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GALOIS AND PATAKI CONNECTIONS FOR FUNCTIONS OF TWO VARIABLES AND RESIDUATED, PREORDERED GROUPOIDS

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Abstract. Having in mind Galois connections and residuated structures, we introduce and investigate the following two basic definitions:

1. Suppose that

(a) X and Y are generalized ordered sets and Z is a set;
(b) F is a function of $X \times Z$ to Y and G is a function of $Z \times Y$ to X.
(c) for all $x \in X$, $y \in Y$ and $z \in Z$, we have

$$F(x, z) \leqslant y \quad \iff \quad x \leqslant G(z, y).$$

Then, we say that the function F is increasingly G–normal.

2. Suppose that

(a) $X(*)$ and $X(\bullet)$ are goupoids;
(b) $X(\leqslant)$ is a generalized ordered set;
(c) for all $x, y, z \in X$, we have

$$x * z \leqslant y \quad \iff \quad x \leqslant z \bullet y.$$

Then, we say that the structure $X(*, \bullet, \leqslant)$ is an increasingly normal, generalized ordered bigroupoid.

1. Introduction

The most important particular case of "Galois connection" was already considered by G. Birkhoff, under the name "polarity", in the first edition of his famous book "Lattice Theory" [**6**, p. 122].

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The observations of Birkhoff were extended to posets (partially ordered sets) by O. Ore [**56**], who having in mind the classical Galois theory of algebraic equations introduced the term "Galois connexion",

The next important step, in the theory of Galois connections, was made by J. Schmidt [**75**]. However, he was mainly interested in the original setting of Birkhoff despite the papers of Everett [**30**], Riquet [**68**] and Pickert [**64**].

Schmidt actually observed that if f is a function of one poset X to another Y, and g is a function of Y to X, then the pair (f, g) may be defined to be an increasing Galois connection between X and Y if for all $x \in X$ and $y \in Y$

$$f(x) \leqslant y \quad \iff \quad x \leqslant g(y).$$

Curiously enough, in [**34**, p. 18] and [**45**], the increasingness of the corresponding functions was also postulated. Namely, it is a consequence of the above equivalence even if X and Y are assumed to be only prosets (preordered sets) [**96**].

Most of the former authors considered decreasing Galois connections by assuming that the functions f and g are decreasing and the functions $\varphi = g \circ f$ and $\psi = f \circ g$ are extensive in the sense that $x \leqslant \varphi(x)$ and $y \leqslant \psi(y)$ for all $x \in X$ and $y \in Y$.

Increasing Galois connections, by using analogous assumptions were frequently studied under the name residuated mappings [**24, 9**]. Their advantage lies mainly in the fact that the compositions of residuated maps are also residuated maps.

Now, if (f, g) is an increasing Galois connection between two gosets (generalized ordered sets) X and Y and $\varphi = g \circ f$, then for all $u, v \in X$

$$f(u) \leqslant f(v) \iff u \leqslant g\big(f(v)\big) \iff u \leqslant (g \circ f)(v) \iff u \leqslant \varphi(v).$$

This shows that before Galois connections it is more convenient to investigate first another, more simple connection which usually lies between closure operations and Galois connections.

Thus, if φ is a function of the goset X to itself such that for all $u, v \in X$

$$f(u) \leqslant f(v) \quad \iff \quad u \leqslant \varphi(v),$$

then the pair (f, φ) may be called an increasing Pataki connection between X and Y [**94**].

Namely, if $\mathfrak{F}$ is a structure (set-valued function) and $\square$ is a unary operation for relators (families of relations), then Pataki [**60**] called the function $\mathfrak{F}$ to be $\square$–increasing if, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X

$$\mathfrak{F}_{\mathcal{R}} \subseteq \mathfrak{F}_{\mathcal{S}} \quad \iff \quad \mathcal{R} \subseteq \mathcal{S}^{\square}.$$

Several particular cases of the above Galois and Pataki connections were formerly also considered by the present author [**84**]. Moreover, he also determined the Galois adjoints of some particular structures for relators [**93**].

For a primary illustration of this situation, we can note that if $\mathcal{R}$ is a relator on X to Y, then for any $B \subseteq Y$ we may naturally define

$$\operatorname{Int}_{\mathcal{R}}(B) = \big\{ A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subseteq B \big\}$$

and $\operatorname{int}_{\mathcal{R}}(B) = \{ x \in X : \quad \{x\} \in \operatorname{Int}_{\mathcal{R}}(B) \}$.

Moreover, if in particular $\mathcal{R}$ is a relator on X, then we may also naturally define $\tau_{\mathcal{R}} = \{ A \subseteq X : \quad A \in \operatorname{Int}_{\mathcal{R}}(A) \}$,

$$\mathcal{T}_{\mathcal{R}} = \{ A \subseteq X : \quad A \subseteq \operatorname{int}_{\mathcal{R}}(A) \} \qquad \text{and} \qquad \mathcal{E}_{\mathcal{R}} = \{ A \subseteq X : \quad \operatorname{int}_{\mathcal{R}}(A) \neq \emptyset \} .$$

Thus, only the most widely used structure $\mathcal{T}$ is not, in general, union-preserving. Moreover, for a relator $\mathcal{R}$ on X, there does not, in general, exist a largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$.

However, Mala [**49, 51**] could still find a projection operation $\Diamond$ for relators such that for any two nonvoid relators $\mathcal{R}$ and $\mathcal{S}$ on X we could have

$$\mathcal{T}_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{S}} \quad \iff \quad \mathcal{R}^{\Diamond} \subseteq \mathcal{S}^{\Diamond} .$$

In the sequel, if (f, g) is an increasing Galois connection, then following a more convenient terminology of [**96**], we shall say that f is increasingly g–normal. While, if (f, φ) is an increasing Pataki connection, then we shall say that f is increasingly φ–regular.

Moreover, we shall introduce and investigate the following two basic definitions.

Definition 1.1. Suppose that

(a) X and Y are gosets and Z is a set;

(b) F is a function of $X \times Z$ to Y and G is a function of $Z \times Y$ to X;

(c) for all $x \in X$, $y \in Y$ and $z \in Z$, we have

$$F(x, z) \leqslant y \quad \iff \quad x \leqslant G(z, y) .$$

Then, we say that the function F is increasingly G–normal.

Definition 1.2. Suppose that

(a) $X(\leqslant)$ is a goset;

(b) $X(*)$ and $X(\bullet)$ are goupoids;

(c) for all $x, y, z \in X$, we have

$$x * z \leqslant y \quad \iff \quad x \leqslant z \bullet y .$$

Then, we say that the structure $X(*, \bullet, \leqslant)$ is an increasingly normal, generalized ordered bigroupoid.

For an easy illustration of the above two definitions, we can state here the following two relational examples.

Example 1.1. Suppose that $\mathcal{R}$ is a relator on X to Y, and for all $A \subseteq X$, $B \subseteq Y$ and $R \in \mathcal{R}$ define

$$F(B, R) = \operatorname{cl}_{R^{-1}}(B) \qquad \text{and} \qquad G(R, A) = \operatorname{int}_R(B) .$$

Then, F is an increasingly G–normal function of $\mathcal{P}(Y) \times \mathcal{R}$ to $\mathcal{P}(X)$.

Example 1.2. Suppose that X is a set, and for all $R, S, T \subseteq X^2$ define

$$R * T = T \circ R \qquad \text{and} \qquad T \bullet B = \big(S^c * T^{-1} \big)^c .$$

Then, $\mathcal{P}(X^2)(*, \bullet, \subseteq)$ is an increasingly normal, partially ordered bigroupoid.

Remark 1.1. Here, we can also note that G and $\bullet$ are uniquely determined by F and $*$, respectively.

Therefore, instead of "increasingly normal" we may write "uniquely increasingly normal" in the above two examples.

2. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a *relation* on X to Y. In particular, a relation F on X to itself is simply called a relation on X. And, $\Delta_X = \{ (x, x) : \ x \in X \}$ is called the *identity relation* on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subseteq X$ the sets $F(x) = \{ y \in Y : \ (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images* or *neighbourhoods* of x and A under F, respectively.

If $(x, y) \in F$, then instead of $y \in F(x)$, we may also write $x\,F\,y$. However, instead of $F[A]$, we cannot write $F(A)$. Namely, it may occur that, in addition to $A \subseteq X$, we also have $A \in X$.

The sets $D_F = \{ x \in X : \ F(x) \neq \emptyset \}$ and $R_F = F[X]$ are called the *domain* and *range* of F, respectively. And, if $D_F = X$, then we say that F is a *relation of X to Y*, or that F is a *non-partial relation on X to Y*.

If F is a relation on X to Y and $U \subseteq D_F$, then the relation $F \,|\, U = F \cap (U \times Y)$ is called the *restriction* of F to U. Moreover, if F and G are relations on X to Y such that $D_F \subseteq D_G$ and $F = G \,|\, D_F$, then G is called an *extension* of F.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ instead of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation* on X. While, a function $*$ of X^2 to X is called a *binary operation* on X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection function* of F if $f(x) \in F(x)$ for all $x \in D_F$. Thus, by the Axiom of Choice [**39**], we can see that every relation is the union of its selection functions.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ_F of X to $\mathcal{P}(Y)$ and Φ_F of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_F(x) = F(x)$ for all $x \in X$ and $\Phi_F(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful objects than relations on X to Y. In [**99, 104, 105**], they were briefly called *corelations on X to Y*.

However, if U is a relation on $\mathcal{P}(X)$ to Y and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that U is a *super relation* and V is a *hyper relation on X to Y* [**109**]. Thus, closures (proximities) [**113**] are super (hyper) relations.

Note that a super relation on X to Y is an arbitrary subset of $\mathcal{P}(X)\times Y$. While, a corelation on X to Y is a particular subset of $\mathcal{P}(X)\times\mathcal{P}(Y)$. Thus, set inclusion is a natural partial order for super relations, but not for corelations.

For a relation F on X to Y, the relation, $F^c = (X\times Y)\setminus F$ is called the *complement* of F. Thus, it can be shown that $F^c(x) = F(x)^c = Y\setminus F(x)$ for all $x\in X$, and $F^c[A]^c = \bigcap_{a\in A} F(a)$ for all $A\subseteq X$.

Moreover, the relation $F^{-1} = \{(y, x):\ (x, y)\in F\}$ is called the *inverse* of F. Thus, it can be shown that $F^{-1}[B] = \{x\in X:\ F(x)\cap B\neq\emptyset\}$ for all $B\subseteq Y$, and in particular $D_F = F^{-1}[Y]$.

If F is a relation on X to Y, then we have $F = \bigcup_{x\in X}\{x\}\times F(x)$. Therefore, the values $F(x)$, where $x\in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x\in X$.

For instance, if G is a relation on Y to Z, then the *composition product relation* $G\circ F$ can be defined such that $(G\circ F)(x) = G[F(x)]$ for all $x\in X$. Thus, it can be shown that $(G\circ F)[A] = G\big[F[A]\big]$ also holds for all $A\subseteq X$.

While, if G is a relation on Z to W, then the *box product relation* $F\boxtimes G$ can be defined such that $(F\boxtimes G)(x, z) = F(x)\times G(z)$ for all $x\in X$ and $z\in Z$. Thus, it can be shown that $(F\boxtimes G)[A] = G\circ A\circ F^{-1}$ for all $A\subseteq X\times Z$ [**98**].

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

3. Some important relational properties

Now, a relation R on X may be defined to be *reflexive* if $R^0 = \Delta_X\subseteq R$, and *transitive* if $R^2 = R\circ R\subseteq R$. Moreover, R may be defined to be *symmetric* if $R^{-1}\subseteq R$, *antisymmetric* if $R\cap R^{-1}\subseteq R^0$, and *total* if $X^2\subseteq R\cup R^{-1}$.

In addition to the above well-known, basic properties, several further remarkable relational properties were also studied in [**87**] with the help of the self closure and interior relations $R^- = R^{-1}\circ R$ and $R^\circ = R^{c-c} = \big(R^{-1}\circ R^c\big)^c$.

In the sequel, as it is usual, a reflexive and transitive (symmetric) relation will be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation will be called an *equivalence (partial order) relation*.

If R is a relation on X, then we may also naturally define $R^n = R\circ R^{n-1}$ for $n\in\mathbb{N}$. Moreover, we may also define $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, R^∞ is the smallest preorder relation on X containing R [**35**].

Now, in contrast to $(R^c)^c = R$ and $(R^{-1})^{-1} = R$, we have $(R^\infty)^\infty = R^\infty$. And, analogously to $(R^c)^{-1} = (R^{-1})^c$, we also have $(R^\infty)^{-1} = (R^{-1})^\infty$. Moreover, R may be briefly defined to be *well-chained* if $X^2\subseteq R^\infty$ [**43**].

For $A\subseteq X$, the *Pervin relation* $R_A = A^2\cup(A^c\times X)$ is an important preorder on X [**63**]. While, for a *pseudometric* d on X, the *Weil surrounding* $B_r^d = \{(x, y)\in X^2:\ d(x, y) < r\}$, with $r > 0$, is only a tolerance on X [**117**].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cup \big(A^c\big)^2$ is already an equivalence relation on X. And, more generally, if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

Now, as a straightforward generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker–Lindgren relation* $R_{(A,B)} = (A \times B) \cup (A^c \times Y)$ [**38**].

However, it is now more important to note that if $\mathcal{A} = (A_n)_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{P}(X)$, then the *Cantor relation* $R_{\mathcal{A}} = \Delta_X \cup \bigcup_{n=1}^{\infty} \big(A_n \times A_n^c\big)$ is also an important preorder on X [**58, 40**].

Note that if R is only reflexive relation on X and $x \in X$, then $\mathcal{A}_R(x) = \big(R^n(x)\big)_{n=1}^{\infty}$ is already an increasing sequence in $\mathcal{P}(X)$. Thus, the preorder relation $R_{\mathcal{A}_R(x)}$ may also be naturally investigated.

Moreover, for a real function φ of X and a quasi-pseudo-metric d on X [**31**], the *Brøndsted relation* $R_{(\varphi,d)} = \{(x, y) \in X^2 : \quad d(x, y) \leqslant \varphi(y) - \varphi(x)\}$ is also an important preorder on X [**14**] .

From this relation, by letting φ and d to be the zero functions, we can obtain the *specialization* and *preference relations* $R_d = \{(x, y) \in X^2 : \quad d(x, y) = 0\}$ and $R_{\varphi} = \{(x, y) \in X^2 : \ \varphi(x) \leqslant \varphi(y)\}$, respectively. (See [**21, 115**].)

In this respect, it is also worth mentioning that the *divisibility relation* on $\mathbb{Z}$, the *subsequence relation* on $X^{\mathbb{N}}$, and the *refines* and *devides relations* for covers, relations and relators are also, in general, only preorder relations [**86**].

For a relation R on X to Y, the ordered pair $(X, Y)(R) = \big((X, Y), R\big)$ is usually called a *formal context* or *context space* [**33**]. However, it is better to call it a *relational space* or a *properly simple relator space* [**59**].

If in particular R is a relation on X, then having in mind a widely used terminology of Birkhoff [**6**] the ordered pair $X(R) = (X, R)$ may be called a *goset* (generalized ordered set) [**101**], instead of a *relational system* [**17, 10, 71**].

If P is a relational property, then the goset $X(R)$ will be said to have property P if the relation R has this property. For instance, the goset $X(R)$ will be called *reflexive* if R is a reflexive relation on X.

In particular, the goset $X(R)$ will be called a *proset* (preordered set) if R is a preorder on X. Moreover, $X(R)$ will be called a *poset* (partially ordered set) if R is a partial order on X.

The terms "goset" and "proset" were perhaps first introduced by the present author [**94**]. However, by Rudeanu [**72**], the abbreviations "toset" and "woset" for totally and well-ordered sets, respectively, were also used.

Thus, every set X is a poset with the identity relation Δ_X. Moreover, X is a proset with the *universal relation* X^2. And, the *power set* $\mathcal{P}(X) = \{A : \ A \subseteq X\}$ of X is a poset with the ordinary set inclusion $\subseteq$, and also with its inverse $\supseteq$.

Several definitions on posets can as well be applied to gosets. For instance, if $X(R)$ is a goset, then for any $Y \subseteq X$ the goset $Y(R \cap Y^2)$ is called a *subgoset* of $X(R)$. While, the goset $X'(R') = X(R^{-1})$ is called the *dual* of $X(R)$.

4. Lower and upper bounds in simple relator spaces

NOTATION 4.1. **In this section, we shall assume that R is a relation on X to Y.**

REMARK 4.1. However, the subsequent definitions can be easily extended to the more general case when R is replaced by a relator $\mathcal{R}$ [**91**].

DEFINITION 4.1. For any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, $y \in Y$, we define

(1) $A \in \operatorname{Lb}_R(B)$ and $B \in \operatorname{Ub}_R(A)$ if $A \times B \subseteq R$;

(2) $x \in \operatorname{lb}_R(B)$ if $\{x\} \in \operatorname{Lb}_R(B)$; (3) $y \in \operatorname{ub}_R(A)$ if $\{y\} \in \operatorname{Ub}_R(A)$.

(4) $B \in \mathfrak{L}_R$ if $\operatorname{lb}_R(B) \neq \emptyset$; (5) $A \in \mathfrak{U}_R$ if $\operatorname{ub}_R(A) \neq \emptyset$.

Thus, we can easily prove the following two theorems.

THEOREM 4.1. *We have*

(1) $\operatorname{Ub}_R = \operatorname{Lb}_{R^{-1}} = \operatorname{Lb}_{\mathcal{R}}^{-1}$; (2) $\operatorname{ub}_{\mathcal{R}} = \operatorname{lb}_{R^{-1}}$; (3) $\mathfrak{U}_R = \mathfrak{L}_{R^{-1}}$.

THEOREM 4.2. *For any $A \subseteq X$ and $B \subseteq Y$, we have*

(1) $A \in \operatorname{Lb}_R(B) \iff A \subseteq \operatorname{lb}_R(B)$; (2) $B \in \operatorname{Ub}_R(A) \iff B \subseteq \operatorname{ub}_R(A)$.

PROOF. For instance, by Definition 4.1, we have

$$A \in \operatorname{Lb}_R(B) \iff A \times B \subseteq R \iff \forall\ x \in A: \ \{x\} \times B \subseteq R \iff$$
$$\forall\ x \in A: \ \{x\} \in \operatorname{Lb}_R(B) \iff \forall\ x \in A: \ x \in \operatorname{lb}_R(B) \iff A \subseteq \operatorname{lb}_R(B).$$

REMARK 4.2. The above two theorems show that the lower and upper bound relations are actually equivalent tools in the simple relator space $(X, Y)(R)$.

Now, as an immediate consequence of Theorems 4.1 and 4.2, we can also state

COROLLARY 4.1. *For any $A \subseteq X$ and $B \subseteq Y$, we have*

$$A \subseteq \operatorname{lb}_R(B) \iff B \subseteq \operatorname{ub}_R(A).$$

PROOF. By Theorems 4.2 and 4.1, it is clear that

$$A \subseteq \operatorname{lb}_R(B) \iff A \in \operatorname{Lb}_R(B) \iff B \in \operatorname{Lb}_R^{-1}(A) \iff$$
$$B \in \operatorname{Ub}_R(A) \iff B \subseteq \operatorname{ub}_R(A).$$

Hence, by identifying singletons with their elements, we can immediately derive

COROLLARY 4.2. *For any $A \subseteq X$ and $B \subseteq Y$, we have*

(1) $\operatorname{lb}_R(B) = \big\{x \in X: \ B \subseteq \operatorname{ub}_R(x)\big\}$; (2) $\operatorname{ub}_R(A) = \big\{y \in X: \ A \subseteq \operatorname{lb}_R(y)\big\}$.

REMARK 4.3. However, it is now more important to note that by defining

$$F(A) = \operatorname{ub}_R(A) \qquad \text{and} \qquad G(B) = \operatorname{lb}_R(B),$$

for all $A \subseteq X$ and $B \subseteq Y$, we can at once see that

$$F(A) \supseteq B \iff B \subseteq \mathrm{ub}_{\mathcal{R}}(A) \iff A \subseteq \mathrm{lb}_{\mathcal{R}}(B) \iff A \subseteq G(B)$$

for all $A \subseteq X$ and $B \subseteq Y$.

