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SHEFFER STROKE BM-ALGEBRAS AND RELATED ALGEBRAS

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Abstract. The aim of this work is to define a Sheffer stroke BM-algebra and to study some of its features. It is indicated that the axioms of a Sheffer stroke BM-algebra are independent. The relationship between a Sheffer stroke BM-algebra and a BM-algebra is stated. By presenting fundamental notions about Sheffer stroke B-algebras, it is proved that every Sheffer stroke BM-algebra is a Sheffer stroke B-algebra. After determining 0-commutative Sheffer stroke B-algebra, a Sheffer stroke Coxeter algebra and an associative Sheffer stroke BM-algebra, the relationships between this algebraic structures are shown.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [**5, 6**]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Hu and Li [**3, 4**] introduced a new class of algebras so-called BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [**11**] introduced the notion of a B-algebra. C. B. Kim and H. S. Kim defined BG-algebras [**7**] and BM-algebras [**8**]. H. S. Kim, Y. H. Kim and J. Neggers [**9**] introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group.

The Sheffer stroke operation was originally introduced by H. M. Sheffer [**19**]. Since any Boolean function or operation can be stated by only this operation [**10**], it engages many researchers' attention. So, many researchers want to use this

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operation on their studies. Also, some applications of this operation has been appeared in algebraic structures such as Sheffer stroke non-associative MV-algebras [**1**] and filters [**14**], (fuzzy) filters of Sheffer stroke BL-algebras [**15**], Sheffer stroke Hilbert algebras [**12**] and filters [**13**], Sheffer stroke UP-algebras [**16**], Sheffer stroke BG-algebras [**17**], Sheffer stroke BCK-algebras [**18**] and Sheffer operation in ortholattices [**2**].

After giving basic definitions and notions about a Sheffer stroke and a BM-algebra, it is defined a Sheffer stroke BM-algebra. It is stated that the axioms of a Sheffer stroke BM-algebra are independent. By presenting fundamental notions about this algebraic structure, it is denoted the connection between a Sheffer stroke BM-algebra and a BM-algebra. It is proved that every Sheffer stroke BM-algebra is a Sheffer stroke B-algebra. A 0-commutative Sheffer stroke B-algebra is determined and it is shown that 0-commutative Sheffer stroke B-algebra is a Sheffer stroke BM-algebra. It is demonstrated that a Sheffer stroke B-algebra is a Sheffer stroke BM-algebra under one condition. It is proved that an algebra $(A, |, 0)$ is a 0-commutative Sheffer stroke B-algebra if and only if it is a Sheffer stroke BM-algebra. Finally, a Sheffer stroke Coxeter algebra and an associative Sheffer stroke BM-algebra are defined and the relationship between this algebraic structures are shown.

2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke, a BM-algebra and related algebras.

Definition 2.1. [**1**] Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be *Sheffer stroke* if it satisfies the following conditions:

$(S1)$ $a_1 \mid a_2 = a_2 \mid a_1$,
$(S2)$ $(a_1 \mid a_1) \mid (a_1 \mid a_2) = a_1$,
$(S3)$ $a_1 \mid ((a_2 \mid a_3) \mid (a_2 \mid a_3)) = ((a_1 \mid a_2) \mid (a_1 \mid a_2)) \mid a_3$,
$(S4)$ $(a_1 \mid ((a_1 \mid a_1) \mid (a_2 \mid a_2))) \mid (a_1 \mid ((a_1 \mid a_1) \mid (a_2 \mid a_2))) = a_1$.

Definition 2.2. [**8**] A BM-algebra is a non-empty set A with a constant 0 and a binary operation "$*$" satisfying the following axioms:
$(BM.1)$ $a_1 * 0 = a_1$,
$(BM.2)$ $(a_3 * a_1) * (a_3 * a_2) = a_2 * a_1$,
for all $a_1, a_2, a_3 \in A$.

A BM-algebra is called bounded if it has the greatest element.

Definition 2.3. [**11**] A B-algebra is a non-empty set A with a constant 0 and a binary operation $*$ satisfying the following axioms:
(i) $a_1 * a_1 = 0$,
(ii) $a_1 * 0 = a_1$,
(iii) $(a_1 * a_2) * a_3 = a_1 * (a_3 * (0 * a_2))$,
for all $a_1, a_2, a_3 \in A$.

Definition 2.4. [**9**] A Coxeter algebra is a non-empty set with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

$(C.1)$ $a_1 * a_1 = 0$,
$(C.2)$ $a_1 * 0 = a_1$,
$(C.3)$ $(a_1 * a_2) * a_3 = a_1 * (a_2 * a_3)$,
for all $a_1, a_2, a_3 \in A$.

Definition 2.5. [**9**] An algebra $(A, *, 0)$ is called a pre-Coxeter algebra if it satisfies the axioms:
(i) $a_1 * a_1 = 0$,
(ii) $a_1 * 0 = a_1$,
(iii) if $a_1 * a_2 = 0 = a_2 * a_1$, then $a_1 = a_2$,
(iv) $a_1 * a_2 = a_2 * a_1$,
for all $a_1, a_2 \in A$.

Definition 2.6. [**17**] A Sheffer stroke B-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$, where A is a non-empty set, 0 is the constant in A and $|$ is Sheffer stroke on A, such that the following identities are satisfied for all $a_1, a_2, a_3 \in A$:
$(sB.1)$ $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1)) = 0$,
$(sB.2)$ $((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3) = (a_1 \mid (a_3 \mid (0 \mid (a_2 \mid a_2))))$.

