac{1}{5}$, $d_2 = \frac{1}{6}$, $d_3 = \frac{1}{4}$, $d_4 = \frac{1}{8}$ and $d_5 = \frac{1}{8}$ with $d_1 + d_2 + d_3 + d_4 + d_5 < 1$.

Hence we conclude that

$$\begin{aligned}\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v) &\leqslant d_1\,\phi(\mathcal{S}(\mathcal{R}u,\mathcal{R}u,\mathcal{T}v))+d_2\,\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u))\\ &\quad +d_3\,\phi(\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v))\\ &\quad +d_4\,\phi\Big(\frac{\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{P}u)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big)\\ &\quad +d_5\,\phi\Big(\frac{\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{T}v)[1+\mathcal{S}(\mathcal{Q}v,\mathcal{Q}v,\mathcal{T}v)]}{[1+\mathcal{S}(\mathcal{P}u,\mathcal{P}u,\mathcal{Q}v)]}\Big).\end{aligned}$$

Case II. Now we show that the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ are weakly compatible. For this, suppose that $\mathcal{T}u=\mathcal{Q}u$ for $u\in X$. Then $u=\frac{u}{4}$. It follows that $u=0$. Now, we consider $\mathcal{T}\mathcal{Q}(u)=\mathcal{T}(\mathcal{Q}u)=\mathcal{T}(0)=0$ and $\mathcal{Q}\mathcal{T}(u)=\mathcal{Q}(\mathcal{T}u)=\mathcal{Q}(0)=0$. Thus, the pair $(\mathcal{Q},\mathcal{T})$ is weakly compatible. Now, let $\mathcal{P}u=\mathcal{R}u$ for $u\in X$. This implies that $\frac{u}{4}=\frac{u}{2}$ and hence $u=0$. Now, we consider $\mathcal{P}\mathcal{R}(u)=\mathcal{P}(\mathcal{R}u)=\mathcal{P}(0)=0$ and $\mathcal{R}\mathcal{P}(u)=\mathcal{R}(\mathcal{P}u)=\mathcal{R}(0)=0$. It follows that the pair $(\mathcal{P},\mathcal{R})$ is also weakly compatible.

Case III. Now we show that the pairs $(\mathcal{P},\mathcal{R})$ and $(\mathcal{Q},\mathcal{T})$ satisfy (CLR_{RT}) property. For this, we choose the sequences $\{u_n\}=\{\frac{1}{n}\}_{n\geqslant 1}$ and $\{v_n\}=\{\frac{1}{2n+3}\}_{n\geqslant 1}$. Clearly the sequences $\{u_n\}$ and $\{v_n\}$ are in X. Then we have

$$\begin{aligned}S(\mathcal{R}u_n,\mathcal{R}u_n,0)&=S\Big(\frac{1}{2n},\frac{1}{2n},0\Big)=\max\Big\{\frac{1}{2n},\frac{1}{2n},0\Big\}\\ &=\frac{1}{2n}\to 0 \text{ as } n\to\infty.\end{aligned}$$

This shows that $\mathcal{R}u_n\to 0$ as $n\to\infty$.

Also we observe that

$$\begin{aligned}S(\mathcal{P}u_n,\mathcal{P}u_n,0)&=S\Big(\frac{1}{4n},\frac{1}{4n},0\Big)=\max\Big\{\frac{1}{4n},\frac{1}{4n},0\Big\}\\ &=\frac{1}{4n}\to 0 \text{ as } n\to\infty.\end{aligned}$$

This shows that $\mathcal{P}u_n\to 0$ as $n\to\infty$.

Similarly, we obtain that

$$\begin{aligned}S(\mathcal{Q}v_n,\mathcal{Q}v_n,0)&=S\Big(\frac{1}{4(2n+3)},\frac{1}{4(2n+3)},0\Big)=\max\Big\{\frac{1}{4(2n+3)},\frac{1}{4(2n+3)},0\Big\}\\ &=\frac{1}{4(2n+3)}\to 0 \text{ as } n\to\infty.\end{aligned}$$

This shows that $\mathcal{Q}v_n\to 0$ as $n\to\infty$.

Also we observe that

$$\begin{aligned}S(\mathcal{T}v_n,\mathcal{T}v_n,0)&=S\Big(\frac{1}{2n+3},\frac{1}{2n+3},0\Big)=\max\Big\{\frac{1}{2n+3},\frac{1}{2n+3},0\Big\}\\ &=\frac{1}{2n+3}\to 0 \text{ as } n\to\infty.\end{aligned}$$

This shows that $\mathcal{T}v_n\to 0$ as $n\to\infty$.

Since $\mathcal{R}(0) = 0 = \mathcal{T}(0)$, we have $0 \in \mathcal{R}(X) \cap \mathcal{T}(X)$. Therefore there exist sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{R}(u_n) = \lim_{n\to\infty} \mathcal{P}(u_n) = \lim_{n\to\infty} \mathcal{T}(v_n) = \lim_{n\to\infty} \mathcal{Q}(v_n).$$

Therefore the pairs $(\mathcal{P}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{T})$ satisfy (CLR_{RT}) property.

Thus all the conditions of Theorem 3.1 and Theorem 3.2 are satisfied and hence the mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{T}$ have a unique common fixed point, namely $u = 0 \in X$.

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