Thus, the functions F and G establish a *decreasing Galois connection* between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Therefore, several properties of the super relations ub_R and lb_R can be derived from the extensive theory of Galois connections [**9, 34, 29, 33, 22, 23, 8**].

Thus, for instance, from Corollary 4.1 we can already derive the following theorem. However, it is frequently more convenient to apply some direct proofs.

THEOREM 4.3. *If* $B \subseteq Y$, *then*

(1) $\mathrm{lb}_R(B) \subseteq \mathrm{lb}_R(C)$ *for all* $C \subseteq B$;

(2) $B \subseteq \mathrm{ub}_R\big(\mathrm{lb}_R(B)\big)$; (3) $\mathrm{lb}_R(B) = \mathrm{lb}_R\Big(\mathrm{ub}_R\big(\mathrm{lb}_R(B)\big)\Big)$.

In addition to Corollary 4.2, it is also worth proving the following

THEOREM 4.4. *For any* $A \subseteq X$ *and* $B \subseteq Y$, *we have*

(1) $\mathrm{ub}_R(A) = \bigcap_{x \in A} \mathrm{ub}_R(x)$; (2) $\mathrm{lb}_R(B) = \bigcap_{y \in B} \mathrm{lb}_R(y)$.

REMARK 4.4. Assertion (1) can be generalized by showing that the relation $F = \mathrm{ub}_R$ is *union-reversing* in the sense that, for any $\mathcal{A} \subseteq \mathcal{P}(X)$, we have $F\left(\bigcup \mathcal{A}\right) = \bigcap_{A \in \mathcal{A}} F(A)$.

Now, by Theorem 4.4 and Corollary 4.2, we can also state the following

COROLLARY 4.3. *For any* $A \subseteq X$ *and* $B \subseteq Y$, *we have*

(1) $\mathrm{ub}_R(A) = \bigcap_{x \in A} R(x)$; (2) $\mathrm{lb}_R(B) = \big\{ x \in X : \; B \subseteq R(x) \big\}$.

REMARK 4.5. Assertion (1) can be reformulated by stating that $\mathrm{ub}_R(A) = R^c[A]^c$ for all $A \subseteq X$.

5. Some further important algebraic tools

NOTATION 5.1. **In this section, we shall already assume that R is a relation on X.**

Now, by using Definition 4.1, we may also naturally introduce the following definition which can also be immediately generalized to relators.

DEFINITION 5.1. For any $A \subseteq X$, we define

(1) $\min_R(A) = A \cap \mathrm{lb}_R(A)$; (2) $\max_R(A) = A \cap \mathrm{ub}_R(A)$;

(3) $\mathrm{Min}_R(A) = \mathcal{P}(A) \cap \mathrm{Lb}_R(A)$; (4) $\mathrm{Max}_R(A) = \mathcal{P}(A) \cap \mathrm{Ub}_R(A)$;

(5) $\inf_R(A) = \max_R\big(\mathrm{lb}_R(A)\big)$; (6) $\sup_R(A) = \min_R\big(\mathrm{ub}_R(A)\big)$;

(7) $\mathrm{Inf}_R(A) = \mathrm{Max}_R\big[\mathrm{Lb}_R(A)\big]$; (8) $\mathrm{Sup}_R(A) = \mathrm{Min}_R\big[\mathrm{Ub}_R(A)\big]$;

(9) $A \in \ell_R$ if $A \in \mathrm{Lb}_R(A)$; (10) $A \in \mathcal{L}_R$ if $A \subseteq \mathrm{lb}_R(A)$.

By using this definition, for instance, we can prove the following theorems.

THEOREM 5.1. *We have*

(1) $\mathrm{Max}_R = \mathrm{Min}_{R^{-1}}$; (2) $\mathrm{Sup}_R = \mathrm{Inf}_{R^{-1}}$; (3) $\ell_R = \ell_{R^{-1}}$

(4) $\max_R = \min_{R^{-1}}$; (5) $\sup_R = \inf_{R^{-1}}$; (6) $\ell_R = \mathcal{L}_R$.

THEOREM 5.2. *For any* $A \subseteq X$, *we have*

(1) $\max_R(A) = \bigcap_{x \in A} \big(A \cap \mathrm{ub}_R(x) \big)$; (2) $\max_R(A) = \big\{ x \in A : \ A \subseteq \mathrm{lb}_R(x) \big\}$.

THEOREM 5.3. *For any* $A \subseteq X$, *we have*

(1) $\sup_R(A) = \mathrm{ub}_R(A) \cap \mathrm{lb}_R\big(\mathrm{ub}_R(A)\big)$;

(2) $\max_R(A) = A \cap \sup_R(A)$; (3) $\sup_R(A) = \inf_R\big(\mathrm{ub}_R(A)\big)$.

PROOF. To prove assertion (3), note that by assertion (1) and Theorem 4.3, and their duals, we have

$$\sup_R(A) = \mathrm{lb}_R\big(\mathrm{ub}_R(A)\big) \cap \mathrm{ub}_R(A) = \mathrm{lb}_R\big(\mathrm{ub}_R(A)\big) \cap \mathrm{ub}_R\Big(\mathrm{lb}_R\big(\mathrm{ub}_R(A)\big)\Big) = \inf_R\big(\mathrm{ub}_R(A)\big) .$$

THEOREM 5.4. *For any* $A \subseteq X$, *we have*

$$\sup_R(A) = \big\{ x \in X : \ \mathrm{ub}_R(x) = \mathrm{ub}_R(A) \big\} = \big\{ x \in \mathrm{ub}_R(A) : \ \mathrm{ub}_R(A) \subseteq \mathrm{ub}_R(x) \big\}.$$

THEOREM 5.5. *For any* $A \subseteq X$ *the following assertions are equivalent:*

(1) $A \in \mathcal{L}_R$; (2) $A \in \mathrm{Ub}_R(A)$; (3) $A \in \mathrm{Min}_R(A)$; (4) $A \in \mathrm{Max}_R(A)$.

COROLLARY 5.1. *For any* $A \subseteq X$ *the following assertions are equivalent:*

(1) $\mathrm{ub}_R(A) \in \mathcal{L}_R$; (2) $\mathrm{ub}_R(A) = \sup_R(A)$; (3) $\mathrm{ub}_R(A) \subseteq \mathrm{lb}_R\big(\mathrm{ub}_R(A)\big)$.

THEOREM 5.6. *We have*

$$\mathcal{L}_R = \big\{ \min_R(A) : \quad A \subseteq X \big\} = \big\{ \max_R(A) : \quad A \subseteq X \big\} .$$

THEOREM 5.7. *If* R *is reflexive, then following assertions are equivalent:*

(1) R *is antisymmetric;* (2) $\mathrm{card}(A) \leqslant 1$ *if* $A \in \mathcal{L}_R$;

(3) $\max_R$ *is a function;* (4) $\sup_R$ *is a function.*

REMARK 5.1. The implications (1) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4) do not require the relation R to be reflexive.

DEFINITION 5.2. The relation R on X, or the goset $X(R)$, will be called

(1) *inf–complete* if $\inf_R(A) \neq \emptyset$ for all $A \subseteq X$;

(2) *min–complete* if $\min_R(A) \neq \emptyset$ for all $\emptyset \neq A \subseteq X$.

Remark 5.2. Thus, for instance, the set $\mathbb{Z}$ of all integers is min-complete, but not inf-complete.

While, the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ of all extended real numbers is inf-complete, but not min–complete.

Now, by letting A to be a singleton, and then a doubleton, we can obtain

Theorem 5.8. *If R is min-complete, then R is reflexive and total.*

Moreover, by using Theorem 5.3, we can also easily prove the following

Theorem 5.9. *The following assertions are equivalent:*

(1) *R is inf–complete;* (2) *R is sup–complete.*

Proof. By Theorem 5.3, we have $\sup_R(A) = \inf_R\big(\mathrm{ub}_R(A)\big)$ for all $A \subseteq X$. Hence, the implication $(1) \Longrightarrow (2)$ immediately follows.

Remark 5.3. For several other reasonable order-theoretic completeness properties, and their relationships, see [**13**] and [**12**].

6. A few basic facts on increasing functions

Notation 6.1. **In this section, we shall assume that f is a function of one goset $X(R)$ to another $Y(S)$.**

Definition 6.1. The function f will be called *increasing* if, for all $u, v \in X$,

$$u\,R\,v \quad \Longrightarrow \quad f(u)\,S\,f(v).$$

Remark 6.1. Now, the function f may be briefly defined to be *decreasing* if it is increasing as a function $X(R)$ to the dual $Y\big(S^{-1}\big)$ of $Y(S)$.

Moreover, the function can, for instance, be briefly defined to be *strictly increasing* if it is increasing as a function of $X(R \setminus \Delta_X)$ to $Y(S \setminus \Delta_Y)$.

However, to define a strict form of the relation R, instead of $R \setminus \Delta_X$, the relation $R \setminus R^{-1}$ can also be well-used (See, for instance, Patrone [**62**].)

The following theorem shows that the strictly increasing functions are closely related to the injective, increasing ones.

Theorem 6.1. *If R is total and S is reflexive, then the following assertions are equivalent:*

(1) *f is strictly increasing;* (2) *f is injective and increasing.*

Remark 6.2. To prove the implication $(2) \Longrightarrow (1)$ we do not need any extra conditions on the relations R and S.

While, if assertion (1) holds and f is onto Y, then to prove that f^{-1} is also strictly increasing, we have to assume that R is total and S is antisymmetric.

Concerning increasing functions, we can also prove the following theorems.

THEOREM 6.2. *The following assertions are equivalent:*

(1) f *is increasing;*

(2) $f\,[\,\mathrm{ub}_R(x)\,] \subseteq \mathrm{ub}_S\big(\,f\,(x)\big)$ *for all* $x \in X$*;*

(3) $f\,[\,\mathrm{ub}_R(A)\,] \subseteq \mathrm{ub}_S\big(\,f\,[\,A\,]\big)$ *for all* $A \subseteq X$.

THEOREM 6.3. *If* R *is reflexive, then the following assertions are equivalent:*

(1) f *is increasing;*

(2) $f\,[\,\max_R(A)\,] \subseteq \mathrm{ub}_S\big(\,f\,[\,A\,]\big)$ *for all* $A \subseteq X$*;*

(3) $f\,[\,\max_R(A)\,] \subseteq \max_S\big(\,f\,[\,A\,]\big)$ *for all* $A \subseteq X$*;*

From Theorem 6.2, by using Theorem 4.3, we can immediately derive

THEOREM 6.4. *If* f *is increasing, then for any* $A \subseteq X$, *we have*

$$\mathrm{lb}_S\,\big(\,\mathrm{ub}_S\big(\,f\,[\,A\,]\,\big)\big) \subseteq \mathrm{lb}_S\big(\,f\,[\,\mathrm{ub}_R(A)\,]\,\big)\,.$$

Moreover, by using Theorems 5.3, 5.7 and 6.4, we can also prove

THEOREM 6.5. *If* f *is increasing and* R *and* S *are antisymmetric and sup–complete, then for any* $A \subseteq X$ *we have*

$$\sup_S\big(f\,[\,A\,]\big)\; S\; f\big(\sup_R\,(A)\big)\,.$$

Finally, we note that, by the results of [**106**], the following theorems are also true. Therefore, instead of "increasing", we may also naturally say "continuous".

THEOREM 6.6. *The following assertions are equivalent:*

(1) f *is increasing;*

(2) $(u,\,v) \in R$ *implies* $\big(\,f\,(u),\;f\,(v)\big) \in S$*;*

(3) $v \in R\,(u)$ *implies* $f\,(v) \in S\,\big(\,f\,(u)\big)$ *for all* $u \in X$.

THEOREM 6.7. *The following assertions are equivalent:*

(1) f *is increasing;*

(2) $f \circ R \subseteq S \circ f$, (3) $R \subseteq f^{-1} \circ S \circ f$,

(4) $f \circ R \circ f^{-1} \subseteq S$, (5) $R \circ f^{-1} \subseteq f^{-1} \circ S$.

REMARK 6.3. By using the box product of relations, assertion (3) can be reformulated in the form that $R \subseteq (f \boxtimes f)^{-1}\,[\,S\,]$.

However, it is now more important to note that, by using the uniform refinement of relators, instead of (3), we may also write that $f^{-1} \circ S \circ f \in \{R\}^*$.

REMARK 6.4. Finally, we note that a relation F on the goset $X\,(R)$ to a set Y may be naturally called *increasing* if the associated set-valued function φ_F is increasing. That is, $u\,R\,v$ implies $F\,(u) \subseteq F\,(v)$ for all $u,\,v \in X$.

However, if F is a relation on $X(R)$ to $Y(S)$, then in addition to the above *inclusion-increasingness* of F, we may also define an *order-increasingness* of F by requiring the implication $u \in \mathrm{lb}_R(v) \implies F(u) \in \mathrm{Lb}_S\big(F(v)\big)$ for all $u, v \in X$.

Thus, we can show that F is inclusion-increasing if and only if $R \circ F^{-1} \subseteq F^{-1}$, or equivalently F^{-1} is ascending-valued. And, F is order-increasing if and only if $F \circ R \circ F^{-1} \subseteq S$, or equivalently $F[R(u)] \subseteq \mathrm{ub}_S\big(F(u)\big)$ for all $u \in X$ [**106**].

7. The induced order and interior relations

NOTATION 7.1. **In this section, we shall assume that f is a function of a set X to a goset $Y(S)$.**

DEFINITION 7.1. For each $u \in X$ and $y \in Y$, we define

$$\mathrm{Ord}_f(u) = \big\{ v \in X : \quad f(u)\, S\, f(v) \big\} \qquad \text{and} \qquad \mathrm{Int}_f(y) = \big\{ x \in X : \quad f(x)\, S\, y \big\}.$$

The relations Ord_f and Int_f will be called the *natural order* and the *proximal interior relations* induced by f, respectively.

REMARK 7.1. If F is a relation on one set X to another Y, then by using the associated set-valued function φ_F, we may also naturally define $\mathrm{Ord}_F = \mathrm{Ord}_{\varphi_F}$ and $\mathrm{Int}_F = \mathrm{Int}_{\varphi_F}$.

Moreover, if U is a super relation on X to Y, then for instance, for any $B \subseteq Y$, we may also naturally define $\mathrm{int}_U(B) = \{x \in X : \ \{x\} \in \mathrm{Int}_U(B)\}$.

Concerning the relations Ord_f and Int_f, we can easily prove the following four theorems.

THEOREM 7.1. *Ord_f is the largest relation on X making the function f to be increasing.*

PROOF. If R is a relation on X making f to be increasing, then

$$v \in R(u) \implies u\,R\,v \implies f(u) S\, f(v) \implies v \in \mathrm{Ord}_f(u),$$

and thus $R(u) \subseteq \mathrm{Ord}_f(u)$ for all $u \in X$. Therefore, $R \subseteq \mathrm{Ord}_f$ also holds.

THEOREM 7.2. *The following assertions hold:*

(1) Ord_f *is a preorder on X if S is a preorder on Y;*

(2) Ord_f *is a partial order on X if f is injective and S is a partial order on Y.*

THEOREM 7.3. *If S is a preorder, then the following assertions are equivalent:*

(1) *f is increasing;* (2) *Ord_f is decreasing;* (3) *Ord_f is ascending valued.*

PROOF. For instance, if $u, v \in X$ such that $u\,R\,v$, and assertion (1) holds, then $f(u)\,S\,f(v)$. Moreover, if $w \in \mathrm{Ord}_f(v)$, then $f(v)\,S\,f(w)$. Hence, by the transitivity of S, we can infer that $f(u)\,S\,f(w)$, and thus $w \in \mathrm{Ord}_f(u)$. Therefore, $\mathrm{Ord}_f(v) \subseteq \mathrm{Ord}_f(u)$, and thus assertion (2) also holds.

THEOREM 7.4. *If R is a relation on X and S is transitive, then*

(1) Int_f *is increasing;* (2) Int_f *is descending valued if f is increasing.*

PROOF. To prove (2), note that if $y \in Y$ and $x \in \operatorname{Int}_f(y)$, then $f(x)\, S\, y$. Moreover, if $u \in X$ such that $u\, R\, x$ and f increasing, then $f(u)\, S\, f(x)$. Thus, by the transitivity of S, we also have $f(u)\, S\, y$, and thus $u \in \operatorname{Int}_f(y)$. Therefore, $\operatorname{Int}_f(y)$ is a descending subset of X.

The next theorem show that the relations Ord_f and Int_f are not independent of each other, and they are also closely related to the relations lb_S and ub_S.

THEOREM 7.5. *We have*

(1) $\operatorname{Int}_f = f^{-1} \circ S^{-1}$; (2) $\operatorname{Ord}_f = f^{-1} \circ \operatorname{Int}_f^{-1}$;

(3) $\operatorname{Int}_f = f^{-1} \circ \operatorname{ub}_S^{-1}$; (4) $\operatorname{Int}_f = f^{-1} \circ \operatorname{lb}_S \circ \Delta_Y$.

PROOF. By the corresponding definitions, for any $x \in X$ and $y \in Y$, we have

$$x \in \operatorname{Int}_f(y) \iff f(x)\, S\, y \iff y \in S\big(f(x)\big) \iff y \in (S \circ f)(x).$$

Therefore, $\operatorname{Int}_f = \big(S \circ f\big)^{-1} = f^{-1} \circ S^{-1}$.

Moreover, we also have

$$x \in \operatorname{Int}_f(y) \iff f(x) \in \operatorname{lb}_S(y) \iff x \in f^{-1}[\operatorname{lb}_S(y)] \iff$$
$$x \in f^{-1}\big[\operatorname{lb}_S\big(\Delta_Y(y)\big)\big] \iff x \in \big(f^{-1} \circ \operatorname{Int}_f \circ \operatorname{lb}_S \circ \Delta_Y\big)(y).$$

Therefore, assertions (4) is also true.

REMARK 7.2. In this respect, it is also worth noticing that

$$y \in \operatorname{ub}_S\big(f[\operatorname{Int}_f(y)]\big)$$

for all $y \in Y$. Namely, for every $x \in \operatorname{Int}_f(y)$, we have $f(x) S\, y$.

Now, we can also easily prove the following

THEOREM 7.6. *If R is a relation on X such that*

$$f[\sup_R(A)] \subseteq \operatorname{lb}_S\big(\operatorname{ub}_S\big(f[A]\big)\big)$$

for all $A \subseteq X$, then

$$\max_R\big(\operatorname{Int}_f(y)\big) = \sup_R\big(\operatorname{Int}_f(y)\big)$$

for all $y \in Y$.

PROOF. If $y \in Y$, then by Theorem 5.3 we have

$$\max_R\big(\operatorname{Int}_f(y)\big) \subseteq \sup_R\big(\operatorname{Int}_f(y)\big).$$

Therefore, we need actually prove only the converse inclusion.

For this, note that if $x \in \sup_R\big(\operatorname{Int}_f(y)\big)$, then by the assumed property of f we have

$$f(x) \in f\big[\sup_R\big(\operatorname{Int}_f(y)\big)\big] \subseteq \operatorname{lb}_S\big(\operatorname{ub}_S\big(f[\operatorname{Int}_f(y)]\big)\big).$$

Moreover, by Remark 7.2, we also have $y \in \mathrm{ub}_S\big(f[\mathrm{Int}_f(y)]\big)$. Therefore, we necessarily have $f(x)\,S\,y$, and thus $x \in \mathrm{Int}_f(y)$. Hence, by Theorem 5.3, we can see that

$$x \in \mathrm{Int}_f(y) \cap \sup_R\big(\mathrm{Int}_f(y)\big) = \max_R\big(\mathrm{Int}_f(y)\big).$$

Therefore, $\sup_R\big(\mathrm{Int}_f(y)\big) \subseteq \max_R\big(\mathrm{Int}_f(y)\big)$, and thus the required equality is also true.

Remark 7.3. Note that, by Theorem 5.3, for a subset A of the goset $X(R)$ we have $\max_R(A) = \sup_R(A)$ if and only if $\sup_R(A) \subseteq A$.