Example 2.1. [**17**] Consider $(A, |, 0)$ with the following Hasse diagram, where $A = \{0, x, y, 1\}$:

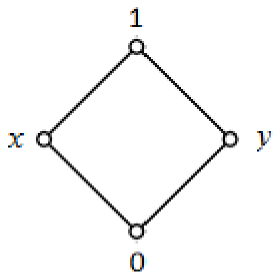

Figure 1

The binary operation $|$ on A has Cayley table as follow:

$\mid$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Then this structure is a Sheffer stroke B-algebra.

Definition 2.7. [**17**] A Sheffer stroke BG-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A and the following axioms are satisfied:
$(sBG.1)$ $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1)) = 0$,

$(sBG.2)$ $(0 \mid (a_2 \mid a_2)) \mid ((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) = a_1 \mid a_1$,
for all $a_1, a_2 \in A$.

3. Sheffer stroke BM-algebras

In this part, we define a Sheffer stroke BM-algebra and give some properties.

DEFINITION 3.1. A Sheffer stroke BM-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A and the following axioms are satisfied:
$(sBM.1)$ $(a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) = a_1$,
$(sBM.2)$ $((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_3 \mid (a_2 \mid a_2)) = a_2 \mid (a_1 \mid a_1)$,
for all $a_1, a_2, a_3 \in A$.

Let A be a Sheffer stroke BM-algebra, unless otherwise is indicated.

LEMMA 3.1. *The axioms (sBM.1) and (sBM.2) are independent.*

PROOF. (1) Independence of (sBM.1):
We construct an example for this axiom which is false while (sBM.2) is true. Let $(\{0, 1\}, |_1)$ be the groupoid defined as follows:

$\vert_1$	0	1
0	0	1
1	0	0

Then $|_1$ satisfies (sBM.2) but not (sBM.1) when $a_1 = 1$.
(2) Independence of (sBM.2):
We construct an example for this axiom which is false while (sBM.1) is true. Let $(\{0, 1\}, |_2)$ be the groupoid defined as follows:

$\vert_2$	0	1
0	0	0
1	1	1

Then $|_2$ satisfies (sBM.1) but not (sBM.2) when $a_1 = 1$, $a_2 = 0$ and $a_3 = 1$. □

LEMMA 3.2. *Let A be a Sheffer stroke BM-algebra. Then the following features hold for all $a_1, a_2, a_3 \in A$:*

(1) $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1)) = 0$,
(2) $(0 \mid (0 \mid (a_1 \mid a_1))) = a_1 \mid a_1$,
(3) $0 \mid (a_1 \mid (a_2 \mid a_2)) = a_2 \mid (a_1 \mid a_1)$,
(4) $((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid (a_3 \mid a_3)) = a_1 \mid (a_2 \mid a_2)$,
(5) $a_1 \mid (a_2 \mid a_2) = 0 \mid 0$ *if and only if* $a_2 \mid (a_1 \mid a_1) = 0 \mid 0$,
(6) $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid a_1) = a_1$,
(7) $a_1 \mid ((a_1 \mid (a_2 \mid a_2)) \mid (a_2 \mid a_2)) \mid ((a_1 \mid (a_2 \mid a_2)) \mid (a_2 \mid a_2))) = 0 \mid 0$,
(8) $(0 \mid 0) \mid (a_1 \mid a_1) = a_1$,
(9) $a_1 \mid ((a_2 \mid (a_3 \mid a_3)) \mid (a_2 \mid (a_3 \mid a_3))) = a_2 \mid ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3)))$,

(10) $((a_1 \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_1 \mid (a_1 \mid (a_2 \mid a_2)))) \mid (a_2 \mid a_2) = 0 \mid 0$.

PROOF. (1) Substituting $[a_1 := 0]$ and $[a_2 := 0]$ in (sBM.2), we obtain $((a_3 \mid (0 \mid 0)) \mid (a_3 \mid (0 \mid 0))) \mid (a_3 \mid (0 \mid 0)) = 0 \mid (0 \mid 0)$. From $(sBM.1)$ and (S2), we get $a_3 \mid (a_3 \mid a_3) = 0 \mid 0$. Then we have $(a_3 \mid (a_3 \mid a_3)) \mid (a_3 \mid (a_3 \mid a_3)) = 0$, for all $a_3 \in A$.

(2) Substituting $[a_3 := 0]$ and $[a_1 := 0]$ in (sBM.2), we obtain $((0 \mid (0 \mid 0)) \mid (0 \mid (0 \mid 0))) \mid (0 \mid (a_2 \mid a_2)) = a_2 \mid (0 \mid 0)$. Applying $(sBM.1)$ and (S2), we have $0 \mid (0 \mid (a_2 \mid a_2)) = a_2 \mid a_2$, for all $a_2 \in A$.

(3) Using $(sBM.2)$ with $[a_3 := a_1]$, we get $((a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1))) \mid (a_1 \mid (a_2 \mid a_2)) = a_2 \mid (a_1 \mid a_1)$. By using (1), we have $0 \mid (a_1 \mid (a_2 \mid a_2)) = a_2 \mid (a_1 \mid a_1)$.

(4) By using $(sBM.2)$ and (3), we obtain

$$\begin{aligned} &((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid (a_3 \mid a_3)) \\ &= ((0 \mid (a_3 \mid (a_1 \mid a_1))) \mid (0 \mid (a_3 \mid (a_1 \mid a_1)))) \mid (0 \mid (a_3 \mid (a_2 \mid a_2))) \\ &= ((a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_1 \mid a_1)) \\ &= a_1 \mid (a_2 \mid a_2). \end{aligned}$$

(5) It is obtained from (3) and $(sBM.1)$.