8. Extensive, involutive, and idempotent operations

Notation 8.1. **In this and the next section, we shall assume that φ is a function of a goset $X(R)$ to itself.**

Definition 8.1. The function φ will be called

(1) extensive if $\Delta_X\,R\,\varphi$;

(2) intensive if $\varphi\,R\,\Delta_X$;

(3) right–semi–involutive if $\Delta_X\,R\,\varphi^2$;

(4) left–semi–involutive if $\varphi^2\,R\,\Delta_X$;

(5) right–semi–idempotent if $\varphi\,R\,\varphi^2$;

(6) left–semi–idempotent if $\varphi^2\,R\,\varphi$.

Remark 8.1. Property (3), in detailed form, means only that $\Delta_X(x)\,R\,\varphi^2(x)$, i. e., $x\,R\,\varphi(\varphi(x))$ for all $x \in X$.

By using Definition 8.1, we can easily establish the following

Theorem 8.1. *The following assertions hold;*

(1) *φ is right-semi-idempotent if φ is extensive;*

(2) *φ is right-semi-involutive if and only if φ^2 is extensive;*

(3) *φ is right-semi-idempotent if and only if $\varphi\,|\,\varphi[X]$ is extensive.*

Proof. If φ is extensive, then $x\,R\,\varphi(x)$ for all $x \in X$. Hence, taking $u \in X$ and writing $\varphi(u)$ in place of x, we can infer that $\varphi(u)\,R\,\varphi^2(u)$. Thus, φ is right-semi-idempotent.

Moreover, if $y \in \varphi[X]$, then there exists $x \in X$ such that $y = \varphi(x)$, and thus $\varphi(y) = \varphi^2(x)$. Moreover, if φ is right-semi-idempotent, then $\varphi(x)\,R\,\varphi^2(x)$, and thus $y\,R\,\varphi(y)$. Therefore, the restriction $\varphi\,|\,\varphi[X]$ is extensive.

Remark 8.2. In addition to the above observations, it is also worth noticing that φ is extensive with respect R if and only if $\varphi(x) \in R(x)$ for all $x \in X$. That is, φ is a selection function of R.

Thus, analogously to a relational reformulation of the Axiom of Choice, the following generalization of a theorem of Bourbaki [**7**, p. 4] may also be considered as a selection theorem.

Theorem 8.2. *If φ is strictly increasing and R is antisymmetric and min-complete, then φ is extensive.*

PROOF. Assume on the contrary that φ is not extensive. Then, by Remark 8.2, φ is not a selection function of R. Thus,

$$A = \{ x \in X : \ \varphi(x) \notin R(x) \} \neq \emptyset .$$

Therefore, by the assumed min-completeness of R, there exists $a \in X$ such that $a \in \min_R(A)$. Hence, by the definition of $\min_R$, we can infer that

$$a \in A \qquad \text{and} \qquad a \in \mathrm{lb}_R(A),$$

and thus $a \, R \, x$ for all $x \in A$.

Now, since $a \in A$, we can also note that $a \, R \, a$, and thus $a \in R(a)$. Moreover, by the definition of A, we can also note that $\varphi(a) \notin R(a)$. Therefore, $\varphi(a) \neq a$. Moreover, from Theorem 5.8, we know that R is total. Thus, since $a \, R \, \varphi(a)$ does not hold, we necessarily have $\varphi(a) \, R \, a$.

Hence, by using that $\varphi(a) \neq a$ and φ is strictly increasing, we can infer that $\varphi\big(\varphi(a)\big) \, R \, \varphi(a)$ and $\varphi\big(\varphi(a)\big) \neq \varphi(a)$. Thus, by the antisymmetry of R, $\varphi(a) \, R \, \varphi\big(\varphi(a)\big)$ cannot hold. This, shows that $\varphi\big(\varphi(a)\big) \notin R\big(\varphi(a)\big)$, and thus $\varphi(a) \in A$. Hence, by using that $a \, R \, x$ for all $x \in A$, we can infer that $a \, R \, \varphi(a)$, and thus $\varphi(a) \in R(a)$. This contradiction shows that φ is extensive.

REMARK 8.3. Note that if φ is extensive, R is antisymmetric and x is a maximal element of $X(R)$ in the sense that $x \, R \, y$ implies $y \, R \, x$ for all $y \in X$, then x is already a fixed point of φ in the sense that $\varphi(x) = x$.

This simple, but important fact was first explicitly stated by Brøndsted [**15**]. And, fixed point theorems for extensive maps (which were sometimes also called expansive, progressive, increasing, or inflationary) were proved by several authors.

9. Involution, projection, and closure operations

DEFINITION 9.1. The function φ will be called

(1) *involution operation* if it is increasing and both left and right semi-involutive;

(2) *projection operation* if it is increasing and both left and right semi-idempotent;

(3) *closure (interior) operation* if it is an extensive (intensive) projection operation.

REMARK 9.1. Moreover, φ may, for instance be called a

(1) *preclosure operation* if it is increasing and extensive;

(2) *semi-closure operation* if it is extensive and left-semi-idempotent;

(3) *left semi-modification operation* if it is increasing and left semi-idempotent.

Note that, by Theorem 8.1, an extensive operation is right-semi-idempotent. Moreover, the corresponding interior operations can be briefly defined by using the dual $X\big(R^{-1}\big)$ of $X(R)$.

In connection with Definition 8.1, it is also worth mentioning if, for instance, φ is both left and right semi-idempotent and R is antisymmetric, then φ is idempotent in the sense that $\varphi^2 = \varphi$. However, if φ is idempotent and R is not reflexive, then φ need not be either left or right semi-idempotent.

Concerning closure operations, for instance, we can prove the following

THEOREM 9.1. *If φ is a closure operation, and R is antisymmetric and inf-complete, then for any $A \subseteq X$ we have*

$$\inf_R\big(\varphi\,[A\,]\big) = \varphi\,\big(\inf_R\big(\varphi\,[A\,]\big)\big)\,.$$

PROOF. By the dual of Theorem 6.5, we have

$$\inf_R\big(\varphi\,[A]\big) \in\, R\,\big(\varphi\big(\inf_R(A)\big)\big)\,.$$

Hence, by writing $\varphi\,[A]$ in place of A, we can see that

$$\inf_R\,(\,\varphi\,[\,\varphi\,[A\,]\,]\,) \in\, R\,\big(\,\varphi\,\big(\inf_R\big(\varphi\,[A\,]\big)\big)\big)\,.$$

Moreover, because of the antisymmetry of R, we can note that φ is now idempotent. Therefore, $\varphi\,[\,\varphi\,[A\,]\,] = (\varphi\circ\varphi)\,[A\,] = \varphi^2[A\,] = \varphi\,[A\,]$. Thus, we actually have

$$\inf_R\,(\varphi\,[A\,]\,) \in\, R\,\big(\,\varphi\,\big(\inf_R\big(\varphi\,[A\,]\big)\big)\big)\,.$$

Moreover, by extensivity of φ, the converse inclusion is also true. Hence, by using the antisymmetry of R, we can see that the required equality is also true.

REMARK 9.2. It can be easily seen that an operation φ on a set X is idempotent if and only if $\varphi\,[X\,]$ is the family of all fixed points of φ.

Therefore, by using Theorem 9.1, we can also prove the following

COROLLARY 9.1. *Under the conditions of Theorem 9.1, for any $A \subseteq \varphi\,[X]$, we have*

$$\inf_R\big(A\big) = \varphi\big(\inf_R(A)\big)\,.$$

PROOF. Now, because of the antisymmetry of R, the operation φ is idempotent. Thus, by Remark 9.2, we have $\varphi(y) = y$ for all $y \in \varphi\,[X\,]$. Hence, by using the assumption $A \subseteq \varphi\,[X\,]$, we can see that $\varphi\,[A\,] = A$. Thus, Theorem 9.1 gives the required equality.

REMARK 9.3. Note that if φ is an extensive and left-semi-idempotent, and R reflexive and antisymmetric, then $\varphi\,[X\,]$ is also the family of all elements x of X which are φ–closed in the sense that $\varphi(x)\,R\,x$.

Therefore, if in addition to the conditions of Theorem 9.1, R is reflexive, then the assertion of Corollary 9.1 can also be expressed by stating that the infimum of any family of φ–closed elements of $X\,(R)$ is also φ–closed.

Now, instead of a counterpart of Theorem 9.1, we can only prove the following

THEOREM 9.2. *If φ is a closure operation, and R is transitive, antisymmetric and sup-complete, then for any $A \subseteq X$ we have*

$$\varphi\big(\sup_R(A)\big) = \varphi\,\big(\sup_R\big(\varphi\,[A\,]\big)\big)\,.$$

PROOF. Define $\alpha = \sup_R(A)$ and $\beta = \sup_R\big(\varphi[A]\big)$. Then, by Theorem 6.5, we have $\beta\, R\, \varphi(\alpha)$. Hence, since φ is increasing, we can infer that $\varphi(\beta)\, R\, \varphi\big(\varphi(\alpha)\big)$. Moreover, since φ is now idempotent, we also have $\varphi\big(\varphi(\alpha)\big) = \varphi(\alpha)$. Therefore, $\varphi(\beta) R\, \varphi(\alpha)$.

On the other hand, since φ is extensive, for any $x \in A$ we have $x\, R\, \varphi(x)$. Moreover, since $\beta \in \mathrm{ub}_R\big(\varphi[A]\big)$, we also have $\varphi(x) R\, \beta$. Hence, by using the transitivity of R, we can infer that $x\, R\, \beta$. Therefore, $\beta \in \mathrm{ub}_R(A)$. Now, by using that $\alpha \in \mathrm{lb}_{RX}\big(\mathrm{ub}_X(A)\big)$, we can see that $\alpha\, R\, \beta$. Hence, by using the increasingness of φ, we can infer that $\varphi(\alpha) R\, \varphi(\beta)$. Therefore, by the antisymmetry of R, we actually have $\varphi(\alpha) = \varphi(\beta)$, and thus the required equality is also true.

By using this theorem, in addition to Theorem 9.1, we can only prove

COROLLARY 9.2. *Under the conditions of Theorem 9.2, for any $A \subseteq X$, the following assertions are equivalent:*

(1) $\sup_R\big(\varphi[A]\big) = \varphi\big(\sup_R(A)\big)$, (2) $\sup_R\big(\varphi[A]\big) = \varphi\left(\sup_R\big(\varphi[A]\big)\right)$.

10. Closures and interiors in simple relator spaces

NOTATION 10.1. **In this section, we shall assume that R is a relation on X to Y.**

REMARK 10.1. However, the subsequent definitions can also be easily extended to the more general case when R is replaced by a relator $\mathcal{R}$, or even a super relator $\mathcal{U}$ [**67, 111**].

DEFINITION 10.1. For any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, we define:

(1) $A \in \mathrm{Int}_R(B)$ if $R[A] \subseteq B$; (2) $A \in \mathrm{Cl}_R(B)$ if $R[A] \cap B \neq \emptyset$;

(3) $x \in \mathrm{int}_R(B)$ if $\{x\} \in \mathrm{Int}_R(B)$; (5) $x \in \mathrm{cl}_R(B)$ if $\{x\} \in \mathrm{Cl}_R(B)$;

(7) $B \in \mathcal{E}_R$ if $\mathrm{int}_R(B) \neq \emptyset$; (8) $B \in \mathcal{D}_R$ if $\mathrm{cl}_R(B) = X$.

REMARK 10.2. The relations Int_R and $\mathrm{int}_{\mathcal{R}}$ are called *the proximal* and *topological interiors* generated by R, respectively. While, the members of the families, $\mathcal{E}_R$ and $\mathcal{D}_R$ are called the *fat and dense subsets* of the simple relator space $(X, Y)(R)$, respectively.

The origins of the relations Cl_R and Int_R go back to Efremović's proximity δ [**27**] and Smirnov's strong inclusion $\Subset$ [**77**], respectively. While, the convenient notations Cl_R and Int_R, and the family $\mathcal{E}_R$, together with its dual $\mathcal{D}_R$, were first explicitly used by the present author in [**79, 82, 83, 92**].

The following theorem indicates that, in a relator space, the closure of a set can be more directly described than in a topological one. Moreover, the corresponding closure and interior relations are equivalent tools.

THEOREM 10.1. *For any $B \subseteq X$, we have*

(1) $\mathrm{cl}_R(B) = R^{-1}[B]$;

(2) $\operatorname{cl}_R(B) = \big(\operatorname{int}_R \circ \mathcal{C}_Y\big)^c = X \setminus \operatorname{int}_R\big(Y \setminus B\big)$;

(3) $\operatorname{Cl}_R(B) = \big(\operatorname{Int}_R \circ \mathcal{C}_Y\big)^c = \mathcal{P}(X) \setminus \operatorname{Int}_R\big(Y \setminus B\big)$.

Remark 10.3. From assertion (2), we can at once see that

(1) $\operatorname{cl}_R = \big(\operatorname{int}_R\big)^\star$; (2) $\operatorname{cl}_R = \big(\operatorname{int}_R\big)^c \circ \mathcal{C}_Y$.

The following theorem shows that, in contrast to their equivalence, the big closure relation is usually a more convenient tool than the big interior one.

Theorem 10.2. *We have*

(1) $\operatorname{Cl}_{R^{-1}} = \operatorname{Cl}_R^{-1}$; (2) $\operatorname{Int}_{R^{-1}} = \mathcal{C}_Y \circ \operatorname{Int}_R^{-1} \circ \mathcal{C}_X$.

In an arbitrary relator space, the small closure and interior relations are usually much weaker tools than the big ones. However, now we can also prove the following

Theorem 10.3. *For any $A \subseteq X$ and $B \subseteq Y$*

(1) $A \in \operatorname{Int}_R(B) \iff A \subseteq \operatorname{int}_R(B)$;

(2) $A \in \operatorname{Cl}_R(B) \iff A \cap \operatorname{cl}_R(B) \neq \emptyset$.

Now, analogously to Corollary 4.1, we can also prove the following

Corollary 10.1. *For any $A \subseteq X$ and $B \subseteq Y$, we have*

$$\operatorname{cl}_{R^{-1}}(A) \subseteq B \iff A \subseteq \operatorname{int}_R(B) .$$

Proof. By Theorems 10.3 and 10.2, it is clear that

$$\begin{aligned} \operatorname{cl}_{R^{-1}}(A) \subseteq B &\iff B^c \cap \operatorname{cl}_{R^{-1}}(A) = \emptyset \iff B^c \notin Cl_{R^{-1}}(A) \iff \\ B^c \notin Cl_R^{-1}(A) &\iff A \notin \operatorname{Cl}_R(B^c) \iff A \in \operatorname{Cl}_R(B^c)^c \iff \\ & A \in \operatorname{Int}_R(B) \iff A \subseteq \operatorname{int}_R(A) . \end{aligned}$$

Remark 10.4. This corollary shows that the functions F and G, defined by

$$F(A) = \operatorname{cl}_{R^{-1}} \qquad \text{and} \qquad G(B) = \operatorname{Int}_R(B)$$

for all $A \subseteq X$ and $B \subseteq Y$, establish an increasing Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Actually, Corollaries 4.1 and 10.1 can be more easily proved directly. Moreover, they can be derived from each other. Namely, we can also prove the following

Theorem 10.4. *We have*

(1) $\operatorname{Lb}_R = \big(\operatorname{Cl}_{R^c}\big)^c = \operatorname{Int}_{R^c} \circ \mathcal{C}_Y$; (2) $\operatorname{lb}_R = \big(\operatorname{cl}_{R^c}\big)^c = \operatorname{int}_{R^c} \circ \mathcal{C}_Y$.

PROOF. By the corresponding definitions, for any $A \subseteq X$ and $B \subseteq Y$ we have

$$A \in \mathrm{Lb}_R(B) \iff A \times B \subseteq R \iff \forall\ (a,b) \in A \times B: \quad (a,b) \notin R^c \iff$$
$$\forall\ a \in A: \ \forall\ b \in B: \quad b \notin R^c(a) \iff R^c[A] \cap B = \emptyset \iff$$
$$A \notin \mathrm{Cl}_{R^c}(B) \iff A \in \mathrm{Cl}_{R^c}(B)^c \iff A \in \big(\mathrm{Cl}_{R^c}\big)^c(B).$$

Therefore, $\mathrm{Lb}_R(B) = \mathrm{Cl}^c_{R^c}(B)$ for all $B \subseteq Y$, and thus the first part of assertion (1) is true. The second part of it now immediate by Theorem 10.1.

Now, by using Theorem 10.1 and Definition 10.1, we can also easily establish

THEOREM 10.5. *We have*

(1) $\mathcal{D}_R = \big\{ B \subseteq Y: \quad X = R^{-1}[B] \big\}$;

(2) $\mathcal{E}_R = \bigcup_{x \in X} \mathcal{U}_R(x)$, *where* $\mathcal{U}_R(x) = \mathrm{int}_R^{-1}(x)$.

REMARK 10.5. Note that thus

$$\mathcal{U}_R(x) = \mathrm{int}_R^{-1}(x) = \big\{ B \subseteq Y: \quad x \in \mathrm{int}_R(B) \big\}$$

is just the family of all *neighbourhoods* of the point x of X in Y.

The following theorem shows that the families of fat and dense sets are also equivalent tools.

THEOREM 10.6. *We have*

(1) $\mathcal{D}_R = \big\{ D \subseteq Y: \quad D^c \notin \mathcal{E}_R \big\}$;

(2) $\mathcal{D}_R = \big\{ D \subseteq Y: \quad \forall\ E \in \mathcal{E}_R: \quad E \cap D \neq \emptyset \big\}$.

REMARK 10.6. By using Theorem 10.4, we can see that $\mathfrak{L}_R = \mathcal{P}(Y) \setminus \mathcal{D}_{R^c}$.

11. Some further important topological tools

NOTATION 11.1. **In this section, we shall already assume that R is a relation on X.**

Now, by using Definition 10.1, we may also naturally introduce the following definition which can also be immediately generalized to relators.

DEFINITION 11.1. *For any $A \subseteq X$, we define:*

(1) $A \in \tau_R$ if $A \in \mathrm{Int}_R(A)$;

(2) $A \in \tau\!\!\!-_R$ if $A^c \notin \mathrm{Cl}_R(A)$;

(3) $A \in \mathcal{T}_\mathcal{R}$ if $A \subseteq \mathrm{int}_R(A)$;

(4) $A \in \mathcal{F}_R$ if $\mathrm{cl}_R(A) \subseteq A$;

(5) $A \in \mathcal{N}_R$ if $\mathrm{cl}_R(A) \notin \mathcal{E}_R$;

(6) $A \in \mathcal{M}_R$ if $\mathrm{int}_\mathcal{R}(A) \in \mathcal{D}_R$.

REMARK 11.1. The members of the families, τ_R, $\mathcal{T}_R$ and $\mathcal{N}_R$ are called the *proximally open, topologically open* and *rare (or nowhere dense) subsets* of the simple relator space $X(R)$, respectively.

The families τ_R and $\bar{\tau}_R$ were first explicitly used by the present author in [**82, 83**]. While, the practical notation $\bar{\tau}_R$ has been suggested by J. Kurdics who first noticed that connectedness is a particular case of well-chainedness [**42, 44, 61**].

By using Definition 11.1 and the corresponding results of Section 10, we can easily establish the following two theorems.

Theorem 11.1. *We have*

(1) $\bar{\tau}_R = \bar{\tau}_{R^{-1}}$; (2) $\bar{\tau}_R = \{ A \subseteq X : \quad A^c \in \tau_R \}$;

(3) $\mathcal{F}_R = \{ A \subseteq X : \quad A^c \in \mathcal{T}_R \}$; (4) $\mathcal{M}_R = \{ A \subseteq X : \quad A^c \in \mathcal{N}_R \}$.

Theorem 11.2. *We have*

(1) $\tau_R = \mathcal{T}_R$; (2) $\mathcal{T}_R \setminus \{\emptyset\} \subseteq \mathcal{E}_R$; (3) $\mathcal{D}_R \cap \mathcal{F}_R \subseteq \{X\}$.

Hint. By Theorem 10.3, for any $A \subseteq X$, we have

$$A \in \tau_R \iff A \in \operatorname{Int}_R(A) \iff A \subseteq \operatorname{int}_R(A) \iff A \in \mathcal{T}_R .$$

Thus, assertion (1) is true.

However, if $\mathcal{R}$ is a relator on X, then we can only prove that $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$.

Remark 11.2. From assertion (3), by using global complementations, we can infer that $\mathcal{F}_{\mathcal{R}} \subseteq \big(\mathcal{D}_{\mathcal{R}}\big)^c \cup \{X\}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \big(\mathcal{F}_{\mathcal{R}}\big)^c \cup \{X\}$.

However, it is now more important to note that we also have the following

Theorem 11.3. *For any* $A \subseteq X$ *we have*

(1) $\mathcal{P}(A) \cap \big(\mathcal{T}_R \setminus \{\emptyset\}\big) \neq \emptyset$ *implies* $A \in \mathcal{E}_{\mathcal{R}}$;

(2) $\bigcup \mathcal{T}_R \cap \mathcal{P}(A) \subseteq \operatorname{int}_R(A)$; (3) $\mathcal{P}\big[\tau_R \cap \mathcal{P}(A)\big] \subseteq \operatorname{Int}_R(A)$.

Remark 11.3. The fat sets are frequently more important tools than the open ones. Namely, for instance, $\mathcal{T}_R$ and $\mathcal{E}_R$ are just the families of all *ascending* and *residual subsets* of the goset $X(R)$, respectively.

This fact, stressed first by the third author in [**81**], can also be well seen from

Example 11.1. If in particular $X = \mathbb{R}$ and

$$R(x) = \{ x - 1 \} \cup [\, x \, , +\infty \, [$$

for all $x \in X$, then R is a reflexive relation on X such that $\mathcal{T}_R = \{ \emptyset \, , X \}$, but $\mathcal{E}_R$ is quite a large family.

Remark 11.4. However, if R is a preorder relation on X, then the converses of the assertions (1)–(3) of Theorem 11.3 can also be proved. Therefore, in this case, the family $\mathcal{T}_R$ is also a quite powerful tool.

12. Increasingly normal and regular functions of gosets

NOTATION 12.1. **In this and the next four sections, we shall assume that**

(a) $X(R)$ **and** $Y(S)$ **are gosets;**

(b) φ **is a function of** X **to itself;**

(c) f **is a function of** X **to** Y**;** **(d)** g **is a function of** Y **to** X**.**

In [**96, 94**], by extending the ideas of Ore [**56**], Schmidt [**75**, p. 209], Blyth and Janowitz [**9**, p. 11], and Pataki [**60**] on Galois connections, residuated mappings, and operation-increasing structures, we have used the following

DEFINITION 12.1. We say that the function f is

(1) *increasingly* g*–normal* if for all $x \in X$ and $y \in Y$ we have

$$f(x)\,S\,y \quad \Longleftrightarrow \quad x\,R\,g(y);$$

(2) *increasingly* φ*–regular* if for all $u, v \in X$ we have

$$f(u)\,S\,f(v) \quad \Longleftrightarrow \quad u\,R\,\varphi(v).$$

REMARK 12.1. Now, the function f may, for instance, be naturally called *increasingly normal* if it is increasingly g–normal for some function g.

And, the function f may, for instance, be naturally called *uniquely increasingly normal* if there exists a unique function g_f such that f is increasingly g_f–normal.

Later, we shall see that the increasingly normal functions are usually increasing. Therefore, the function f may, for instance, be naturally called *decreasingly normal* if it is increasing normal as a function of $X(R)$ to $Y\big(S^{-1}\big)$.

To clarify the relationship between normal and regular functions, we can easily prove the following two theorems.

THEOREM 12.1. *If* f *is increasingly* g*–normal and* $\varphi = g \circ f$*, then* f *is increasingly* φ*–regular.*

THEOREM 12.2. *If* f *is increasingly* φ*–regular,* f *is onto* Y*, and* $\varphi = g \circ f$*, then* f *is increasingly* g*–normal.*

PROOF. Suppose that $x \in X$ and $y \in Y$. Then, since $Y = f[X]$, there exists $v \in X$ such that $y = f(v)$.

Now, we can easily see that

$$f(x)\,S\,y \iff f(x)\,S\,f(v) \iff x\,R\,\varphi(v) \iff$$
$$x\,R\,(g \circ f)(v) \iff x\,R\,g\big(f(v)\big) \iff x\,R\,g(y).$$

Therefore, f is increasingly g–normal.

Remark 12.2. From Theorem 12.1, we can see that several properties of the increasingly normal functions can be immediately derived from those of the increasingly regular ones. Therefore, the latter ones have to studied before the former ones.

Moreover, from Theorem 12.2, we can feel that the increasing regular functions are still less general objects than the increasingly normal ones. Later, we shall see that they are usually strictly between closure operations and increasingly normal functions.

By using Definition 12.1, we can also easily prove the following three theorems.

Theorem 12.3. *If f is an increasingly g–normal function of $X(R)$ to $Y(S)$, then g is an increasingly f–normal function of $Y(S^{-1})$ to $X(R^{-1})$.*

Proof. By the corresponding definitions, for any $y \in Y$ and $x \in X$, we have

$$y\, S^{-1} f(x) \iff f(x)\, S\, y \iff x\, R\, g(y) \iff g(y)\, R^{-1} x\,.$$

Therefore, the required assertion is true.

Remark 12.3. Thus, the properties of the functions g and $f \circ g$ can, in principle, be immediately derived from those of f and $g \circ f$. However, it may sometimes be more convenient to apply some direct proofs.

Theorem 12.4. *If f is an increasingly g–normal function of $X(R)$ to $Y(S)$ and h is an increasingly k–normal function of $Y(S)$ to a further goset $Z(T)$, then $h \circ f$ is an increasingly $g \circ k$–normal function of $X(R)$ to $Z(T)$.*

Proof. By the corresponding definitions, for any $x \in X$ and $z \in Z$, we have

$$(h \circ f)(x)\, T\, z \iff h\big(f(x)\big)\, T\, z \iff$$
$$f(x)\, S\, k(z) \iff z\, R\, g\big(k(z)\big) \iff z\, R\, (g \circ k)(z)\,.$$

Therefore, the required assertion is true.

Remark 12.4. Hence, we can see that the family of all increasingly normal functions of $X(R)$ to itself, with composition, forms a monoid (semigroup with identity).

Unfortunately, an analogue of Theorems 12.3 and cannot be proved for increasingly regular functions. Moreover, an analogue of Theorem 12.4 is not also true for decreasingly normal functions.

Theorem 12.5. *If f is increasingly regular and R and S are preorders, then the family of all functions φ of X to itself such that f is φ-regular is also a semigroup with respect to composition.*

Proof. If f is increasingly φ–regular and ψ–regular, then by Definition 12.1 for any $u, v \in X$, we have

$$u\, R\, (\psi \circ \varphi)(v) \iff u\, R\, \psi\big(\varphi(v)\big) \iff f(u)\, S\, f\big(\varphi(v)\big) \iff f(u)\, S\, (f \circ \varphi)(v).$$

Moreover, by the forthcoming Theorem 13.1, we have

$$(f \circ \varphi)(v)\, S\, f(v) \qquad \text{and} \qquad f(v)\, S\, (f \circ \varphi)(v)\,.$$

Hence, by using the transitivity of S, we can infer that

$$u\,R\,(\psi\circ\varphi)(v) \iff f\,(u)\,S\,f\,(v)\,.$$

Therefore, f is also increasingly $(\psi\circ\varphi$–regular.

REMARK 12.5. Note that f is increasingly Δ_X–regular if and only if, for any $u,\,v\in X$, we have $u\,R\,v \iff f\,(u)\,S\,f\,(v)\,.$

While, a function f of $X\,(R)$ to itself is increasingly Δ_X–normal if and only if, for any $u,\,v\in X$, we have $u\,R\,v \iff f\,(u)\,R\,v\,.$

13. Some basic properties of increasingly regular and normal functions

The following theorems have been proved in some former papers [**96, 107, 1**]. Therefore, most of the proofs will be omitted.

THEOREM 13.1. *If f is increasingly φ–regular, R is a preorder and S is reflexive, then*

(1) *φ is extensive;* (2) *f is increasing;* (3) *$f\circ\varphi\,S\,f$ and $f\,S\,f\circ\varphi$.*

REMARK 13.1. If in addition S is antisymmetric, then instead of assertion (3) we may simply write $f=f\circ\varphi\,.$

THEOREM 13.2. *If R is a preorder, then the following assertions are equivalent:*

(1) *φ is a closure operation;* (2) *φ is increasingly φ–regular;*

(3) *there exists an increasingly φ–regular function h of $X\,(R)$ to a proset $Z\,(T)$.*

HINT. If assertion (3) holds, then by Theorem 13.1, φ is extensive and $h\circ\varphi\,T\,h$. Hence, we can infer that $h\circ\varphi^2\,T\,h\circ\varphi$. Therefore, by the transitivity of T, we also have $h\circ\varphi^2\,T\,h$. Thus, for any $u\in X$, we have $h\big(\varphi^2(u)\big)\,T\,h\,(u)$. Hence, by using the increasing φ–regularity of h, we can infer that $\varphi^2(u)\,R\,\varphi\,(u)$. Therefore, $\varphi^2\,R\,\varphi$. Moreover, since φ is extensive, we also have $\varphi\,R\,\varphi^2$.

On the other hand, if $u,\,v\in X$ such that $u\,R\,v$, then Theorem 13.1 we also have $h\,(u)\,T\,h\,(v)$ and $h\big(\varphi\,(u)\big)\,T\,h\,(u)$. Therefore, by the transitivity of T, we also have $h\big(\varphi\,(u)\big)\,T\,h\,(v)$. Hence, by using the increasing φ–regularity of h, we can already infer that $\varphi\,(u)\,R\,\varphi\,(v)$. Therefore, φ is increasing, and thus assertion (1) also holds.

REMARK 13.2. Thus, in the case of prosets, inreasingly regular functions are natural generalizations of closure operations. Moreover, all closure operations can be obtained from increasingly regular functions.

From Theorem 13.2, by using the corresponding definitions, we can easily derive

COROLLARY 13.1. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is increasingly φ–regular;*

(2) *φ is a closure operation and* $\mathrm{Ord}_\varphi=\mathrm{Ord}_f$.

HINT. If $\mathrm{Ord}_\varphi = \mathrm{Ord}_f$ holds, then by Definition 7.1, for any $u, v \in X$, we have $\varphi(u) R \varphi(v) \iff f(u) S f(v)$.

Moreover, if φ is a closure operation on X, then by Theorem 13.2, for any $u, v \in X$, we have $\varphi(u) R \varphi(v) \iff u R \varphi(v)$.

Therefore, in contrast to the implication $(1) \Longrightarrow (2)$, the converse implication $(2) \Longrightarrow (1)$ does not need any particular property of S.

THEOREM 13.3. *If f is g–normal and R and S are preorders, then*

(1) *f and g are increasing;*

(2) *$g \circ f$ is a closure operation;* (3) *$f \circ g$ is an interior operation;*

(4) *$f \circ g \circ f \, S \, f$ and $f \, S \, f \circ g \circ f$;* (5) *$g \circ f \circ g \, S \, g$ and $g \, S \, g \circ f \circ g$.*

REMARK 13.3. If in addition R and S are antisymmetric, then we can we can simply state that that

$$f = f \circ g \circ f \qquad \text{and} \qquad g = g \circ f \circ g.$$

By using these equalities, we can easily prove that

$$g[Y] = \mathrm{Fix}(g \circ f) \qquad \text{and} \qquad f[X] = \mathrm{Fix}(f \circ g).$$

Namely, if for instance $x \in g[Y]$, then there exists $y \in Y$ such that $x = g(y)$. Therefore,

$$(g \circ f)(x) = g\big(f(x)\big) = g\Big(f\big(g(y)\big)\Big) = (g \circ f \circ g)(y) = g(y) = x,$$

and thus $g[Y] \subseteq \mathrm{Fix}(g \circ f)$.

THEOREM 13.4. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is increasingly g–normal;*

(2) *f and g are increasing, $g \circ f$ is extensive and $f \circ g$ is intensive.*

REMARK 13.4. This theorem shows that the recent definition of Galois connections [**22**, p. 155], suggested by Schmidt [**75**, p. 209], is equivalent to the old one given by Ore [**56**].

THEOREM 13.5. *If R is a preorder, then the following assertions are equivalent:*

(1) *φ is an involution operation;* (2) *φ is increasingly φ–normal.*

HINT. If assertion (1) holds, then φ is increasing and

$$u \, R \, \varphi\big(\varphi(u)\big) \qquad \text{and} \qquad \varphi\big(\varphi(v)\big) \, R \, v$$

for all $u, v \in X$. Hence, by using the increasingness of φ and the transitivity of R, we can see that

$$\varphi(u) \, R \, v \implies \varphi\big(\varphi(u)\big) \, R \, \varphi(v) \implies u \, R \varphi(v)$$

and

$$u R \varphi(v) \implies \varphi(u) R \varphi\big(\varphi(v)\big) \implies \varphi(u) R v .$$

Thus, assertion (2) also holds.

REMARK 13.5. Thus, if R is a preorder, then every involution operation on $X(R)$ can be obtained from increasingly normal functions.

14. Some very particular properties of increasingly regular and normal functions

THEOREM 14.1. *If f is increasingly φ–regular and R and S are partial orders, then the following assertions are equivalent:*

(1) $\varphi = \Delta_X$; (2) f *is injective.*

PROOF. By Theorem 13.1 and the antisymmetry of S, for any $x \in X$, we have $f\big(\varphi(x)\big) = f(x)$. Hence, if assertion (2) holds, we can infer that $\varphi(x) = x = \Delta_X(x)$. Thus, assertion (1) also holds.

To prove the converse implication, suppose now that $u, v \in X$ such that $f(u) = f(v)$. Then, by the reflexivity of S, we also have $f(u) S f(v)$ and $f(v) S f(u)$. Hence, by using the increasing φ–regularity of f, we can infer that $u R \varphi(v)$ and $v R \varphi(u)$. Hence, if assertion (1) holds, then we can infer that $u R v$ and $v R u$. Thus, by the antisymmetry of R, we also have $u = v$. Therefore, assertion (2) also holds.

REMARK 14.1. By the corresponding definitions, f is increasing if and only if f is increasingly left Δ_X–regular.

THEOREM 14.2. *If f is increasingly g–normal, and R and S are partial orders, then the following assertions are equivalent:*

(1) f *is injective;* (2) $g \circ f = \Delta_X$; (3) g *is onto* X.

PROOF. By Theorem 12.1, the function f is $g \circ f$–regular. Hence, by Theorem 14.1, we can see that assertions (1) and (2) are equivalent.

Moreover, by Theorem 13.3 and the antisymmetry of R, we have

$$g\Big(f\big(g(y)\big)\Big)(x) = g(y)$$

for all $y \in Y$. Hence, if assertion (3) holds, i. e., $g[Y] = X$, then we can infer that

$$g\big(f(x)\big) = x$$

for all $x \in X$. Therefore, assertion (2) also holds.

Conversely, if assertion (2) holds, then we can at once see that

$$X = \Delta_X[X] = g\big[f[X]\big] \subseteq g[Y] .$$

Therefore, $X = g[Y]$, and thus assertion (3) also holds.

From this theorem, by using Theorem 12.3, we can immediately derive

Corollary 14.1. *If f is g–normal, and R and S are partial orders, then the following assertions are equivalent:*

(1) *f is onto Y;* (2) *$f \circ g = \Delta_Y$;* (3) *g is injective.*

Now, by Theorem 13.2 and Corollary 14.1, we can also state the following

Corollary 14.2. *If f is g–normal, injective and onto Y, and R and S are partial orders, then $g = f^{-1}$.*

Remark 14.2. Thus, if f is g–normal, then g may be considered as a certain generalized inverse function of f.

Moreover, we can also easily prove the following two theorems.

Theorem 14.3. *If f is increasingly g–normal, R and S are partial orders,*

(a) $\varphi = g \circ f$, $Z = \varphi[X]$, $N = R \,|\, Z$, $h = f \,|\, Z$;

(b) $\psi = f \circ g$, $W = \psi[Y]$, $M = S \,|\, W$, $k = g \,|\, W$;

then

(1) *$Z(N)$ and $W(M)$ are subposets of $X(R)$ and $Y(S)$, respectively;*

(3) *h is an injective, increasing function of $Z(N)$ onto $W(M)$ such that $k = h^{-1}$.*

Proof. It is clear that Z and W are subsets of X and Y, respectively, and thus assertion (1) is true.

Moreover, if $z \in Z$, then there exists $x \in X$ such that $z = \varphi(z)$, and thus $z = (g \circ f)(x)$. Hence, we can see that

$$h(z) = f(z) = f\big((g \circ f)(x)\big) = (f \circ g)\big(f(x)\big) = \psi\big(f(x)\big) \in \psi[Y] = W.$$

Therefore, h is a function of Z to W. Quite similarly, we can also see that k is a function of W to Z. Hence, it is clear that h is an increasingly k–normal function of $Z(N)$ to $W(M)$. Thus, by Theorem 13.3, the functions h and k are increasing.

Furthermore, if $z \in Z$, then by choosing $x \in X$ such that $z = \varphi(x)$ and using Theorem 13.3, we can see that

$$(k \circ h)(z) = k\big(h(z)\big) = g\big(f(z)\big) = g\Big(f\big(\varphi(x)\big)\Big) = g\Big(f\big((g \circ f)(x)\big)\Big) =$$
$$\big(g \circ (f \circ g \circ f)\big)(x) = (g \circ f)(x) = \varphi(x) = z = \Delta_Z(z).$$

Therefore, $k \circ h = \Delta_Z$. Moreover, quite similarly, we can also see that $h \circ k = \Delta_W$. Hence, it is clear that assertion (2) is also true.

Theorem 14.4. *If φ is a closure operation on $X(R)$, R is a preorder,*

$$Z = \varphi[X] \qquad \text{and} \qquad T = R \cap Z^2,$$

then φ is an increasingly Δ_Z–normal function of $X(R)$ onto $Z(T)$ such that $\varphi = \Delta_Z \circ \varphi$.

PROOF. From Theorem 13.2, we can see that φ is an increasingly φ–regular function of $X(R)$ to itself. That is, for any $u, v \in X$ we have

$$\varphi(u) R \varphi(v) \quad \iff \quad u R \varphi(v).$$

Hence, since $Z = \varphi[X]$ and $\varphi(u), \varphi(v) \in Z$, we can see that φ is a φ–regular function of $X(R)$ onto $Z(T)$. Now, since $\varphi = \Delta_Z \circ \varphi$, by Theorem 12.2 we can see that the required assertion is also true.

REMARK 14.3. Thus, if R is a preorder, then every closure operation on $X(R)$ can also be obtained from increasingly normal functions.

15. Characterizations of normal and regular functions

The following theorems have also been proved in our former papers [**96, 107, 1**]. Therefore, most of the proofs will again be omitted.

THEOREM 15.1. *The following assertions are equivalent:*

(1) *f is an increasingly g–normal,* (2) $\operatorname{Int}_f(y) = \operatorname{lb}\big(g(y)\big)$ *for all* $y \in Y$.

HINT. If assertion (1) holds, then for any $x \in X$ and $y \in Y$, we have

$$x \in \operatorname{lb}\big(g(y)\big) \iff x R g(y) \iff f(x) S y \iff x \in \operatorname{Int}_f(y).$$

Therefore, assertion (2) also holds.

COROLLARY 15.1. *If f is increasingly g–normal and R is reflexive, then for any $y \in Y$ we have*

$$g(y) \in \max\big(\operatorname{Int}_f(y)\big).$$

PROOF. By the reflexivity of R and Theorem 15.1, for any $y \in Y$, we evidently have $g(y) \in \operatorname{lb}\big(g(y)\big) = \operatorname{Int}_f(y)$.

Moreover, from the inclusion $\operatorname{Int}_f(y) \subseteq \operatorname{lb}\big(g(y)\big)$, by using Corollary 4.1, we can infer that $\{g(y)\} \subseteq \operatorname{ub}\big(\operatorname{Int}_f(y)\big)$, and thus $g(y) \in \operatorname{ub}\big(\operatorname{Int}_f(y)\big)$.

Thus, by Definition 5.1, the required assertion is also true.