(6) Substituting $[a_2 := (a_1 \mid a_1)]$ in (S2), we obtain

$$(a_1 \mid a_1) \mid (a_1 \mid (a_1 \mid a_1)) \quad = \quad a_1.$$

By using (S1), we get $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid a_1) = a_1$.

(7) In (S3), by substituting $[a_2 := a_1 \mid (a_2 \mid a_2)]$ and $[a_3 := a_2 \mid a_2]$ and applying (S1), (S2), (S3) and (1), we obtain

$$\begin{aligned} &a_1 \mid ((a_1 \mid (a_2 \mid a_2)) \mid (a_2 \mid a_2)) \mid ((a_1 \mid (a_2 \mid a_2)) \mid (a_2 \mid a_2)) \\ &= a_1 \mid (((a_2 \mid a_2) \mid (a_1 \mid (a_2 \mid a_2))) \mid ((a_2 \mid a_2) \mid (a_1 \mid (a_2 \mid a_2)))) \\ &= ((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_1 \mid (a_2 \mid a_2)) \\ &= (a_1 \mid (a_2 \mid a_2)) \mid ((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \\ &= 0 \mid 0. \end{aligned}$$

(8) $(0 \mid 0) \mid (a_1 \mid a_1) = (a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid a_1) = a_1$ from (1) and (6).

(9) By using (S1) and (S3), we have

$$\begin{aligned} a_1 \mid ((a_2 \mid (a_3 \mid a_3)) \mid (a_2 \mid (a_3 \mid a_3))) &= (((a_1 \mid a_2) \mid (a_1 \mid a_2)) \mid (a_3 \mid a_3)) \\ &= (((a_2 \mid a_1) \mid (a_2 \mid a_1)) \mid (a_3 \mid a_3)) \\ &= a_2 \mid ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))). \end{aligned}$$

(10) It is obtained from (7) and (S3).

□

THEOREM 3.1. *Let $(A, |, 0)$ be a Sheffer stroke BM-algebra. If we define*

$$a_1 * a_2 := (a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2)),$$

then $(A, *, 0)$ is a BM-algebra.

PROOF. By using (sBM.1), (sBM.2) and (S2) , we have
$(BM.1): a_1 * 0 = (a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) = a_1$.

$(BM.2)$:

$$\begin{aligned}(a_3 * a_1) * (a_3 * a_2) &= (((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid ((a_3 \mid (a_2 \mid a_2)) \mid \\ &\quad (a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2)))) \mid \\ &\quad (((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid ((a_3 \mid (a_2 \mid a_2)) \\ &\quad \mid (a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2)))) \\ &= (((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_3 \mid (a_2 \mid a_2))) \\ &\quad \mid (((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_3 \mid (a_2 \mid a_2))) \\ &= (a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1)) \\ &= a_2 * a_1.\end{aligned}$$

Then $(A, *, 0)$ is a BM-algebra. □

THEOREM 3.2. *Let $(A, *, 0, 1)$ be a bounded BM-algebra. If we define $a_1 \mid a_2 := (a_1 * a_2^0)^0$ and $a_1^0 = 1 * a_1$, where $a_1 * (1 * a_1) = a_1$ and $1 * (1 * a_1) = a_1$, then $(A, \mid, 0)$ is a Sheffer stroke BM-algebra.*

PROOF. $(sBM.1)$: By using (BM.1), we have

$$\begin{aligned}(a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) &= (a_1 \mid 0^0) \mid (a_1 \mid 0^0) \\ &= (a_1 * 0)^0 \mid (a_1 * 0)^0 \\ &= ((a_1 * 0)^0)^0 \\ &= a_1 * 0 \\ &= a_1.\end{aligned}$$

$(sBM.2)$: By using (BM.2), we obtain

$$\begin{aligned}((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_3 \mid (a_2 \mid a_2)) &= (((a_3 * a_1)^0) \mid ((a_3 * a_1)^0)) \mid \\ &\quad (a_3 * a_2)^0 \\ &= (((a_3 * a_1)^0)^0) \mid (a_3 * a_2)^0 \\ &= (a_3 * a_1) \mid (a_3 * a_2)^0 \\ &= ((a_3 * a_1) * (a_3 * a_2))^0 \\ &= (a_2 * a_1)^0 \\ &= a_2 \mid (a_1 \mid a_1).\end{aligned}$$

Then $(A, \mid, 0)$ is a Sheffer stroke BM-algebra. □

LEMMA 3.3. *Let $(A, \mid, 0)$ be a Sheffer stroke B-algebra. Then the following features hold for all $a_1, a_2, a_3 \in A$:*

(1) $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid a_1) = a_1$,
(2) $(0 \mid 0) \mid (a_1 \mid a_1) = a_1$,
(3) $(a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) = a_1$,
(4) $((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (0 \mid (a_2 \mid a_2)) = a_1 \mid a_1$,
(5) $a_1 \mid (a_3 \mid a_3) = a_2 \mid (a_3 \mid a_3)$ *implies* $a_1 = a_2$,
(6) $a_1 \mid (a_2 \mid a_2) = 0 \mid 0$ *implies* $a_1 = a_2$,
(7) $0 \mid (a_1 \mid a_1) = 0 \mid (a_2 \mid a_2)$ *implies* $a_1 = a_2$,

(8) $0 \mid (0 \mid (a_1 \mid a_1)) = a_1 \mid a_1$.

PROOF. (1) Substituting $[a_2 := (a_1 \mid a_1)]$ in (S2), we obtain $(a_1 \mid a_1) \mid (a_1 \mid (a_1 \mid a_1)) = a_1$. Then $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid a_1) = a_1$ from (S1).

(2) $(0 \mid 0) \mid (a_1 \mid a_1) = (a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid a_1) = a_1$ from (1), (S2) and (sB.1).