REMARK 15.1. If in addition R is antisymmetric, then by Theorem 5.7 we may write $g(y) = \max\big(\operatorname{Int}_f(y)\big)$ in the above corollary.

Therefore, by Corollary 15.1 and Theorem 5.7, we can also state

COROLLARY 15.2. *If f is increasingly normal and R is reflexive and antisymmetric, then f is uniquely increasingly normal.*

However, it is now more important to note that, by using our former results, we can also prove the following

THEOREM 15.2. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is increasingly g–normal;*

(2) *f is increasing and* $g(y) \in \max\big(\operatorname{Int}_f(y)\big)$ *for all* $y \in Y$.

PROOF. If assertion (1) holds, then by Theorems 12.1 and 13.1, and Corollary 15.1 we can see that assertion (2) also holds even if S is only reflexive.

To prove the converse implication, suppose now that assertion (2) holds, and $x \in X$ and $y \in Y$. Then, by the definition of maximum, we have

$$g(y) \in \mathrm{Int}_f(y), \qquad \text{and thus} \qquad f\big(g(y)\big)\, S\, y.$$

Moreover, we also have $g(y) \in \mathrm{ub}\big(\mathrm{Int}_f(y)\big)$. Hence, we can already see that

$$f(x)\, S\, y \implies x \in \mathrm{Int}_f(y) \implies x\, R\, g(y).$$

Moreover, by using the increasigness of f and the transitivity of S, we can also see that

$$x\, R\, g(y) \implies f(x)\, S\, f\big(g(y)\big) \implies f(x)\, S\, y.$$

Thus, assertion (1) also holds even if R is arbitrary and S is transitive.

REMARK 15.2. From Theorems 15.1 and 15.2, by using Theorem 13.5, we can immediately derive some useful characterizations of involution operations.

Moreover, from Theorem 15.2, by the Axiom of Choice, we can derive

COROLLARY 15.3. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is increasingly g–normal;*

(2) *f is increasing and $\max\big(\mathrm{Int}_f(y)\big) \neq \emptyset$ for all $y \in Y$.*

Hence, we can easily derive the the following

COROLLARY 15.4. *If R and S are preorders, and $X(R)$ is* max*-complete, then the following assertions are equivalent:*

(1) *f is increasingly normal;*

(2) *f is increasing and $f[X]$ is cofinal in $Y\big(S^{-1}\big)$.*

Thus, in particular, we can also state the following

COROLLARY 15.5. *If R and S are preorders, $X(R)$ is* max*–complete and f is onto Y, then the following assertions are equivalent:*

(1) *f is increasing;* (2) *f is increasingly normal.*

Now, analogously to the previous results, we can also easily establish the following two theorems and their corollaries.

THEOREM 15.3. *The following assertions are equivalent:*

(1) *f is increasingly φ–regular;*

(2) $\mathrm{Int}_f\big(f(x)\big) = \mathrm{lb}\big(\varphi(x)\big)$ *for all $x \in X$.*

COROLLARY 15.6. *If f is increasingly φ–regular and R is reflexive, then for any $x \in X$ we have*

$$\varphi(x) \in \max\Big(\mathrm{Int}_f\big(f(x)\big)\Big).$$

REMARK 15.3. If in addition R is antisymmetric, then by Theorem 5.7 we may write $\varphi(x) = \max\big(\operatorname{Int}_f\big(f(x)\big)\big)$ in the above corollary.

COROLLARY 15.7. *If f is increasingly regular and R is reflexive and antisymmetric, then f is uniquely increasingly regular.*

THEOREM 15.4. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is increasingly φ–regular;*

(2) *f is increasing and $\varphi(x) \in \max\big(\operatorname{Int}_f\big(f(x)\big)\big)$ for all $x \in X$.*

REMARK 15.4. From Theorems 15.3 and 15.4, by using Theorem 13.2, we can immediately derive some useful characterizations of closure operations.

COROLLARY 15.8. *If R and S are preorders, then the following assertions are equivalent:*

(1) *f is increasingly regular;*

(2) *f is increasing and $\max\big(\operatorname{Int}_f\big(f(x)\big)\big) \neq \emptyset$ for all $x \in X$.*

COROLLARY 15.9. *If R and S are preorders and f is onto Y, then the following assertions are equivalent:*

(1) *f is increasingly regular;* (2) *f is increasingly normal.*

PROOF. If assertion (1) holds, then by Corollary 15.8 $\max\big(\operatorname{Int}_f\big(f(x)\big)\big) \neq \emptyset$ for all $x \in X$. Hence, since $Y = f[X]$, we can infer that $\max\big(\operatorname{Int}_f(y)\big) \neq \emptyset$ for all $y \in Y$. Therefore, by Corollary 15.3, assertion (2) also holds.

Moreover, by Theorem 12.1, the converse implication (2) $\Longrightarrow$ (1) is always true.

16. Supremum properties of normal and regular functions

The following theorems have also been proved in our former papers [**96, 107, 1**]. Therefore, most of the proofs will again be omitted.

THEOREM 16.1. *If f is increasingly normal, then for any $A \subseteq X$ we have*

$$f\big[\operatorname{lb}\big(\operatorname{ub}(A)\big)\big] \subseteq \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big).$$

PROOF. If $y \in f\big[\operatorname{lb}\big(\operatorname{ub}(A)\big)\big]$, then there exists $x \in \operatorname{lb}\big(\operatorname{ub}(A)\big)$ such that $y = f(x)$. Moreover, if $b \in \operatorname{ub}\big(f[A]\big)$, then for any $a \in A$ we have $f(a) S b$. Hence, by using that f is increasingly h–normal, for some function h of Y to X, we can infer that $a R h(b)$. Therefore, $h(b) \in \operatorname{ub}(A)$, and thus by $x \in \operatorname{lb}\big(\operatorname{ub}(A)\big)$ we have $x R h(b)$. Hence, by using that f is increasingly h–normal, we can infer that $f(x) S b$, and thus $y S b$. Therefore, $y \in \operatorname{lb}\big(\operatorname{ub}\big(f[A]\big)\big)$ also holds.

COROLLARY 16.1. *If f is increasingly normal, then for any $y \in Y$ we have*

$$\max\big(\operatorname{Int}_f(y)\big) = \sup\big(\operatorname{Int}_f(y)\big).$$

PROOF. By Theorems 5.3 and 16.1, we have

$$f\left[\sup(A)\right]=f\left[\operatorname{ub}(A)\cap\operatorname{lb}\big(\operatorname{ub}(A)\big)\right]\subseteq f\left[\operatorname{lb}\big(\operatorname{ub}(A)\big)\right]\subseteq\operatorname{lb}\Big(\operatorname{ub}\big(f\,[A\,]\big)\Big).$$

for all $A\subseteq X$. Therefore, Theorem 7.6 can be applied.

Now, by using our former results, we can also prove the following

THEOREM 16.2. *If R and S are preorders and $X(R)$ is sup–complete, then the following assertions are equivalent:*

(1) *f is increasingly normal;*

(2) *$f\,[\sup(A)\,]\subseteq\sup\big(f\,[A\,]\big)$ for all $A\subseteq X$;*

(3) *f is increasing and $\sup\big(\operatorname{Int}_f(y)\big)\subseteq\operatorname{Int}_f(y)\big)$ for all $y\in Y$;*

(4) *f is increasing and $\max\big(\operatorname{Int}_f(y)\big)=\sup\big(\operatorname{Int}_f(y)\big)$ for all $y\in Y$;*

(5) *f is increasing and $f\left[\sup(A)\right]\subseteq\operatorname{lb}\Big(\operatorname{ub}\big(f\,[A\,]\big)\Big)$ for all $A\subseteq X$;*

(6) *f is increasing and $f\left[\operatorname{lb}\big(\operatorname{ub}(A)\big)\right]\subseteq\operatorname{lb}\Big(\operatorname{ub}\big(f\,[A\,]\big)\Big)$ for all $A\subseteq X$.*

HINT. If assertion (1) holds, then from Theorem 13.3 we know that f is increasing. Moreover, by using Theorems 5.3, 6.2 and 16.1 we can see that

$$\begin{aligned}f\left[\sup(A)\right]=f\left[\operatorname{ub}(A)\cap\operatorname{lb}\big(\operatorname{ub}(A)\big)\right]&\subseteq f\left[\operatorname{ub}(A)\right]\cap f\left[\operatorname{lb}\big(\operatorname{ub}(A)\big)\right]\\&\subseteq\operatorname{ub}\big(f\,[A\,]\big)\cap\operatorname{lb}\Big(\operatorname{ub}\big(f\,[A\,]\big)\Big)=\sup\big(f\,[A\,]\big).\end{aligned}$$

Therefore, assertion (2) also holds even if $X(R)$ is not assumed to be sup-complete.

While, if assertion (2) holds, then then by using Theorem 5.3 we can see that

$$f\,[\max(A)\,]\subseteq f\,[\sup(A)\,]\subseteq\sup\big(f\,[A\,]\big)\subseteq\operatorname{ub}\big(f\,[A\,]\big)$$

for all $A\subseteq X$. Thus, by Theorem 6.3, f is increasing.

Moreover, by Theorem 5.3, we can also note that

$$f\,[\sup(A)\,]\subseteq\sup\big(f\,[A\,]\big)\subseteq\operatorname{lb}\Big(\operatorname{ub}\big(f\,[A\,]\big)\Big)$$

for all $A\subseteq X$. Hence, by using Theorem 7.6 and the sup-completeness of $X(R)$ we can already infer that

$$\max\big(\operatorname{Int}_f(y)\big)=\sup\big(\operatorname{Int}_f(y)\big)\neq\emptyset$$

for all $y\in Y$. Thus, by Corollary 15.3, assertion (1) also holds.

REMARK 16.1. If in addition both R and S are antisymmetric, then instead of assertion (2) we may simply write that $f\big(\sup(A)\big)=\sup\big(f\,[A\,]\big)$ for all $A\subseteq X$.

Analogously, to Theorem 16.2, we can also prove the following

THEOREM 16.3. *If R and S are preorders, $X(R)$ is a sup–complete and f is onto Y, then the following assertions are equivalent:*

(1) *f is regular;*

(2) *$f\,[\sup(A)\,]\subseteq\sup\big(f\,[A\,]\big)$ for all $A\subseteq X$;*

(3) f *is increasing and* $\sup\big(\operatorname{Int}_f\big(f(x)\big)\big) \subseteq \operatorname{Int}_f\big(f(x)\big)$ *for all* $x \in X$.

(4) f *is increasing and* $\max\big(\operatorname{Int}_f\big(f(x)\big)\big) = \sup\big(\operatorname{Int}_f\big(f(x)\big)\big)$ *for all* $x \in X$.

Remark 16.2. In this theorem, we may also write $\operatorname{Ord}_f^{-1}(x)$ in place of $\operatorname{Int}_f\big(f(x)\big)$.

From Theorem 16.3, by using Theorem 13.2, we can immediately derive

Corollary 16.2. *If* $X(R)$ *is a* sup*–complete proset and* φ *is onto* X, *then the following assertions are equivalent:*

(1) φ *is a closure operation;* (2) $\varphi[\sup(A)] \subseteq \sup\big(\varphi[A]\big)$ *for all* $A \subseteq X$.

Remark 16.3. If in addition R is antisymmetrics, then instead of assertion (2) we may simply write that $\varphi\big(\sup(A)\big) = \sup\big(\varphi[A]\big)$ for all $A \subseteq X$.

17. Relational characterizations of increasingly normal and regular functions

Analogously to Theorem 6.7, we can also prove the following

Theorem 17.1. *The following assertions are equivalent:*

(1) f *is increasingly* g*–normal;*

(2) $S \circ f = g^{-1} \circ R$; (3) $g \circ S \circ f \subseteq R$ *and* $g^{-1} \circ R \circ f^{-1} \subseteq S$.

Proof. For any $x \in X$ and $y \in Y$, the following assertions are equivalent:

$$f(x) S y \iff x R g(y),$$
$$y \in S\big(f(x)\big) \iff g(y) \in R(x),$$
$$y \in S\big(f(x)\big) \iff y \in g^{-1}[R(x)],$$
$$S\big(f(x)\big) = g^{-1}[R(x)]$$
$$\big(S \circ f\big)(x) = \big(g^{-1} \circ R\big)(x).$$

Therefore, by Definition 12.1, assertions (1) and (2) are equivalent.

Moreover, for instance, by using the inclusions $f \circ f^{-1} \subseteq \Delta_Y$ and $\Delta_X \subseteq f^{-1} \circ f$, we can also see that

$$g^{-1} \circ R \subseteq S \circ f \implies g^{-1} \circ R \circ f^{-1} \subseteq S \circ f \circ f^{-1} \implies$$
$$g^{-1} \circ R \circ f^{-1} \subseteq S \circ \Delta_Y \implies g^{-1} \circ R \circ f^{-1} \subseteq S$$

and

$$g^{-1} \circ R \circ f^{-1} \subseteq S \implies g^{-1} \circ R \circ f^{-1} \circ f \subseteq S \circ f \implies$$
$$g^{-1} \circ R \circ \Delta_X \subseteq S \circ f \implies g^{-1} \circ R \subseteq S \circ f.$$

Therefore, $g^{-1} \circ R \subseteq S \circ f \iff g^{-1} \circ R \circ f^{-1} \subseteq S$, and thus the second halves of assertions (2) and (3) are equivalent.

Moreover, by using Theorem 15.2, we can also prove the following

THEOREM 17.2. *If* R *and* S *are preorders, then the following assertions are equivalent:*

(1) f *is increasingly* g*–normal;*

(2) $f \circ R \subseteq S \circ f$ *and* $g \subseteq (S \circ f)^{-1} \setminus R^c \circ (S \circ f)^{-1}$.

PROOF. By Theorem 15.2, assertion (1) is equivalent to the statement that

(a) f is increasing and $g(y) \in \max\big(\operatorname{Int}_f(y)\big)$ for all $y \in Y$.

Moreover, from Theorem 6.7, we can see that

(b) f is increasing if and only if $f \circ R \subseteq S \circ f$.

Furthermore, from Theorem 7.5 and Remark 4.5, we can see that $\operatorname{Int}_f = (S \circ f)^{-1}$ and

$$\max(A) = A \cap \operatorname{ub}(A) = A \cap R^c[A]^c = A \setminus R^c[A]$$

for all $A \subseteq X$.

Therefore, for any $y \in Y$, we have

$$\begin{aligned} g(y) \in \max\big(\operatorname{Int}_f(y)\big) \iff \\ g(y) \in (S \circ f)^{-1}(y) \setminus R^c\big[(S \circ f)^{-1}(y)\big] \iff \\ g(y) \in (S \circ f)^{-1}(y) \setminus \big(R^c \circ (S \circ f)^{-1}\big)(y) \iff \\ g(y) \in \big((S \circ f)^{-1} \setminus R^c \circ (S \circ f)^{-1}\big)(y). \end{aligned}$$

Thus, assertions (1) and (2) are equivalent.

REMARK 17.1. From Theorems 17.1 and 17.2, by using Theorem 13.5, we can immediately derive some useful characterizations of involution operations.

Moreover, from Theorem 17.1 we can easily derive the following

THEOREM 17.3. *The following assertions are equivalent:*

(1) f *is increasingly normal;*

(2) *for each* $y \in Y$, *there exists* $x \in X$ *such that* $f^{-1}[S^{-1}(y)] = R^{-1}(x)$.

HINT. If assertion (2) holds, then by the Axiom of Choice there exists a function k of Y to X such that

$$f^{-1}[S^{-1}] = R^{-1}\big(k(y)\big).$$

Hence, we can infer that

$$f^{-1} \circ S^{-1} = R^{-1} \circ k, \qquad \text{and thus} \qquad S \circ f = k^{-1} \circ R.$$

Therefore, by Theorem 17.1, f is increasingly k–normal, and thus assertion (1) also holds.

Now, analogously to the above three theorems, we can also prove the following three theorems.

THEOREM 17.4. *The following assertions are equivalent:*

(1) f *is increasingly* φ*–regular;*

(2) $\varphi^{-1} \circ R = f^{-1} \circ S \circ f$; (3) $f \circ \varphi^{-1} \circ R \subseteq S \circ f$ *and* $\varphi \circ f^{-1} \circ S \circ f \subseteq R$.

HINT. For any $u, v \in X$, the following assertions are equivalent:

$$f(u) S f(v) \iff u R \varphi(v),$$
$$f(v) \in S\big(f(u)\big) \iff \varphi(v) \in R(u),$$
$$v \in f^{-1}\big[S\big(f(u)\big)\big] \iff v \in \varphi^{-1}[R(u)],$$
$$f^{-1}\big[S\big(f(u)\big)\big] = \varphi^{-1}[R(u)]$$
$$\big(f^{-1} \circ S \circ f\big)(u) = \big(\varphi^{-1} \circ R\big)(u).$$

Therefore, by Definition 12.1, assertion (1) and (2) are equivalent.

REMARK 17.2. In addition to assertion (3), we can also prove that

$$f \circ \varphi^{-1} \circ R \subseteq S \circ f \iff \varphi^{-1} \circ R \circ f^{-1} \subseteq f^{-1} \circ S.$$

THEOREM 17.5. *The following assertions are equivalent:*

(1) f *is increasingly* φ*–regular;*

(2) $f \circ R \subseteq S \circ f$ *and* $\varphi \subseteq (S \circ f)^{-1} \circ f \setminus R^c \circ (S \circ f)^{-1} \circ f$.

REMARK 17.3. From Theorems 17.4 and 17.5, by using Theorem 13.2, we can immediately derive some useful characterizations of closure operations.

THEOREM 17.6. *The following assertions are equivalent:*

(1) f *is increasingly regular;*

(2) *for each* $x \in X$, *there exists* $u \in X$ *such that* $f^{-1}\big[S^{-1}\big(f(x)\big)\big] = R^{-1}(u)$.

REMARK 17.4. From the results of this section, by using some basic theorems on the box product of relations [**98**], we can easily derive some further characterizations of increasingly normal and reqular functions.

18. Increasingly normal and regular functions of power sets

NOTATION 18.1. **In this and the next two sections, we shall assume that**

(a) X **and** Y **are sets;**

(b) Φ **is a function of** $\mathcal{P}(X)$ **to itself;**

(c) F **is a function of** $\mathcal{P}(X)$ **to** $\mathcal{P}(Y)$ **and** G **is a function of** $\mathcal{P}(Y)$ **to** $\mathcal{P}(X)$ **.**

REMARK 18.1. Here, instead of *corelations* (functions of power sets to power sets), it would also be more convenient to consider *super relations* (ordinary relations on power sets to sets) [**1**].

However, our former definitions and results on normal and regular functions can be more directly applied to corelations. For instance, by specializing Definition 12.1, we may naturally use the following

DEFINITION 18.1. We say that the function F is

(1) *increasingly G–normal* if for all $A \subseteq X$ and $B \subseteq Y$ we have

$$F(A) \subseteq B \qquad \Longleftrightarrow \qquad A \subseteq G(B);$$

(2) *increasingly Φ–regular* if for all $A_1, A_2 \subseteq X$ we have

$$F(A_1) \subseteq F(A_2) \qquad \Longleftrightarrow \qquad A_1 \subseteq \Phi(A_2).$$

Now, by identifying singletons with their elements, we may also introduce

DEFINITION 18.2. For the function F, we define a two functions G_F and Φ_F such that

$$G_F(B) = \{ x \in X : \quad F(x) \subseteq B \}$$

for all $B \subseteq Y$, and

$$\Phi_F(A) = \{ x \in X : \quad F(x) \subseteq F(A) \}$$

for all $A \subseteq X$.

REMARK 18.2. By the corresponding definitions, for any $x \in X$ and $B \subseteq Y$, we have

$$x \in G_F(B) \iff F(\{x\}) \subseteq B \iff \{x\} \in \mathrm{Int}_F(B) \iff x \in \mathrm{int}_F(B).$$

Thus, by the usual identification of relations with set-valued functions, we can actually state that $G_F = \mathrm{int}_F$.

Moreover, for any $A \subseteq X$, we can also note that

$$\Phi_F(A) = G_F\big(F(A)\big) = \mathrm{int}_F\big(F(A)\big).$$

Therefore, we have $\Phi_F = G_F$, and we can actually also state that $\Phi_F = \mathrm{int}_F \circ F$.

However, our present notation G_F is more convenient for

THEOREM 18.1. *If F is increasingly G–normal, then $G = G_F$.*

PROOF. By the corresponding definitios, for any $x \in X$ and $B \subseteq Y$, we have

$$x \in G_F(B) \iff F(\{x\}) \subseteq B \iff \{x\} \subseteq G(B) \iff x \in G(B).$$

Therefore, $G_F(B) = G(B)$ for all $B \subseteq Y$, and thus $G_F = G$.

Thus, in particular, we can also state the following three corollaries.

COROLLARY 18.1. *There exists at most one function G of $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ such that F is increasingly G–normal.*

COROLLARY 18.2. *If* F *is increasingly normal, then* $G = G_F$ *is the unique function of* $\mathcal{P}(Y)$ *to* $\mathcal{P}(X)$ *such that* F *is increasingly* G*–normal.*

COROLLARY 18.3. *The following assertions are equivalent:*

(1) F *is increasingly normal;* (2) F *is increasingly* G_F*–normal.*

By using the definition of G_F, we can also easily prove the following

THEOREM 18.2. *The function* G_F *is increasing.*

PROOF. If $B_1 \subseteq B_2 \subseteq Y$, then

$$x \in G_F(B_1) \implies F(x) \subseteq B_1 \implies F(x) \subseteq B_2 \implies x \in G_F(B_2).$$

Therefore, $G_F(B_1) \subseteq G_F(B_2)$.

REMARK 18.3. Hence, we can infer that the inverse of the relation associated with G_F is ascending valued.

However, it is now more important to note that, by identifying singletons with their elements, we can also prove the following

THEOREM 18.3. *If* X *is a goset, then the following assertions are equivalent:*

(1) $F \,|\, X$ *is increasing;* (2) G_F *is descending valued.*

PROOF. Suppose that $B \subseteq Y$, $x \in G_F(B)$ and $u \leqslant x$. Then, if assertion (1) holds, then we also have $F(u) \subseteq F(x)$. Moreover, since $x \in G_F(B)$, we also have $F(x) \subseteq B$. Hence, we can infer that $F(u) \subseteq B$, and thus $u \in F_G(B)$. This shows that $G_F(B)$ is a descending subset of X, and thus assertion (2) also holds.

To prove the converse implication, suppose now that $x_1, x_2 \in X$ such that $x_1 \leqslant x_2$. Then, because of $F(x_2) \subseteq F(x_2)$, we have $x_2 \in G_F\big(F(x_2)\big)$. Hence, if assertion (2) holds, and thus $G_F\big(F(x_2)\big)$ is a descending subset of X, we can infer that $x_1 \in G_F\big(F(x_2)\big)$, and thus $F(x_1) \subseteq F(x_2)$. Thus, assertion (1) also holds.

19. Characterizations of increasingly normal set-functions

From Theorem 15.1, by using Corollary 18.1, we can immediately derive

THEOREM 19.1. *The following assertions assertions are equivalent:*

(1) F *is increasingly normal;*

(2) $\operatorname{Int}_F(B) = \mathcal{P}\big(G_F(B)\big)$ *for all* $B \subseteq Y$.

PROOF. By Corollary 18.3 and Theorem 15.1, assertion (1) is equivalent to the statement that:

(a) $\operatorname{Int}_F(B) = \operatorname{lb}\big(G_F(B)\big)$ for all $B \subseteq Y$.

Moreover, by the corresponding definitions, for any $A \subseteq X$ and $B \subseteq Y$, we have

$$A \in \operatorname{lb}\big(G_F(B)\big) \iff A \subseteq G_F(B) \iff A \in \mathcal{P}\big(G_F(B)\big),$$

and thus $\mathrm{lb}\big(G_F(y)\big) = \mathcal{P}\big(G_F(y)\big)$.

Therefore, statement (a) is equivalent to assertion (2), and thus assertions (1) and (2) are also equivalent.

From the above theorem, by using Theorems 13.3 and 7.4, we can easily derive

Corollary 19.1. *The following assertions are equivalent:*

(1) F *is increasingly normal;*

(2) F *is increasing and* $G_F(B) \in \mathrm{Int}_F(B)$ *for all* $B \subseteq Y$.

Proof. From Theorems 13.3 and 19.1, we can see that $(1) \Longrightarrow (2)$. Therefore, we need actually prove the converse implication.

For this, note that if F is increasing, then by Theorem 7.4 the relation Int_F is descending valued. Therefore, for any $B \subseteq Y$,

$$G_F(B) \in \mathrm{Int}_F(B) \quad \Longrightarrow \quad \mathcal{P}\big(G_F(B)\big) \subseteq \mathrm{Int}_F(B).$$

Moreover, if F is increasing, we can also see that

$$\begin{aligned} A \in \mathrm{Int}_F(B) \quad &\Longrightarrow \quad F(A) \subseteq B \quad \Longrightarrow \quad \forall\ x \in A: \quad F(x) \subseteq B \quad \Longrightarrow \\ &\forall\ x \in A: \quad x \in G_F(B) \quad \Longrightarrow \quad A \subseteq G_F(B) \quad \Longrightarrow \quad A \in \mathcal{P}\big(G_F(B)\big). \end{aligned}$$

and thus $\mathrm{Int}_F(B) \subseteq \mathcal{P}\big(G_F(B)\big)$. Therefore, if F is increasing, then

$$G_F(B) \in \mathrm{Int}_F(B) \quad \Longrightarrow \quad \mathrm{Int}_F(B) = \mathcal{P}\big(G_F(B)\big).$$

Thus, Theorem 19.1 can be used to see that $(2) \Longrightarrow (1)$.

Remark 19.1. Note that if $B \subseteq Y$, then by Remark 18.2, we have $G_F(B) = \mathrm{int}_F(B)$.

Therefore, for any $x \in X$, we have

$$x \in G_F(B) \iff x \in \mathrm{int}_F(B) \iff \{x\} \in \mathrm{Int}_F(B).$$

Thus, the inclusion $G_F(B) \subseteq \bigcup \mathrm{Int}_F(B)$ is always true.

However, it is now more important to note that, by using Theorems 15.2 and 16.2 and Corollary 18.3, we can also prove the following two theorems.

Theorem 19.2. *The the following assertions are equivalent:*

(1) F *is increasingly normal;*

(2) F *is increasing and* $\max\big(\mathrm{Int}_F(B)\big) \neq \emptyset$ *for all* $B \subseteq Y$*;*

(3) F *is increasing and* $G_F(B) = \max\big(\mathrm{Int}_F(B)\big)$ *for all* $B \subseteq Y$*,*

(4) F *is increasing and* $G_F(B) \in \mathrm{Int}_F(B) \subseteq \mathcal{P}\big(G_F(B)\big)$ *for all* $B \subseteq Y$*;*

(5) F *is increasing and* $G_F(B) = \bigcup \mathrm{Int}_F(B) \in \mathrm{Int}_F(B)$ *for all* $B \subseteq Y$.

HINT. To prove the equivalence of assertions (3)–(5), note that

$$G_F(B) = \max\big(\mathrm{Int}_F(B)\big) \iff G_F(B) \in \mathrm{Int}_F(B)\,, \quad G_F(B) \in \mathrm{ub}\,\big(\mathrm{Int}_F(B)\big)\,.$$

Moreover, we also have

$$G_F(B) \in \mathrm{ub}\,\big(\mathrm{Int}_F(B)\big) \iff \forall\ A \in \mathrm{Int}_F(B): \quad A \subseteq G_F(B) \iff$$
$$\mathrm{Int}_F(B) \subseteq \mathcal{P}\big(G_F(B)\big) \iff \bigcup \mathrm{Int}_F(B) \subseteq G_F(B) \iff \bigcup\ \mathrm{Int}_F(B) = G_F(B).$$

THEOREM 19.3. *The the following assertions are equivalent:*

(1) F *is increasingly normal;*

(2) $F\,(\bigcup \mathcal{A}\,) = \bigcup F\,[\,\mathcal{A}\,]$ *for all* $\mathcal{A} \subseteq \mathcal{P}(X)$ *;*

(3) F *is increasing and* $\bigcup \mathrm{Int}_F(B) \in \mathrm{Int}_F(B)$ *for all* $B \subseteq Y$*,*

(4) F *is increasing and* $F\,(\bigcup \mathcal{A}\,) \in \mathcal{P}\,(\bigcup F\,[\,\mathcal{A}\,]\,)$ *for all* $\mathcal{A} \subseteq \mathcal{P}(X)$ *;*

(5) F *is increasing and* $F\big[\,\mathcal{P}\big(\bigcup \mathcal{A}\,\big)\big] \in \mathcal{P}\,(\bigcup F\,[\,\mathcal{A}\,]\,)$ *for all* $\mathcal{A} \subseteq \mathcal{P}(X)$ *.*

HINT. To derive this theorem from Theorem 16.2, note that if $A \subseteq \mathcal{P}\,(X)$, then for any $B \subseteq X$, we have

$$B \in \mathrm{ub}\,(\mathcal{A}) \iff \forall\ A \in \mathcal{A}: \quad A \subseteq B \iff \bigcup \mathcal{A} \subseteq B\,,$$

and thus

$$\mathrm{ub}\,(\mathcal{A}) = \mathcal{P}^{-1}\big(\bigcup \mathcal{A}\,\big) \qquad \text{and} \qquad \bigcap \mathrm{ub}\,(\mathcal{A}) = \bigcup \mathcal{A}\,.$$

Moreover, for any $C \subseteq X$, we have

$$C \in \mathrm{lb}\big(\,\mathrm{ub}\,(\mathcal{A})\big) \iff \forall\ B \in \mathrm{ub}\,(\mathcal{A}): \quad C \subseteq B \iff$$
$$C \subseteq \bigcap \mathrm{ub}\,(\mathcal{A}) \iff C \subseteq \bigcup \mathcal{A} \iff C \in \mathcal{P}\,\big(\bigcup \mathcal{A}\big)\,,$$

and thus

$$\mathrm{lb}\big(\,\mathrm{ub}\,(\mathcal{A})\big) = \mathcal{P}\,\big(\bigcup \mathcal{A}\big)\,.$$

Therefore,

$$\sup\,(\mathcal{A}) = \max\,\big(\mathrm{ub}\,(\mathcal{A})\big) = \mathrm{ub}\,(\mathcal{A}) \cap \mathrm{lb}\big(\,\mathrm{ub}\,(\mathcal{A})\big) = \mathcal{P}^{-1}\big(\bigcup \mathcal{A}\big) \cap \mathcal{P}\,\big(\bigcup \mathcal{A}\big) = \bigcup \mathcal{A}\,.$$

Now, by using our former results, we can also easily prove the following

THEOREM 19.4. *The following assertions are equivalent:*

(1) F *is increasingly normal;*

(2) $F\,(A) = \bigcup_{x \in A} F\,(x)$ *for all* $A \subseteq X$*;*

(3) *there exists a relation* R *on* X *to* Y *such that, for all* $A \subseteq X$*, we have*

$$F\,(A) = \mathrm{cl}_{R^{-1}}(A)\,.$$

Proof. From Theorems 19.3 and [**99**, Theorem 3], we can see that

$$(1) \quad \Longleftrightarrow \quad F \text{ is union-preserving} \quad \Longleftrightarrow \quad (2).$$

Moreover, if assertion (3) holds and $G(B) = \operatorname{int}_R(B)$ for all $B \subseteq Y$, then from Corollary 10.1 we can see that F is increasingly G–normal. Thus, in particular assertion (1) also holds.

Furthermore, if assertion (2) holds, then defining a relation R on X to Y such that $R(x) = F(x)$ for all $x \in X$, we can at once see that

$$F(A) = F(A) = \bigcup_{x \in A} F(x) = \bigcup_{x \in A} R(x) = R[A] = \operatorname{cl}_{R^{-1}}(A)$$

for all $A \subseteq X$. Thus, assertion (3) also holds.

Remark 19.2. If R is as in assertion (3), then we necessarily have

$$R(x) = R[\{x\}] = \operatorname{cl}_{R^{-1}}\big(\{x\}\big) = F\big(\{x\}\big) = F(x)$$

for all $x \in X$. Therefore, the relation R is uniquely determined by F.

20. Normality properties of complement and dual set-functions

Definition 20.1. For the function F, the functions F^c and $F^\star$, defined by

$$F^c(A) = F(A)^c \qquad \text{and} \qquad F^\star(A) = F(A^c)^c$$

for all $A \subseteq X$, will be called the *complement* and *dual* of F.

Remark 20.1. Hence, we can at once see that

$$F^c = \mathcal{C}_Y \circ F \qquad \text{and} \qquad F^\star = F^c \circ \mathcal{C}_X .$$

Thus, in contrast to c, the operation $*$ not only involutive, but also increasing.

Moreover, to illustrate the appropriateness of Definition 20.1, we can state

Example 20.1. If R is a relation on X to Y and

$$F_R(B) = \operatorname{cl}_R(B)$$

for all $B \subseteq Y$, then by Theorem 10.1 we can see that

$$F_R^\star(B) = F_R(B^c)^c = \operatorname{cl}_R(B^c)^c = \operatorname{int}_R(B)$$

for all $B \subseteq Y$.

Now, by using Definition 20.1, we can also easily prove the following

Theorem 20.1. *The following assertions are equivalent:*

(1) F^c *is increasingly* G^c*–normal;*

(2) $G \circ \mathcal{C}_Y$ *is increasingly* $F \circ \mathcal{C}_X$*–normal.*

Proof. By the corresponding definitions, the following assertions are equivalent:

(a) $F^c(A) \subseteq B \iff A \subseteq G^c(B)$ for all $A \subseteq X$ and $B \subseteq Y$;

(b) $F(A)^c \subseteq B \iff A \subseteq G(B)^c)$ for all $A \subseteq X$ and $B \subseteq Y$;

(c) $B^c \subseteq F(A) \iff G(B) \subseteq A^c$ for all $A \subseteq X$ and $B \subseteq Y$;

(d) $B \subseteq F(A^c) \iff G(B^c) \subseteq A$ for all $A \subseteq X$ and $B \subseteq Y$;

(e) $B \subseteq (F \circ \mathcal{C}_X)(A) \iff (G \circ \mathcal{C}_Y)(B) \subseteq A$ for all $A \subseteq X$ and $B \subseteq Y$.

Moreover, by Definition 18.1, assertion (1) is equivalent to assertion (a), and assertion (2) is equivalent to assertion (e). Thus, assertions (1) and (2) are also equivalent.

From this theorem, by writing G^c in place of G, we can immediately derive

Corollary 20.1. *The following assertions are equivalent:*

(1) F^c *is increasingly* G*–normal;* (2) $G^\star$ *is increasingly* $F \circ \mathcal{C}_X$*–normal.*

Moreover, from Theorem 20.1, by writing F^c and G^c in place of F and G, respectively, we can also derive the following

Theorem 20.2. *The following assertions are equivalent:*

(1) F *is increasingly* G*–normal;* (2) $G^\star$ *is increasingly* $F^\star$*–normal.*

Proof. By using Theorem 20.1, we can see that

$$(1) \iff (F^c)^c \text{ is increasingly } (G^c)^c\text{–normal} \iff$$
$$G^c \circ \mathcal{C}_Y \text{ is increasingly } F^c \circ \mathcal{C}_X\text{–normal} \iff (2).$$

From this theorem, by writing $F^\star$ in place of F, we can immediately derive

Corollary 20.2. *The following assertions are equivalent:*

(1) $F^\star$ *is increasingly* G*–normal;* (2) $G^\star$ *is increasingly* F*–normal.*

Moreover, from Theorem 20.2, by writing $F^\star$ and $G^\star$ in place of F and G, respectively, we can also derive the following

Theorem 20.3. *The following assertions are equivalent:*

(1) G *is increasingly* F*–normal;* (2) $F^\star$ *is increasingly* $G^\star$*–normal.*

Hence, by writing $F^\star$ in place of F, we can immediately derive

Corollary 20.3. *The following assertions are equivalent:*

(1) G *is* $F^\star$*–normal;* (2) F *is* $G^\star$*–normal.*

Finally, we note that by using Theorem 20.1, we can also prove the following

Theorem 20.4. *The following assertions are equivalent:*

(1) F *is decreasingly* G*–normal;*

(2) F^c *is increasingly* $G \circ \mathcal{C}_Y$*–normal;*

(3) G^c *is increasingly* $F \circ \mathcal{C}_X$*–normal.*

Proof. By the corresponding definitions, the following assertions are equivalent:

(a) $B \subseteq F(A) \iff A \subseteq G(B)$ for all $A \subseteq X$ and $B \subseteq Y$;

(b) $F(A)^c \subseteq B^c \iff A \subseteq G(B)$ for all $A \subseteq X$ and $B \subseteq Y$;

(c) $F^c(A) \subseteq B^c \iff A \subseteq (G \circ C_Y)(B^c)$ for all $A \subseteq X$ and $B \subseteq Y$;

(d) $F^c(A) \subseteq B \iff A \subseteq (G \circ C_Y)(B)$ for all $A \subseteq X$ and $B \subseteq Y$.

Hence, since assertion assertion (1) is to assertion (a), and assertion (d) is equivalent assertion (2), we can see that assertions (1) and (2) are equivalent.

Moreover, from Theorem 20.1, by writing $G^\star$ in place of G, we can see that assertion (2) and (3) are also equivalent.

Remark 20.2. To obtain some more instructive reformulations of the corresponding results of this section, we can use that $F^c = \mathcal{C}_Y \circ F$ and $G^c = \mathcal{C}_X \circ G$.

21. Increasingly normal and regular functions of two variables

Notation 21.1. **In this section, we shall assume that**

(a) $X(R)$ **and** $Y(S)$ **are gosets;**

(b) Φ **is a function of** $X \times Z$ **to** X **for some set** Z**;**

(c) F **is a function of** $X \times Z$ **to** Y**;** **(d)** G **is a function of** $Z \times Y$ **to** X**.**

Remark 21.1. An important case will be when

$$\Phi(x, z) = G\big(z, F(x, z)\big)$$

for all $x \in X$ and $z \in Z$.

Now, analogously to Definition 12.1, we may also naturally introduce

Definition 21.1. We say that the function F is

(1) *increasingly* G*–normal* if for all $x \in X$, $y \in Y$ and $z \in Z$ we have

$$F(x, z)\, S\, y \iff x\, R\, G(z, y);$$

(2) *increasingly* Φ*–regular* if for all $u, v \in X$ and $z \in Z$ we have

$$F(u, z)\, S\, F(u, z) \iff u\, R\, \Phi(v, z).$$

Remark 21.2. Thus, the function F may, for instance, be naturally called *increasingly normal* if it is increasingly G–normal for some function G.

And, the function F may, for instance, be naturally called *uniquely increasingly normal* if there exists a unique function G_F such that F is increasingly G_F–normal.

Moreover, the function F may, for instance, be naturally called *decreasingly normal* if it is increasing normal as a function of $X(R) \times Z$ to $Y\big(S^{-1}\big)$.

The study of increasingly normal and regular functions of two variables can be easily traced back to that of those functions of one variable by using the following

DEFINITION 21.2. For any $x \in X$, $y \in Y$ and $z \in Z$, we define

$$f_z(x) = F(x, z), \qquad g_z(y) = G(z, y) \qquad \text{and} \qquad \varphi_z(x) = \Phi(x, z).$$

REMARK 21.3. Thus, if Φ is as in Remark 21.1, then we have

$$\varphi_z(x) = \Phi(x, z) = G\big(z, F(x, z)\big) = G\big(z, f_z(x)\big) = g_z\big(f_z(x)\big) = (g_z \circ f_z)(x)$$

for all $x \in X$, $y \in Y$ and $z \in Z$. Therefore, $\varphi_z = g_z \circ f_z$ for all $z \in Z$.

Now, as a useful consequence of our former definitions, we can easily establish

THEOREM 21.1. *The following assertions are true:*

(1) *F is increasingly G–normal if and only if f_z is increasingly g_z–normal for all $z \in Z$;*

(2) *F is increasingly Φ–regular if and only if f_z is increasingly φ_z–regular for all $z \in Z$.*

PROOF. For instance, if F is increasingly G–normal, then for any $x \in X$, $y \in Y$ and $z \in Z$, we have

$$f_z(x)\, S\, y \iff F(x, z)\, S\, y \iff x\, R\, G(z, y) \iff x\, R\, g_z(y).$$

Therefore, for any $z \in Z$, the function f_z is increasingly g_z–normal.