(3) By using (S1), (S2) and (2),

$$\begin{aligned}(a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) &= ((0 \mid 0) \mid ((a_1 \mid a_1) \mid (a_1 \mid a_1))) \mid ((0 \mid 0) \mid \\ &\quad ((a_1 \mid a_1) \mid (a_1 \mid a_1))) \\ &= (a_1 \mid a_1) \mid (a_1 \mid a_1) \\ &= a_1.\end{aligned}$$

(4) Substituting $a_3 = (0 \mid (a_2 \mid a_2)) \mid (0 \mid (a_2 \mid a_2))$ in $(sB.2)$ and by using (3), (sB.1) and (S2), we obtain
$((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (0 \mid (a_2 \mid a_2))$
$= a_1 \mid (((0 \mid (a_2 \mid a_2)) \mid (0 \mid (a_2 \mid a_2))) \mid (0 \mid (a_2 \mid a_2)))$
$= a_1 \mid ((0 \mid (a_2 \mid a_2)) \mid ((0 \mid (a_2 \mid a_2)) \mid (0 \mid (a_2 \mid a_2))))$
$= a_1 \mid (0 \mid 0)$
$= a_1 \mid a_1$.

(5) If $a_1 \mid (a_3 \mid a_3) = a_2 \mid (a_3 \mid a_3)$, then
$((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (0 \mid (a_3 \mid a_3)) = ((a_2 \mid (a_3 \mid a_3)) \mid (a_2 \mid (a_3 \mid a_3))) \mid (0 \mid (a_3 \mid a_3))$. From (4), we get $a_1 \mid a_1 = a_2 \mid a_2$. By (S2), $a_1 = a_2$.

(6) Since $a_1 \mid (a_2 \mid a_2) = 0 \mid 0$ and by using (5), (sB.1) and (S2), we have

$$a_1 \mid (a_2 \mid a_2) \quad = \quad a_2 \mid (a_2 \mid a_2)$$

Then, we get $a_1 = a_2$.

(7) If $0 \mid (a_1 \mid a_1) = 0 \mid (a_2 \mid a_2)$, then

$$\begin{aligned}0 \mid 0 &= (a_1 \mid (a_1 \mid a_1)) \\ &= ((a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1))) \mid (0 \mid 0) \\ &= a_1 \mid (0 \mid (0 \mid (a_1 \mid a_1))) \\ &= a_1 \mid (0 \mid (0 \mid (a_2 \mid a_2))) \\ &= ((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (0 \mid 0) \\ &= (a_1 \mid (a_2 \mid a_2)),\end{aligned}$$

from (sB.1), (sB.2), (S2) and (3). Therefore, $a_1 = a_2$ from (6).

(8) For any $a_1 \in A$,

$$\begin{aligned}0 \mid (a_1 \mid a_1) &= ((0 \mid (a_1 \mid a_1)) \mid (0 \mid (a_1 \mid a_1))) \mid (0 \mid 0) \\ &= 0 \mid (0 \mid (0 \mid (a_1 \mid a_1))),\end{aligned}$$

from (sB.2), (S2) and (3). Then $0 \mid (0 \mid (a_1 \mid a_1)) = a_1 \mid a_1$ from (7).

□

THEOREM 3.3. $(A, |, 0)$ *is a Sheffer stroke B-algebra if and only if it satisfies the axioms:*

(i) $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1)) = 0$,
(ii) $0 \mid (0 \mid (a_1 \mid a_1)) = a_1 \mid a_1$,
(iii) $((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid (a_3 \mid a_3)) = a_1 \mid (a_2 \mid a_2)$,
for any $a_1, a_2, a_3 \in A$.

Proof. ($\Rightarrow$) Suppose that A is a Sheffer stroke B-algebra. Then
- *(i)* is obtained from $(sB.1)$.
- *(ii)* is obtained from Lemma 3.3 (8).
- *(iii)* By using $(sB.1)$, $(sB.2)$, (S2) and Lemma 3.3 (3) we obtain

$$\begin{aligned}((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid (a_3 \mid a_3)) &= (a_1 \mid (((a_2 \mid (a_3 \mid a_3)) \mid (a_2 \mid (a_3 \mid a_3))) \mid (0 \mid (a_3 \mid a_3)))) \\ &= a_1 \mid (a_2 \mid (((0 \mid (a_3 \mid a_3)) \mid (0 \mid (a_3 \mid a_3))) \mid (0 \mid (a_3 \mid a_3)))) \\ &= a_1 \mid (a_2 \mid (0 \mid 0)) \\ &= a_1 \mid (a_2 \mid a_2).\end{aligned}$$

$\Leftarrow$ $(sB.1)$: It is ontained from (i).
In *(iii)*, substituting $[a_3 := a_2]$ and $[a_2 := 0]$, we have

$$\begin{aligned}((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (0 \mid (a_2 \mid a_2)) &= a_1 \mid (0 \mid 0) \\ &= a_1 \mid a_1.\end{aligned}$$

$(sB.2)$ By using *(iii)*, we get

$$\begin{aligned}a_1 \mid (a_3 \mid (0 \mid (a_2 \mid a_2))) &= ((((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (0 \mid (a_2 \mid a_2))) \mid \\ &\quad (((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (0 \mid (a_2 \mid a_2)))) \\ &\quad \mid (a_3 \mid (0 \mid (a_2 \mid a_2))) \\ &= ((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3).\end{aligned}$$

□

Theorem 3.4. *Every Sheffer stroke BM-algebra is a Sheffer stroke B-algebra.*

Proof. It is obtained from Lemma 3.2 (1), (2), (4) and Theorem 3.3. □

Remark 3.1. The converse of Theorem 3.4 does not hold in general. In Example 2.1, $(A, |, 0)$ is a Sheffer stroke B-algebra but not a Sheffer stroke BM-algebra. Since $((x \mid (0 \mid 0)) \mid (x \mid (0 \mid 0))) \mid (x \mid (y \mid y)) = 1 \neq x = y \mid (0 \mid 0)$.