Now, by using this theorem, from our former results on increasingly normal and regular functions of one variable we can easily derive several statements for those functions of two variables.

THEOREM 21.2. *If F is increasingly G–normal and Φ is as in Remark 21.1, then F is increasingly Φ–regular.*

THEOREM 21.3. *If F is increasingly Φ–regular, $Y = F[X \times \{z\}]$ for all $z \in Z$, and Φ is as in Remark 21.1, then F is increasingly G–normal.*

PROOF. By Theorem 21.1, for each $z \in Z$, the function f_z is φ_z–regular and onto Y. Moreover, by Remark 21.3, we have $\varphi_z = g_z \circ f_z$. Therefore, by Theorem 12.2, the function f_z is increasingly g_z–normal. Thus, by Theorem 21.1, the required assertion is also true.

THEOREM 21.4. *If F is increasingly normal (regular) and R is reflexive and antisymmetric, then F is uniquely increasingly normal (regular).*

PROOF. For instance, if F is increasingly G–normal, then by Theorem 21.1, for each $z \in Z$, the function f_z is increasingly g_z–normal. Thus, by Corollary 15.1 and Remark 15.1, we have

$$G(z, y) = g_z(y) = \max\big(\operatorname{Int}_{f_z}(y)\big) = \max\big(\{x \in X : \ f_z(x)\, S\, y\}\big) = \max\big(\{x \in X : \ F(x, z)\, S\, y\}\big)$$

for all $x \in X$, $y \in Y$ and $z \in Z$. Thus, G is uniquely determined by F.

REMARK 21.4. Because of the above theorems, to illustrate the appropriateness of our present definitions, it is enough to provide some important examples only for normal functions of two variables.

22. Illustrating examples for normal functions of two variables

NOTATION 22.1. **In this section, we shall assume that $\mathcal{U}$ is a relator on X to Y.**

EXAMPLE 22.1. For all $A \subseteq X$, $B \subseteq Y$ and $U \in \mathcal{U}$, define

$$F(A, U) = \operatorname{cl}_{U^{-1}}(A) \qquad \text{and} \qquad G(U, B) = \operatorname{int}_U(B).$$

Then, F is a uniquely increasingly G–normal function of $\mathcal{P}(X)(\subseteq) \times \mathcal{U}$ to $\mathcal{P}(Y)(\subseteq)$.

To prove this, note that, by Corollary 10.1, we have

$$F(A, U) \subseteq B \iff \operatorname{cl}_{U^{-1}}(A) \subseteq B \iff A \subseteq \operatorname{int}_U(B) \iff A \subseteq G(U, B)$$

for all $A \subseteq X$, $B \subseteq Y$ and $U \in \mathcal{U}$. Therefore, the function F is increasingly G–normal. Moreover, by Theorem 21.4, it is also uniquely increasingly normal.

REMARK 22.1. The $\left(cl_{U^{-1}}, \operatorname{int}_U\right)$ increasing Galois connection can be used to unify several particular theorems on relational inclusions [**103**].

REMARK 22.2. If in particular U is a function of X to Y, then by the usual identification of singletons with their elements, for any $x \in X$ and $B \subseteq Y$, we have

$$x \in \operatorname{int}_U(B) \iff U(x) \in B \iff x \in U^{-1}[B] \iff x \in \operatorname{cl}_U(B).$$

Therefore, in this very particular case, the equality

$$\operatorname{int}_U(B) = \operatorname{cl}_U(B), \qquad \text{and thus} \qquad G(U, B) = \operatorname{cl}_U(B)$$

also holds for all $B \subseteq Y$.

Therefore, if U is a function of X to Y, then for any $A \subseteq X$ and $B \subseteq Y$, we have not only

$$U[A] \subseteq B \iff A \subseteq U^{-1}[B^c]^c, \quad \text{but also} \quad U[A] \subseteq B \iff A \subseteq U^{-1}[B].$$

Of course, the latter equivalence can be more easily proved directly, without using the induced closures and interiors. However, the use of these basic tools puts this equivalence also a better perspective.

From Example 22.1, by using Theorems 21.1 and 20.2, we can also derive

EXAMPLE 22.2. For all $A \subseteq X$, $B \subseteq Y$ and $U \in \mathcal{U}$, define

$$F(B, U) = \operatorname{cl}_U(B) \qquad \text{and} \qquad G(U, A) = \operatorname{int}_{U^{-1}}(A).$$

Then, F is a uniquely increasingly G–normal function of $\mathcal{P}(Y)(\subseteq) \times \mathcal{U}$ to $\mathcal{P}(X)(\subseteq)$.

From Example 22.1, by Theorem 21.1, we can see that, for any $U \in \mathcal{U}$,

(a) $f_{U^{-1}}$ is increasingly $g_{U^{-1}}$–normal.

Hence, by using Theorem 20.2, we can infer that

(b) $g^{\star}_{U^{-1}}$ is increasingly $f^{\star}_{U^{-1}}$–normal.

Moreover, by using Definition 20.1 and Theorem 10.1, we can see that

$$g^{\star}_{U^{-1}}(B) = g_{U^{-1}}(B^c)^c = \operatorname{int}_U(B^c)^c = \operatorname{cl}_U(B) = f_U(B)$$

and

$$f^{\star}_{U^{-1}}(A) = f_{U^{-1}}(A^c)^c = \operatorname{cl}_{U^{-1}}(A^c)^c = \operatorname{int}_{U^{-1}}(A) = g_U(A)$$

for all $B \subseteq Y$ and $A \subseteq X$. Therefore,

$$g^{\star}_{U^{-1}} = f_U \qquad \text{and} \qquad f^{\star}_{U^{-1}} = g_U\,,$$

and thus

(c) f_U is increasingly g_U–normal.

Therefore, by Theorem 21.1, we can also state that F is increasingly G–normal. Moreover, by Theorem 21.4, it is also uniquely increasingly normal.

However, it is now more important to note that we can also state the following

Example 22.3. For all $A \subseteq X$, $B \subseteq Y$ and $U \in \mathcal{U}$, define

$$F(A, U) = \operatorname{ub}_U(A) \qquad \text{and} \qquad G(U, B) = \operatorname{lb}_U(B)\,.$$

Then, F is a uniquely decreasingly G–normal function of $\mathcal{P}(X)(\subseteq) \times \mathcal{U}$ to $\mathcal{P}(Y)(\subseteq)$.

To prove this, note that, by Corollary 4.1, we have

$$F(A, U) \supseteq B \iff \operatorname{ub}_U(A) \supseteq B \iff A \subseteq \operatorname{lb}_U(A) \iff A \subseteq G(U, B)$$

for all $A \subseteq X$, $B \subseteq Y$ and $U \in \mathcal{U}$. Thus, F is an increasingly G–normal function of $\mathcal{P}(X)(\subseteq)$ to $\mathcal{P}(X)(\supseteq)$. Moreover, by Theorem 21.4, it is also uniquely increasingly normal. Thus, the required assertion is also true.

Remark 22.3. The $\big(\operatorname{ub}_U, \operatorname{lb}_U\big)$ decreasing Galois connection can be used to construct the Dedekind–MacNeille completions of posets [**22, 72**].

Remark 22.4. This decreasing Galois connection is not independent of the former increasing one $\big(cl_{U^{-1}}, \operatorname{int}_U\big)$.

Namely, by Theorems 4.1 and 10.4, we have

$$\operatorname{ub}_U = \operatorname{lb}_{U^{-1}} \qquad \text{and} \qquad \operatorname{lb}_U = \operatorname{cl}^c_{U^c} = \operatorname{int}_U \circ\, \mathcal{C}_Y\,.$$

From Example 22.2, by using Theorems 21.1 and 20.4, we can easily derive the following two further examples.

Example 22.4. For all $A \subseteq X$, $B \subseteq Y$ and $U \in \mathcal{U}$, define

$$F(U, A) = \operatorname{ub}_U(A)^c \qquad \text{and} \qquad G(U, B) = \operatorname{lb}_U(B^c)\,.$$

Then, F is a uniquely increasingly G–normal function of $\mathcal{P}(X)(\subseteq) \times \mathcal{U}$ to $\mathcal{P}(Y)(\subseteq)$.

EXAMPLE 22.5. For all $A \subseteq X$, $B \subseteq Y$ and $U \in \mathcal{U}$, define

$$F(U, B) = \mathrm{lb}_U(B)^c \qquad \text{and} \qquad G(U, A) = \mathrm{ub}_U(A^c).$$

Then, F is a uniquely increasingly G–normal function of $\mathcal{P}(Y)(\subseteq) \times \mathcal{U}$ to $\mathcal{P}(X)(\subseteq)$.

REMARK 22.5. However, it is now more important to note that if F and G are as in Notation 21.1, then by using the plausible notations

$$x * z = F(x, z) \qquad \text{and} \qquad z \bullet y = G(z, y),$$

the increasing G–normality of the function F can be expressed in the form that

$$(x * z)\, S\, y \iff x\, R\, (z \bullet y)$$

for all $x \in X$, $y \in Y$ and $z \in Z$.

Therefore, Definition 21.1 can be used to provide some reasonable generalizations of the *residuated algebraic structures* of Bonzio and Chajda [**10**], for instance.

23. An interesting, preordered bigroupoid

Because of Remark 22.5, we may naturally consider the following

NOTATION 23.1. **In this and the next two sections, we shall assume that**

(a) $X(\leqslant)$ **is a proset;**

(b) $X(*)$ **and** $X(\bullet)$ **are groupoids;**

(c) for all $x, y, z \in X$, **we have**

$$x * z \leqslant y \iff x \leqslant z \bullet y.$$

REMARK 23.1. In this case, we shall say that the structure $X(*, \bullet, \leqslant)$ is an *increasingly normal, preordered bigroupoid*.

To provide a more direct motivation for the above assumptions, we shall show that if in particular $X(*, \leqslant)$ is a preordered group and $z \bullet y = y * z^{-1}$ for all $y, z \in X$, then assumption (c) also holds.

Thus, our present notion of an increasingly normal, preordered bigroupoid is a natural generalization of that of a preordered group.

REMARK 23.2. Under assumptions (a)–(c), by defining

$$f_z(x) = x * z \qquad \text{and} \qquad g_z(y) = z \bullet y$$

and

$$\varphi_z(x) = (g_z \circ f_z)(x) = g_z\big(f_z(x)\big) = g_z\big(x * z\big) = z \bullet (x * z)$$

for all $x, y, z \in X$, we can see that f_z, g_z and φ_z are functions of $X(\leqslant)$ to itself, for each $z \in X$, such that:

(1) f_z is increasingly g_z–normal; (2) f_z is increasingly φ_z–regular.

Therefore, by using our former results on regular and normal functions, we can easily establish several basic properties of the structure $X(*, \bullet, \leqslant)$.

For instance, from the corresponding results of Sections 12–16, by using Remark 23.1, we can easily derive the following theorems, and also their certain duals.

Theorem 23.1. *For each* $x_1, x_2, z \in X$,

(1) $x_1 \leqslant x_2$ *implies* $x_1 * z \leqslant x_2 * z$;

(2) $x_1 \leqslant x_2$ *implies* $z \bullet x_1 \leqslant z \bullet x_2$;

(3) $x_1 * z \leqslant x_2 * z$ *if and only if* $z \bullet (x_1 * z) \leqslant z \bullet (x_2 * z)$.

Proof. By Theorem 13.3, the functions f_z and g_z are is increasing. Thus, assertions (1) and (2) are true.

Moreover, from Corollary 13.1, we can see that $\mathrm{Ord}_{f_z} = \mathrm{Ord}_{\varphi_z}$. Therefore, for any $x_1, x_2 \in X$, we have

$$f_z(x_1) \leqslant f_z(x_2) \quad \Longleftrightarrow \quad \varphi_z(x_1) \leqslant \varphi_z(x_2).$$

Thus, assertion (3) is also true.

Theorem 23.2. *For each* $x, y, z \in X$, *we have*

(1) $x \leqslant z \bullet (x * z)$; (2) $(z \bullet y) * z \leqslant y$;

(3) $\big(z \bullet (x * z)\big) * z \leqslant x * z$ *and* $x * z \leqslant \big(z \bullet (x * z)\big) * z$.

Proof. By Theorem 13.3, we also have $x \leqslant \varphi_z(x)$,

$$f_z\big(\varphi_z(x)\big) \leqslant f_z(x) \qquad \text{and} \qquad f_z(x) \leqslant f_z\big(\varphi_z(x)\big).$$

Thus, assertions (1) and (3) are true. Assertion (2) also follows from Theorem 13.3. Moreover, it is also immediate from the inequality $z \bullet y \leqslant z \bullet y$ by property (c).

Theorem 23.3. *For each* $y, z \in X$, *we have*

(1) $z \bullet y \in \max\big(\big\{ x \in X : \ x * z \leqslant y \big\}\big)$;

(2) $\big\{ x \in X : \ x * z \leqslant y \big\} = \big\{ x \in X : \ x \leqslant z \bullet y \big\}$.

Proof. By Theorems 15.2 and 15.1, we have

$$g_z(y) \in \max\big(\mathrm{Int}_{f_z}(y)\big) \qquad \text{and} \qquad \mathrm{Int}_{f_z}(y) = \mathrm{lb}\big(g_z(y)\big).$$

Hence, by using that

$$\mathrm{Int}_{f_z}(y) = \big\{ x \in X : \ f_z(x) \leqslant y \big\} \quad \text{and} \quad \mathrm{lb}\big(g_z(y)\big) = \big\{ x \in X : \ x \leqslant g_z(y) \big\},$$

we can see that assertions (1) and (2) are true.

Remark 23.3. If in particular $X(\leqslant)$ is a poset, then in assertion (1) the equality also holds.

Therefore, in this particular case, the operation $\bullet$ is uniquely determined by the operation $*$.

THEOREM 23.4. *If $X(\leqslant)$ is a poset, then for each $z \in X$, the following assertions are equivalent:*

(1) $X = z \bullet X$;

(2) $x = z \bullet (x * z)$ *for all* $x \in X$;

(3) $x_1 * z = x_2 * z$ *implies* $x_1 = x_2$ *for all* $x_1, x_2 \in X$.

PROOF. By Theorem 14.2, the following assertions are equivalent:

(a) $X = g_z [X]$;

(b) $x = \varphi_z(x)$ for all $x \in X$;

(c) $f_z(x_1) = f_z(x_1)$ implies $x_1 = x_2$.

Hence, since

$$g_z [A] = \big\{ g_z(y) : \quad y \in A \big\} = \{ z \bullet y : \quad y \in A \big\} = z \bullet A,$$

for all $A \subseteq X$, we can see that assertions (1)–(3) are also equivalent.

THEOREM 23.5. *For each $z \in X$, we have*

(1) $\operatorname{lb}\big(\operatorname{ub}(A)\big) * z \subseteq \operatorname{lb}\,\big(\operatorname{ub}\big(\{ x * z : \quad x \in A \}\big)\big)$ *for all* $A \subseteq X$;

(2) $\max\big(\{ x \in X : \ x * z \leqslant y \}\big) = \sup\big(\{ x \in X : \ x * z \leqslant y \}\big)$ *for all* $y \in X$.

PROOF. By Theorem 16.1 and its corollary, we have

(a) $f_z\big[\operatorname{lb}\big(\operatorname{ub}(A)\big)\big] \subseteq \operatorname{lb}\big(\operatorname{ub}\big(f_z [A]\big)\big)$ for all $A \subseteq X$;

(b) $\max\big(\operatorname{Int}_{f_z}(y)\big) = \sup\big(\operatorname{Int}_{f_z}(y)\big)$ for all $y \in X$.

Hence, by using that

$$f_z [A] = \big\{ f_z(x) : \quad x \in A \big\} = \big\{ x * z : \quad x \in A \big\} = A * z$$

and $\operatorname{Int}_{f_z}(y) = \{ x \in X : \ x * z \leqslant y \}$, we can see that assertions (1) and (2) are true.

REMARK 23.4. If $X(\leqslant)$ is sup–complete, then by Theorem 16.2, we can see that, for each $z \in X$ and $A \subseteq X$,

$$\sup(A) * z \subseteq \sup(A * z).$$

If in particular $X(\leqslant)$ is poset then the equality also holds.

24. Some further theorems on increasingly normal bigroupoids

THEOREM 24.1. *If 1 is a left identity of $X(*)$, then for any $x, y \in X$ we have*

$$x \leqslant y \iff 1 \leqslant x \bullet y.$$

Proof. By the corresponding definitions, we have

$$x \leqslant y \iff 1 * x \leqslant y \iff 1 \leqslant x \bullet y\,.$$

Corollary 24.1. *If* 1 *is a left identity of* $X(*)$, *then for any* $x, y \in X$ *we have*

(1) $1 \leqslant 1 \bullet 1$;

(1) $1 \leqslant x \bullet x$;

(3) $x \leqslant 1 \iff 1 \leqslant x \bullet 1$;

(4) $1 \leqslant y \iff 1 \leqslant 1 \bullet y$.

Theorem 24.2. *If* 1 *is a left identity of* $X(*)$, *then for any* $x, y, z \in X$

(1) $x \leqslant 1 \implies x \leqslant y \bullet y \implies x * y \leqslant y$;

(2) $1 \leqslant x \bullet y$ *and* $z \leqslant 1 \implies x * z \leqslant y \implies x \leqslant z \bullet y$ *if* $*$ *is commutative*.

Proof. By Corollary 24.1, we have $1 \leqslant y \bullet y$. Hence, if $x \leqslant 1$, then by using the transitivity of $\leqslant$, we can infer $x \leqslant y \bullet y$. This implies $x * y \leqslant y$, and thus assertion (1) is true.

Moreover, by Theorems 24.1 and 23.1,

$$1 \leqslant x \bullet y \implies x \leqslant y \implies x * z \leqslant y * z\,,$$

and

$$z \leqslant 1 \implies z * y \leqslant 1 * y \implies z * y \leqslant y \implies y * z \leqslant y$$

if $*$ is commutative. Hence, by using the transitivity of $\leqslant$, we can infer that $x * z \leqslant y$. This implies $x \leqslant z \bullet y$, and thus assertion (2) is also true.

Remark 24.1. If $X(\leqslant)$ is a poset and 1 is a left identity of $X(*)$ such that $x \leqslant 1$ for all $x \in X$, then by Theorem 24.1, for any $x, y \in X$, we have

$$x \leqslant y \iff x \bullet y = 1\,.$$

Theorem 24.3. *If* 1 *is a right identity of* $X(*)$, *then for any* $x, y \in X$ *we have*

$$x \leqslant y \iff x \leqslant 1 \bullet y\,.$$

Proof. By the corresponding definitions, we have

$$x \leqslant y \iff x * 1 \leqslant y \iff x \leqslant 1 \bullet y\,.$$

Corollary 24.2. *If* 1 *is a right identity of* $X(*)$, *then for any* $x, y \in X$ *we have*

(2) $1 \leqslant 1 \bullet 1$;

(1) $x \leqslant 1 \bullet x$;

(3) $x \leqslant 1 \iff x \leqslant 1 \bullet 1$;

(4) $1 \leqslant y \iff 1 \leqslant 1 \bullet y$.

Theorem 24.4. *If* 1 *is a right identity of* $X(*)$, *then for any* $x, y, z \in X$

(1) $1 \bullet x \leqslant 1 \implies 1 \leqslant x$;

(2) $x \leqslant 1 \bullet y$ *and* $z \leqslant 1 \implies x * z \leqslant y \implies x \leqslant z \bullet y$ *if* $*$ *is commutative*.

Proof. By Corollary 24.2, we have $x \leqslant 1 \bullet x$. Hence, if $1 \bullet x \leqslant 1$, then by using the transitivity of $\leqslant$, we can infer $1 \leqslant x$. Therefore, assertion (1) is true.