Proposition 3.1. *If* $(A, |, 0)$ *is a Sheffer stroke BM-algebra, then*

$$((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3) = ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid a_2),$$

for any $a_1, a_2, a_3 \in A$.

PROOF. By Theorem 3.4, (sBM.2), (sB.2), Lemma 3.2 (3), we get

$((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3)$
$= ((((a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_1 \mid a_1))) \mid (((a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_1 \mid a_1)))) \mid (a_3 \mid a_3)$
$= ((a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2))) \mid (a_3 \mid (0 \mid (a_3 \mid (a_1 \mid a_1))))$
$= ((0 \mid (a_3 \mid (a_1 \mid a_1))) \mid (0 \mid (a_3 \mid (a_1 \mid a_1)))) \mid (a_2 \mid a_2)$
$= ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid a_2).$

□

LEMMA 3.4. *If $(A, |, 0)$ is a Sheffer stroke B-algebra, then* $0 \mid (a_1 \mid (a_2 \mid a_2)) = a_2 \mid (a_1 \mid a_1)$, *for any* $a_1, a_2 \in A$.

DEFINITION 3.2. A Sheffer stroke B-algebra $(A, |, 0)$ is said to be 0-commutative if $a_1 \mid (0 \mid (a_2 \mid a_2)) = a_2 \mid (0 \mid (a_1 \mid a_1))$, for any $a_1, a_2 \in A$.

THEOREM 3.5. *If $(A, |, 0)$ is a 0-commutative Sheffer stroke B-algebra, then A is a Sheffer stroke BM-algebra.*

PROOF. $(sBM.1)$: Since $(A, |, 0)$ is a Sheffer stroke B-algebra, $(a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) = a_1$ from Lemma 3.3 (3), i.e., $(sBM.1)$ holds.
$(sBM.2)$: By using Lemma 3.4, Definition 3.2, (S2), Theorem 3.3 (ii) and (iii), we obtain

$((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_3 \mid (a_2 \mid a_2))$
$= ((0 \mid (a_1 \mid (a_3 \mid a_3))) \mid (0 \mid (a_1 \mid (a_3 \mid a_3)))) \mid (0 \mid (a_2 \mid (a_3 \mid a_3)))$
$= ((a_2 \mid (a_3 \mid a_3)) \mid (a_2 \mid (a_3 \mid a_3))) \mid (0 \mid (0 \mid (a_1 \mid (a_3 \mid a_3))))$
$= ((a_2 \mid (a_3 \mid a_3)) \mid (a_2 \mid (a_3 \mid a_3))) \mid (a_1 \mid (a_3 \mid a_3))$
$= a_2 \mid (a_1 \mid a_1).$

Thus $(A, |, 0)$ is a Sheffer stroke BM-algebra. □

REMARK 3.2. Let $(A, |, 0)$ be a Sheffer stroke B-algebra with $a_1 \mid (a_2 \mid a_2) = a_2 \mid (a_1 \mid a_1)$, for any $a_1, a_2 \in A$. Then A is a Sheffer stroke BM-algebra.

PROOF. Since $a_1 \mid (a_2 \mid a_2) = a_2 \mid (a_1 \mid a_1)$ for any $a_1, a_2 \in A$, we obtain

$$\begin{aligned} a_1 \mid (0 \mid (a_2 \mid a_2)) &= a_1 \mid (a_2 \mid (0 \mid 0)) \\ &= a_1 \mid (a_2 \mid a_2) \\ &= a_2 \mid (a_1 \mid a_1) \\ &= a_2 \mid (a_1 \mid (0 \mid 0)) \\ &= a_2 \mid (0 \mid (a_1 \mid a_1)), \end{aligned}$$

for any $a_1, a_2 \in A$. Thus $(A, |, 0)$ is a 0-commutative Sheffer stroke B-algebra. Hence $(A, |, 0)$ is a Sheffer stroke BM-algebra by Theorem 3.5. □

PROPOSITION 3.2. *An algebra $(A, |, 0)$ is a 0-commutative Sheffer stroke B-algebra if and only if it satisfies the following axioms:*
(i) $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1)) = 0$,
(ii) $a_2 \mid (a_2 \mid (a_1 \mid a_1)) = a_1 \mid a_1$,
(iii) $((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid (a_3 \mid a_3)) = a_1 \mid (a_2 \mid a_2)$,
for any $a_1, a_2, a_3 \in A$.

THEOREM 3.6. *Let $(A, |, 0)$ be a Sheffer stroke BM-algebra. Then A is a 0-commutative Sheffer stroke B-algebra.*

PROOF. Let A be a Sheffer stroke BM-algebra. Then, by Theorem 3.4, it is a Sheffer stroke B-algebra. From Theorem 3.3, we obtain that it satisfies Proposition 3.2 (i) and (iii). Substituting $[a_1 := 0]$ in $(sBM.2)$, we get $((a_3 \mid (0 \mid 0)) \mid (a_3 \mid (0 \mid 0))) \mid (a_3 \mid (a_2 \mid a_2) = a_2 \mid (0 \mid 0)$.
By using $(sBM.1)$, we have $a_3 \mid (a_3 \mid (a_2 \mid a_2)) = a_2 \mid a_2$, for any $a_2, a_3 \in A$. Then Proposition 3.2 (ii) holds in A. Therefore, A is a 0-commutative Sheffer stroke B-algebra. □

COROLLARY 3.1. *An algebra $(A, |, 0)$ is a 0-commutative Sheffer stroke B-algebra if and only if it is a Sheffer stroke BM-algebra.*

THEOREM 3.7. *Let $(A, |, 0)$ be a Sheffer stroke BM-algebra. Then the following features hold for $a_1, a_2, a_3, a_4 \in A$:*
(i) $a_1 \mid (a_1 \mid (a_2 \mid a_2)) = a_2 \mid a_2$,
(ii) $a_1 \mid (a_2 \mid a_2) = 0 \mid 0$ *implies* $a_1 = a_2$,
(iii) $a_1 \mid (a_2 \mid (a_3 \mid a_3)) = a_3 \mid (a_2 \mid (a_1 \mid a_1))$,
(iv) $((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_4 \mid a_4)) = ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid (a_4 \mid a_4))$,
(v) $((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (0 \mid (a_2 \mid a_2)) = a_1 \mid a_1$.

PROOF. *(i)* By using $(sBM.1)$ and $(sBM.2)$, we get

$$\begin{aligned} a_1 \mid (a_1 \mid (a_2 \mid a_2)) &= ((a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0))) \mid (a_1 \mid (a_2 \mid a_2)) \\ &= a_2 \mid (0 \mid 0) \\ &= a_2 \mid a_2. \end{aligned}$$

(ii) Let $a_1 \mid (a_2 \mid a_2) = 0 \mid 0$. By using (S2), (i) and $(sBM.1)$, we obtain

$$\begin{aligned} a_1 &= (a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) \\ &= (a_1 \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_1 \mid (a_1 \mid (a_2 \mid a_2))) \\ &= (a_2 \mid a_2) \mid (a_2 \mid a_2) \\ &= a_2. \end{aligned}$$

(iii) By using (i) and Proposition 3.1, we obtain

$$\begin{aligned} a_1 \mid (a_2 \mid (a_3 \mid a_3)) &= ((a_2 \mid (a_2 \mid (a_1 \mid a_1))) \mid (a_2 \mid (a_2 \mid (a_1 \mid a_1)))) \mid \\ &\quad (a_2 \mid (a_3 \mid a_3)) \\ &= ((a_2 \mid (a_2 \mid (a_3 \mid a_3))) \mid (a_2 \mid (a_2 \mid (a_3 \mid a_3)))) \mid \\ &\quad (a_2 \mid (a_1 \mid a_1)) \\ &= ((a_3 \mid a_3) \mid (a_3 \mid a_3)) \mid (a_2 \mid (a_1 \mid a_1)) \\ &= a_3 \mid (a_2 \mid (a_1 \mid a_1)). \end{aligned}$$

(iv) From (iii), we have

$$\begin{aligned}((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_4 \mid a_4)) &= a_4 \mid (a_3 \mid (a_1 \mid (a_2 \mid a_2))) \\ &= a_4 \mid (a_2 \mid (a_1 \mid (a_3 \mid a_3))) \\ &= ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid \\ &\quad (a_3 \mid a_3))) \mid (a_2 \mid (a_4 \mid a_4)).\end{aligned}$$

(v) It is obtained from (iv), Lemma 3.2 (1) and (sBM.1). □

Definition 3.3. A Sheffer stroke BM-algebra $(A, |, 0)$ is said to be associative if it satisfies

$$a_1 \mid (a_2 \mid (a_3 \mid a_3)) = ((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3),$$

for all $a_1, a_2, a_3 \in A$.

Definition 3.4. A Sheffer stroke Coxeter algebra is a non-empty set with a constant 0 satisfying the following axioms:
$(sC.1)$ $((a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1))) = 0$,
$(sC.2)$ $((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3) = a_1 \mid (a_2 \mid (a_3 \mid a_3))$,
for all $a_1, a_2, a_3 \in A$.

Proposition 3.3. *Let $(A, |, 0)$ is a Sheffer stroke Coxeter algebra. Then*
(i) $((a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0))) = a_1$,
(ii) $0 \mid (a_1 \mid a_1) = a_1 \mid a_1$,
(iii) $((a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1))) \mid (a_2 \mid a_2) = a_1 \mid a_1$,
(iv) $a_1 \mid (a_2 \mid a_2) = a_2 \mid (a_1 \mid a_1)$,
for any $a_1, a_2 \in A$.

Proof. Substituting $[a_2 := (a_1 \mid a_1)]$ in (S2), we obtain $(a_1 \mid a_1) \mid (a_1 \mid (a_1 \mid a_1)) = a_1$. Then $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid a_1) = a_1$ from (S1). $(0 \mid 0) \mid (a_1 \mid a_1) = (a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid a_1) = a_1$ from (S2) and (sC.1).
(i) By using (S1), (S2),

$$\begin{aligned}(a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) &= ((0 \mid 0) \mid ((a_1 \mid a_1) \mid (a_1 \mid a_1))) \mid ((0 \mid 0) \mid ((a_1 \mid a_1) \mid \\ &\quad (a_1 \mid a_1))) \\ &= (a_1 \mid a_1) \mid (a_1 \mid a_1) \\ &= a_1.\end{aligned}$$

(ii) By using $(sC.1)$, $(sC.2)$ and (i), we have

$$\begin{aligned}0 \mid (a_1 \mid a_1) &= ((a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1))) \mid (a_1 \mid a_1) \\ &= a_1 \mid (a_1 \mid (a_1 \mid a_1)) \\ &= a_1 \mid (0 \mid 0) \\ &= a_1 \mid a_1.\end{aligned}$$