Moreover, by Theorems 24.3 and 23.1,

$$x \leqslant 1 \bullet y \quad \Longrightarrow \quad x \leqslant y \quad \Longrightarrow \quad x * z \leqslant y * z,$$

and

$$z \leqslant 1 \quad \Longrightarrow \quad z * y \leqslant 1 * y \quad \Longrightarrow \quad y * z \leqslant y * 1 \quad \Longrightarrow \quad y * z \leqslant y$$

if $*$ is commutative. Hence, by using the transitivity of $\leqslant$, we can infer that $x * z \leqslant y$. This implies that $x \leqslant z \bullet y$, and thus assertion (2) is also true.

Theorem 24.5. *If* $1, x, x^{-1} \in X$ *such that* $x^{-1} * x = 1$, *then for any* $y \in X$ *we have*

$$1 \leqslant y \quad \Longleftrightarrow \quad x^{-1} \leqslant x \bullet y.$$

Proof. By the corresponding assumptions, we have

$$1 \leqslant y \quad \Longleftrightarrow \quad x^{-1} * x \leqslant y \quad \Longleftrightarrow \quad x^{-1} \leqslant x \bullet y.$$

Corollary 24.3. *Under the assumptions of Theorem 24.5,* $x^{-1} \leqslant x \bullet 1$.

Corollary 24.4. *If* $X(\leqslant)$ *is a poset, then under the assumptions of Theorem 24.5, for any* $y \in X$ *with* $y \leqslant 1$, *we have* $y = 1 \iff x^{-1} \leqslant x \bullet y$.

Theorem 24.6. *If* $1, x, x^{-1} \in X$ *such that* $x * x^{-1} = 1$, *then for any* $y \in X$

$$1 \leqslant y \quad \Longleftrightarrow \quad x \leqslant x^{-1} \bullet y.$$

Proof. By the corresponding assumptions, we have

$$1 \leqslant y \quad \Longleftrightarrow \quad x * x^{-1} \leqslant y \quad \Longleftrightarrow \quad x \leqslant x^{-1} \bullet y.$$

Corollary 24.5. *Under the assumptions of Theorem 24.6,* $x \leqslant x^{-1} \bullet 1$.

Corollary 24.6. *If* $X(\leqslant)$ *is a poset, then under the assumptions of Theorem 24.6, for any* $y \in X$ *with* $y \leqslant 1$, *we have* $y = 1 \iff x \leqslant x^{-1} \bullet y$.

Theorem 24.7. *If* x *is an idempotent of* $X(*)$, *then for any* $y \in X$ *we have*

$$x \leqslant y \quad \Longleftrightarrow \quad x \leqslant x \bullet y.$$

Proof. By the corresponding definitions, we have

$$x \leqslant y \quad \Longleftrightarrow \quad x * x \leqslant y \quad \Longleftrightarrow \quad x \leqslant x \bullet y.$$

Corollary 24.7. *If* x *is an idempotent of* $X(*)$, *then* $x \leqslant x \bullet x$.

Theorem 24.8. *If* 0 *is a left zero of* $X(*)$, *then for any* $x, y \in X$ *we have*

$$0 \leqslant y \quad \Longleftrightarrow \quad 0 \leqslant x \bullet y$$

Proof. By the corresponding definitions, we have

$$0 \leqslant y \quad \Longleftrightarrow \quad 0 * x \leqslant y \quad \Longleftrightarrow \quad 0 \leqslant x \bullet y.$$

Corollary 24.8. *If* 0 *is a left zero of* $X(*)$, *then for any* $x, y \in X$ *we have*

(1) $0 \leqslant 0 \bullet 0$; (2) $0 \leqslant x \bullet 0$; (3) $0 \leqslant y \iff 0 \leqslant 0 \bullet y$.

THEOREM 24.9. *If* 0 *is a right zero of* $X(*)$, *then for any* $x, y \in X$ *we have*

$$0 \leqslant y \quad \Longleftrightarrow \quad 0 \leqslant x \bullet y$$

PROOF. By the corresponding definitions, we have

$$0 \leqslant y \quad \Longleftrightarrow \quad x * 0 \leqslant y \quad \Longleftrightarrow \quad x \leqslant 0 \bullet y\,.$$

COROLLARY 24.9. *If* 0 *is a right zero of* $X(*)$, *then for any* $x, y \in X$ *we have*

(1) $0 \leqslant 0 \bullet 0$; (2) $x \leqslant 0 \bullet 0$; (3) $0 \leqslant y \iff 0 \leqslant 0 \bullet y$.

REMARK 24.2. Note that only a very few statements needed the reflexivity and the transitivity of the relation $\leqslant$.

25. Some consequences of the commutativity and associativity of $*$

THEOREM 25.1. *If* $X(*)$ *is commutative, then for any* $x, y \in X$ *we have*

$$x \leqslant y \bullet (y * x)\,.$$

PROOF. By the corresponding assumptions, it is clear that

$$x * y = y * x \quad \Longrightarrow \quad x * y \leqslant y * x \quad \Longrightarrow \quad x \leqslant y \bullet (y * x)\,,$$

and thus the required assertion is true.

THEOREM 25.2. *If* $X(*)$ *is a semigroup, then for any* $x, y, z, w \in X$ *we have*

(1) $x * y \leqslant z \bullet w \iff x \leqslant (y * z) \bullet w$;

(2) $x \bullet (y \bullet z) \leqslant (x * y) \bullet z$; (3) $(x * y) \bullet z \leqslant x \bullet (y \bullet z)$.

PROOF. By the corresponding assumptions, we have

$$x * y \leqslant z \bullet w \iff (x * y) * z \leqslant w \iff x * (y * z) \leqslant w \iff x \leqslant (y * z) \bullet w\,.$$

Thus, assertion (1) is true.

Because of the reflexivity of $\leqslant$, we have

$$x \bullet (y \bullet z) \leqslant x \bullet (y \bullet z)\,.$$

Hence, by using assumption (c) in Notation 23.1, we can infer that

$$\big(x \bullet (y \bullet z)\big) * x \leqslant y \bullet z\,, \quad \text{and thus also} \quad \Big(\big(x \bullet (y \bullet z)\big) * x\Big) * y \leqslant z\,.$$

Hence, by using the corresponding assumptions, we can infer that

$$\big(x \bullet (y \bullet z)\big) * (x * y) \leqslant z\,, \quad \text{and thus also} \quad x \bullet (y \bullet z) \leqslant (x * y) \bullet z\,.$$

Therefore, assertion (2) is also true.

On the other hand, by assertion (2) in Theorem 23.2 and the associativity of $*$, we have

$$\big((x * y) \bullet z\big) * (x * y) \leqslant z\,, \quad \text{and thus also} \quad \Big(\big((x * y) \bullet z\big) * x\Big) * y \leqslant z\,.$$

Hence, by using assumption (c) in Notation 23.1, we can infer that

$$\big((x*y)\bullet z\big)*x \leqslant y\bullet z\,, \qquad \text{and thus also} \qquad (x*y)\bullet z \leqslant x\bullet(y\bullet z)\,.$$

Therefore, assertion (3) is also true.

Corollary 25.1. *If $X(*)$ is a commutative semigroup, then for any $x, y, z \in X$ we have*

$$x\bullet(y\bullet z) \leqslant y\bullet(x\bullet z)\,.$$

Proof. By using assertion (2) in Theorem 25.2 and the commutativity of $*$, we can see that

$$x\bullet(y\bullet z) \leqslant (x*y)\bullet z = (y*x)\bullet z \leqslant y\bullet(x\bullet z)\,.$$

Therefore, by the transitivity of $\leqslant$, the required inequality is also true.

Theorem 25.3. *If $X(\leqslant)$ is a poset, then the following assertions are equivalent:*

(1) $X(*)$ *is a semigroup;*

(2) $(x*y)\bullet z = x\bullet(y\bullet z)$ *for all* $x, y, z \in X$.

Proof. If assertion (1) holds, then by assertions (2) and (3) in Theorem 25.2 and the antisymmetry of $\leqslant$, we can at once see that assertion (2) also holds. Therefore, we need only prove the converse implication.

For this, note that if assertion (2) holds, then by Remark 23.2 we have

$$g_{x*y}(z) = (x*y)\bullet z = x\bullet(y\bullet z) = x\bullet g_y(z) = g_x\big(g_y(z)\big) = (g_x\circ g_y)(z)$$

for all $x, y, z \in X$, and thus

$$g_{x*y} = g_x \circ g_y$$

for all $x, y \in X$.

Moreover, by Remark 23.2 and Theorem 12.3, we can see that, for any $x, y \in X$,

(a) g_x is increasingly f_x–normal;

(b) g_y is increasingly f_y–normal; (c) g_{x*y} increasingly f_{x*y}–normal;

as functions of $X(\geqslant)$ to itself.

On the other hand, from assertions (a) and (b), by using Theorem 12.4, we can infer that

(d) $g_{x*y} = g_x\circ g_y$ is increasingly $f_y\circ f_x$–normal,

as a function of $X(\geqslant)$ to itself;

Therefore, by Corollary 15.2, we necessarily have

$$f_{x*y} = f_y\circ f_x$$

for all $x, y \in X$, and thus also

$$f_{x*y}(z) = (f_y\circ f_x)(z) = f_y\big(f_x(z)\big)$$

for all $x, y, z \in X$. Hence, by using Remark 23.1, we can infer that

$$z * (x * y) = f_{x*y}(z) = f_y\big(f_x(z)\big) = f_y(z * x) = (z * x) * y$$

for all $x, y, z \in X$. Thus, assertion (1) also holds.

Remark 25.1. Note that the implication (1) $\Longrightarrow$ (2) can also be proved by using a similar argument.

However, it would now be more important to give a shorter direct proof for the implication (2) $\Longrightarrow$ (1) too.

Theorem 25.4. *If $X(\leqslant)$ is a poset and $X(*)$ is a semigroup such that*

(1) *$X(*)$ has a right identity 1;*

(2) *each $x \in X$ has a right inverse x^{-1} in $X(*)$;*

then for any $y, z \in X$ we have

$$z \bullet y = y * z^{-1}.$$

Proof. By assertion (1) in Theorem 23.1 and the corresponding assumptions, for any $x, y, z \in X$,

$$x * z \leqslant y \quad\Longrightarrow\quad (x * z) * z^{-1} \leqslant y * z^{-1} \quad\Longrightarrow$$
$$x * (z * z^{-1}) \leqslant y * z^{-1} \quad\Longrightarrow\quad x * 1 \leqslant y * z^{-1} \quad\Longrightarrow\quad x \leqslant y * z^{-1}.$$

Moreover, by using that

$$z^{-1} * z = \big(z^{-1} * z\big) * 1 = (z^{-1} * z) * \Big(z^{-1} * \big(z^{-1}\big)^{-1}\Big) =$$
$$\big(z^{-1} * \big(z * z^{-1}\big)\big) * \big(z^{-1}\big)^{-1} = \big(z^{-1} * 1\big) * \big(z^{-1}\big)^{-1} = z^{-1} * \big(z^{-1}\big)^{-1} = 1,$$

we can also see that

$$x \leqslant y * z^{-1} \quad\Longrightarrow\quad x * z \leqslant \big(y * z^{-1}\big) * z \quad\Longrightarrow$$
$$x * z \leqslant y * \big(z^{-1} * z\big) \quad\Longrightarrow\quad x * z \leqslant y * 1 \quad\Longrightarrow\quad x * z \leqslant y.$$

Therefore, we actually have

$$x * z \leqslant y \qquad\Longleftrightarrow\qquad x \leqslant y * z^{-1}.$$

Hence, by using Remark 23.2 and Corollary 15.2, we can infer that $z \bullet y = y * z^{-1}$.

Remark 25.2. If $X(*)$ is a semigroup such that assumptions (1) and (2) hold, then $X(*)$ is actually a group [**76**, p. 6].

Namely, if $z \in X$, then addition to $z^{-1} * z = 1$ we also have

$$1 * z = \big(z * z^{-1}\big) * z = z * \big(z^{-1} * z\big) = z * 1 = z.$$

26. Illustrating examples for increasingly normal bigroupoids

In addition to Theorem 25.4, we can also easily establish the following

EXAMPLE 26.1. Suppose that $X(\leqslant)$ is a goset and $X(*)$ is a semigroup such that

(1 $X(*)$ has a right identity 1;

(2) each $x \in X$ has a right inverse x^{-1} in $X(*)$;

(3) $x_1 \leqslant x_2$ implies $x_1 * y \leqslant x_1 * y$ for all $x_1, x_2, y \in X$.

For any $y, z \in Z$, define

$$z \bullet y = y * z^{-1}.$$

Then, $X(*, \bullet, \leqslant)$ is an increasingly normal, generalized ordered bigroupoid.

Namely, by the above assumptios, for any $x, y, z \in X$

$$x * z \leqslant y \implies (x * z) * z^{-1} \leqslant y * z^{-1} \implies x * \big(z * z^{-1}\big) \leqslant y * z^{-1} \implies$$
$$x * 1 \leqslant y * z^{-1} \implies x \leqslant y * z^{-1}. \implies x \leqslant z \bullet y.$$

Moreover, by using the former observation that $z^{-1} * z = 1$, we can also see that

$$x \leqslant z \bullet y \implies x \leqslant y * z^{-1} \implies x * z \leqslant \big(y * z^{-1}\big) * z \implies$$
$$x * z \leqslant y * \big(z^{-1} * z\big) \implies x * z \leqslant y * 1 \implies x * z \leqslant y.$$

Therefore, the equivalence $x * z \leqslant y \iff x \leqslant z \bullet y$ also holds.

REMARK 26.1. The operation $\bullet$ has several useful additional properties. However, of course, it need not be either commutative or associative even if the operation $*$ is commutative.

To see this, for instance, take $X = \,]\,0, +\infty\,[$, with the usual multiplication and inequality. Moreover, for all $y, z \in X$, define

$$z \bullet y = y \cdot z^{-1}.$$

Then, we have

$$x \bullet 1 = 1 \cdot x^{-1} = x^{-1} \qquad \text{and} \qquad 1 \bullet x = x \cdot 1^{-1} = x \cdot 1 = x$$

for all $x \in X$. Therefore, for instance, $2 \bullet 1 = 2^{-1}$ and $1 \bullet 2 = 2$, and thus $2 \bullet 1 \neq 1 \bullet 2$.

Moreover, for instance, we have

$$2 \bullet (3 \bullet 1) = 2 \bullet 3^{-1} = 3^{-1} \cdot 2^{-1}$$

and

$$(2 \bullet 3) \bullet 1 = \big(2 \bullet 3\big)^{-1} = \big(3 \cdot 2^{-1}\big)^{-1} = 2 \cdot 3^{-1},$$

and thus $2 \bullet (3 \bullet 1) \neq (2 \bullet 3) \bullet 1$.

Example 26.2. Suppose that $X(*)$ is a groupoid. For all $A, B, C \subseteq X$, define

$$A * C = \{ a * c : \quad a \in A , \quad c \in C \} \qquad \text{and} \qquad C \bullet B = \big\{ x \in X : \quad x * C \subseteq B \big\} .$$

Then, $\mathcal{P}(X)(*, \bullet, \subseteq)$ is an increasingly normal, partially ordered bigroupoid.

To prove this, for all $A, C \subseteq X$, define

$$F_C(A) = A * C \qquad \text{and} \qquad G_C(B) = C \bullet B$$

Then, for any $A, C \subseteq X$ we have

$$F_C(A) = A * C = \bigcup_{x \in A} x * C = \bigcup_{x \in A} F_C(x) .$$

Thus, by Theorem 19.4 and Corollary 18.3, the function F_C is increasingly G_{F_C}–normal.

Moreover, by Definition 18.2, for any $B, C \subseteq X$, we have

$$G_{F_C}(B) = \big\{ x \in X : \quad F_C(x) \subseteq B \big\} = \big\{ x \in X : \quad x * C \subseteq B \big\} = G_C(B) .$$

Therefore, $G_{F_C} = G_C$, and thus F_C is also increasingly G_C–normal. That is, for any $A, B, C \subseteq X$, we have

$$A * C \subseteq B \iff F_C(A) \subseteq B \iff A \subseteq G_C(B) \iff A \subseteq C \bullet B .$$

Example 26.3. Suppose that X is a set. For all $A, B, C \subseteq X$, define

$$A * C = A \cap C \qquad \text{and} \qquad C \bullet B = B \cup C^c .$$

Then, $\mathcal{P}(X)(*, \bullet, \subseteq)$ is an increasingly normal, partially ordered bigroupoid.

To prove this, for all $A, C \subseteq X$, define

$$F_C(A) = A * C \qquad \text{and} \qquad G_C(B) = C \bullet B$$

Then, for any $A, C \subseteq X$, we have

$$F_C(A) = A \cap C = \big(\bigcup_{x \in A} \{x\} \big) \cap C = \bigcup_{x \in A} \big(\{x\} \cap C \big) = \bigcup_{x \in A} F_C(x) .$$

Thus, by Theorem 19.4 and Corollary 18.3, the function F_C is increasingly G_{F_C}–normal.

Moreover, by Definition 18.2, for any $B, C \subseteq X$, we have

$$G_{F_C}(B) = \big\{ x \in X : \quad F_C(x) \subseteq B \big\} = \\ \big\{ x \in X : \quad \{x\} \cap C \subseteq B \big\} = (C \cap B) \cup C^c = B \cup C^c = G_C(B) .$$

Namely, we have $B = X \cap B = (C \cup C^c) \cap B = (C \cap B) \cup (C^c \cap B)$, and thus also $B \cup C^c = (C \cap B) \cup \big((C^c \cap B) \cup C^c \big) = (C \cap B) \cup C^c$.

Therefore, $G_{F_C} = G_C$, and thus F_C is also increasingly G_C–normal. That is, for any $A, B, C \subseteq X$, we have

$$A * C \subseteq B \iff F_C(A) \subseteq B \iff A \subseteq G_C(B) \iff A \subseteq C \bullet B .$$

EXAMPLE 26.4. Suppose that X is a set. For all $R, S, T \subseteq X^2$, define

$$R * T = T \circ R \qquad \text{and} \qquad T \bullet B = \big(S^c * T^{-1} \big)^c .$$

Then, $\mathcal{P}(X^2)(*, \bullet, \subseteq)$ is an increasingly normal, partially ordered bigroupoid.

To prove this, by using Theorem 10.1 and Corollary 10.1, we can note that, for any $x \in X$,

$$\begin{aligned}(R*T)(x) \subseteq S(x) \iff (T \circ R)(x) \subseteq S(x) \iff T[R(x)] \subseteq S(x) \iff \\ \mathrm{cl}_{T^{-1}}\big(R(x)\big) \subseteq S(x) \iff R(x) \subseteq \mathrm{int}_T\big(S(x)\big) \iff \\ R(x) \subseteq \mathrm{cl}_T\big(S(x)^c\big)^c \iff R(x) \subseteq T^{-1}[S(x)^c]^c \iff \\ R(x) \subseteq \big(T^{-1} \circ S^c\big)^c(x) \iff R(x) \subseteq \big(S * T^{-1}\big)^c(x) .\end{aligned}$$

Therefore,

$$R * T \subseteq S \iff R \subseteq \big(S * T^{-1}\big)^c \iff R \subseteq T \bullet S .$$

Thus, the required assertion is also true.

REMARK 26.2. By taking $T \subseteq X^2$ and defining

$$F_T(R) = R * T \qquad \text{and} \qquad G_T = T \bullet S$$

for all $R, S \subseteq X^2$, we can note that

$$G_{F_T}(S) = \big\{ (x, y) \in X^2 : \quad F_T(x, y) \subseteq S \big\} = \big\{ (x, y) \in X^2 : \quad T \circ \{(x, y)\} \subseteq S \big\}$$

for all $S \subseteq X^2$, and moreover F_T is increasingly G_T–normal.

Therefore, the equality

$$\big(S^c * T^{-1}\big)^c = T \bullet S = G_T(S) = G_{F_T}(S) = \big\{ (x, y) \in X^2 : \quad T \circ \{(x, y)\} \subseteq S \big\}$$

has to be true.

To prove it directly, by our former computations, it is enough to show only that

$$y \in \mathrm{int}_T\big(S(x)\big) \iff T(y) \subseteq S(x) \iff T \circ \{(x, y)\} \subseteq S$$

for all $x, y \in X$.

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