(iii) By using (i), (ii), (sC.1), (sC.2), (S1), we obtain

$$\begin{aligned}
a_1 \mid a_1 &= 0 \mid (a_1 \mid a_1) \\
&= (((a_2 \mid (a_1 \mid a_1)) \mid ((a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1)))) \mid ((a_2 \mid (a_1 \mid a_1)) \mid \\
&\quad ((a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1))))) \mid (a_1 \mid a_1) \\
&= ((a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1))) \mid (((a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1))) \mid \\
&\quad (a_1 \mid a_1)) \\
&= ((a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1))) \mid (a_2 \mid (a_1 \mid (a_1 \mid a_1))) \\
&= ((a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1))) \mid (a_2 \mid (0 \mid 0)) \\
&= ((a_2 \mid (a_1 \mid a_1)) \mid (a_2 \mid (a_1 \mid a_1))) \mid (a_2 \mid a_2).
\end{aligned}$$

(iv) By using (i), (iii), (sC.1), (sC.2), we get

$$\begin{aligned}
a_2 \mid (a_1 \mid a_1) &= ((((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_1 \mid a_1)) \mid \\
&\quad (((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_1 \mid a_1))) \mid (a_1 \mid a_1) \\
&= ((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_1 \mid (a_1 \mid a_1)) \\
&= ((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (0 \mid 0) \\
&= a_1 \mid (a_2 \mid a_2).
\end{aligned}$$

□

THEOREM 3.8. *Every Sheffer stroke Coxeter-algebra is a Sheffer stroke BM-algebra.*

PROOF. It is enough to show that the axiom $(sBM.2)$ holds in A. By using $(sC.2)$, (S2), Proposition 3.3 (iii) and (iv), we obtain

$$\begin{aligned}
&(((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_3 \mid (a_2 \mid a_2)) \\
&= (((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_2 \mid (a_3 \mid a_3)) \\
&= ((((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_2 \mid a_2)) \mid (((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid \\
&a_1))) \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3) \\
&= (((a_3 \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_1 \mid (a_2 \mid a_2)))) \mid (a_3 \mid a_3)) \\
&= a_1 \mid (a_2 \mid a_2) \\
&= a_2 \mid (a_1 \mid a_1).
\end{aligned}$$

□

PROPOSITION 3.4. *Every associative Sheffer stroke BM-algebra is a Sheffer stroke Coxeter algebra.*

PROOF. It is obtained from Lemma 3.2 (1) and Definition 3.3. □

THEOREM 3.9. *Let* $(A, |, 0)$ *be an algebra of type* $(2, 0)$ *satisfying*
(i) $(a_1 \mid (a_1 \mid a_1) \mid (a_1 \mid (a_1 \mid a_1) = 0$,
(ii) $((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3) = ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid a_2)$.
Then the following statements are equivalent:
(a) A *satisfies* $(sBG.2)$,

(b) $0 \mid (0 \mid (a_1 \mid a_1)) = a_1 \mid a_1$*, for all* $a_1 \in A$*,*
(c) $a_1 \mid (a_2 \mid a_2) = 0 \mid 0$ *implies* $a_1 = a_2$ *for all* $a_1, a_2 \in A$*,*
(d) A *is a Sheffer stroke BM-algebra.*

PROOF. $(a) \Rightarrow (b)$: By using $(sBG.2)$, $(0 \mid (a_1 \mid a_1)) \mid ((a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1))) = a_1 \mid a_1$. Hence $0 \mid (0 \mid (a_1 \mid a_1)) = a_1 \mid a_1$ from (i) and $(S1)$.
$(b) \Rightarrow (c)$: Let $a_1 \mid (a_2 \mid a_2) = 0 \mid 0$. By using $(b), (i)$ and (ii), we obtain

$$\begin{aligned} a_1 \mid a_1 &= 0 \mid (0 \mid (a_1 \mid a_1)) \\ &= 0 \mid (((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_1 \mid a_1)) \\ &= 0 \mid (((a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1))) \mid (a_2 \mid a_2) \\ &= 0 \mid (0 \mid (a_2 \mid a_2)) \\ &= a_2 \mid a_2. \end{aligned}$$

By $(S2)$, we get $a_1 = a_2$.
$(c) \Rightarrow (d)$: $(sBM.2)$: By using (i) and (ii),
$((a_3 \mid (a_3 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_3 \mid (a_2 \mid a_2)))) \mid (a_2 \mid a_2)$
$= ((a_3 \mid (a_2 \mid a_2)) \mid (a_3 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_2 \mid a_2))$
$= 0 \mid 0$.
By (c),

$$((a_3 \mid (a_3 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_3 \mid (a_2 \mid a_2)))) = a_2. \tag{3.1}$$

By using the equation (3.1) and (ii),
$((a_3 \mid (a_1 \mid a_1)) \mid (a_3 \mid (a_1 \mid a_1))) \mid (a_3 \mid (a_2 \mid a_2))$
$= ((a_3 \mid (a_3 \mid (a_2 \mid a_2))) \mid (a_3 \mid (a_3 \mid (a_2 \mid a_2)))) \mid (a_1 \mid a_1)$
$= a_2 \mid (a_1 \mid a_1)$.
$(sBM.1)$: By using the equation (3.1) and (i), we get

$$\begin{aligned} (a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0)) &= ((a_1 \mid (a_1 \mid (a_1 \mid a_1))) \mid (a_1 \mid (a_1 \mid (a_1 \mid a_1)))) \\ &= a_1. \end{aligned}$$

Therefore, A is a Sheffer stroke BM-algebra.
$(d) \Rightarrow (a)$: Let A be a Sheffer stroke BM-algebra. By Theorem 3.7 (v), A satisfies $(sBG.2)$. □

THEOREM 3.10. *Let* $(A, |, 0)$ *be an algebra of type* $(2, 0)$*. Then the following statements are equivalent:*
(i) A *is a Sheffer stroke BM-algebra.*
(ii) A *is a Sheffer stroke BG-algebra with condition*

$$((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3) = ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid (a_2 \mid a_2).$$

PROOF. $(i) \Rightarrow (ii)$: It is obtained from Lemma 3.2 (1), Proposition 3.1 and Theorem 3.7 (v).
$(ii) \Rightarrow (i)$: It is obtained from Theorem 3.9. □

THEOREM 3.11. *Every Sheffer stroke Coxeter algebra is a 0-commutative Sheffer stroke B-algebra.*

PROOF. It is obtained from Theorem 3.6 and Theorem 3.8. □

THEOREM 3.12. *Let* $(A, |, 0)$ *be a Sheffer stroke BM-algebra with* $0 \mid (a_1 \mid a_1) = a_1 \mid a_1$*, for all* $a_1 \in A$*. Then* A *is a Sheffer stroke Coxeter algebra.*

PROOF. It is enough to show $(sC.2)$. By using Proposition 3.1, Theorem 3.4, Definition 2.6 and (sB.2), we have

$$\begin{aligned}((a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))) \mid (a_3 \mid a_3) &= ((a_1 \mid (a_3 \mid a_3)) \mid (a_1 \mid (a_3 \mid a_3))) \mid \\ &\quad (a_2 \mid a_2) \\ &= a_1 \mid (a_2 \mid (0 \mid (a_3 \mid a_3))) \\ &= a_1 \mid (a_2 \mid (a_3 \mid a_3)).\end{aligned}$$

Therefore, A is a Sheffer stroke Coxeter algebra. □

COROLLARY 3.2. *An algebra* $(A, |, 0)$ *is a Sheffer stroke Coxeter algebra if and only if* A *is a Sheffer stroke BM-algebra with* $0 \mid (a_1 \mid a_1) = a_1 \mid a_1$*, for all* $a_1 \in A$.

DEFINITION 3.5. An algebra $(A, |, 0)$ is called a Sheffer stroke pre-Coxeter algebra if it satisfies the axioms:
(i) $(a_1 \mid (a_1 \mid a_1)) \mid (a_1 \mid (a_1 \mid a_1)) = 0$,
(ii) If $a_1 \mid (a_2 \mid a_2) = 0 \mid 0 = a_2 \mid (a_1 \mid a_1)$, then $a_1 = a_2$,
(iii) $a_1 \mid (a_2 \mid a_2) = a_2 \mid (a_1 \mid a_1)$,
for all $a_1, a_2 \in A$.

THEOREM 3.13. *Every Sheffer stroke BM-algebra* A *with* $0 \mid (a_1 \mid a_1) = a_1 \mid a_1$ *is a Sheffer stroke pre-Coxeter algebra.*

PROOF. We must show that (ii) and (iii) hold in A. Assume that $a_1 \mid (a_2 \mid a_2) = 0 \mid 0 = a_2 \mid (a_1 \mid a_1)$. By using $(sBM.1)$, $(sBM.2)$ and (S2),

$$\begin{aligned}a_1 \mid a_1 &= a_1 \mid (0 \mid 0) \\ &= ((a_1 \mid (0 \mid 0)) \mid (a_1 \mid (0 \mid 0))) \mid (a_1 \mid (a_2 \mid a_2)) \\ &= a_2 \mid (0 \mid 0) \\ &= a_2 \mid a_2.\end{aligned}$$

From $(S2)$, $a_1 = a_2$, for all $a_1, a_2 \in A$. It follows from Theorem 3.12 and Proposition 3.3 (iv), that $a_1 \mid (a_2 \mid a_2) = a_2 \mid (a_1 \mid a_1)$, for any $a_1, a_2 \in A$. □

4. Conclusion

In this study, a Sheffer stroke BM-algebra, a (0-commutative) Sheffer stroke B-algebra and a Sheffer stroke (pre)- Coxeter algebra are investigated. By presenting definitions of Sheffer stroke and BM-algebra, a Sheffer stroke BM-algebra is introduced and related concepts are given. Then it is shown that a Sheffer stroke BM-algebra is a BM-algebra if $a_1 * a_2 := (a_1 \mid (a_2 \mid a_2)) \mid (a_1 \mid (a_2 \mid a_2))$, and also that a bounded BM-algebra is a Sheffer stroke BM-algebra where $a_1 \mid a_2 := (a_1 * a_2^0)^0$, for any elements a_1 and a_2. A Sheffer stroke B-algebra is given. It is shown that every Sheffer stroke BM-algebra is a Sheffer stroke B-algebra. Then, 0-commutative Sheffer stroke B-algebra is identified and it is proved that an algebra $(A, |, 0)$ is a 0-commutative Sheffer stroke B-algebra if and only if it is a Sheffer stroke BM-algebra. Finally, a Sheffer stroke Coxeter algebra and an associative Sheffer stroke BM-algebra are defined and the relationship between this algebraic structures are shown.

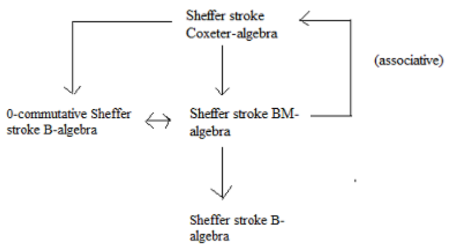

FIGURE 2

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