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QUASI-ORDERED RESIDUATED SYSTEM, A REVIEW

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ABSTRACT. The concept of quasi-ordered residuated systems was introduced in 2018 by S. Bonzio and I. Chajda. This author designed the concepts of ideals and filters in such systems as implicative, associated, comparative, weak implicative, shift and normal filters. The introduction of the concept of strong quasi-ordered residuated systems has made it possible to create the concept of irreducible filters and concepts of three types of prime filters in them. This article provides an overview of the results obtained about the created ideals, filters and their interrelationships in quasi-ordered residuated systems and in strong quasi-ordered residuated systems.

1. Introduction

Algebraic systems derived from logical systems are the focus of many studies (for example, see [**6, 15**]). These algebraic systems and their substructures have their own importance in mathematics. The concept of filters is mostly studied by the researchers working in logical algebras and in systems related to them [**5, 15**]. Maddux suggests that the text [**31**] written by Tarski in 1941, is probably one of the first articles which relates to 'The calculus of relations' (see [**11**], page 438). The approach outlined in [**31**] is worked out in more detail in [**32**]. In addition, according to Madduox, the first definition of relation algebras appears in [**9**] (cited by [**11**], page 441). Some authors (for example, see [**4**]) associate the study of binary relational systems with Riguet's article [**16**]. Also, some authors (according

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to [**4**]) link Mal'cev's article [**12**] with the first attempts to research of relational systems into the algebraic frameworks.

The concept of residuated relational systems ordered under a quasi-order relation, or quasi-ordered residuated systems (briefly, QRS), was introduced in 2018 by S. Bonzio and I. Chajda [**4**]. Previously, this idea was discussed in [**8**] (2014) and [**3**] (2015). Quasi-ordered residauted system, generally speaking, differs from the commutative residuated lattice $\langle A, \cdot, 0, 1, \sqcap, \sqcup, R\rangle$ where R is a lattice quasi-order. First, our observed system does not have to be limited from below. Second, the observed system does not have to be a lattice. However, the difference between a quasi-ordered relational system and a commutative residuated po-monoid ([**14**], briefly, a CRPM) (Example 2.5) is only in order relations since a quasi-order relation does not have to be antisymmetric. Some more about this last-mentioned algebraic structure it can be found in [**13**].

The author introduced and developed the concepts of ideals [**24**] (2021) and filters [**17**] (2020) in this algebraic structure as well as several types of filters such as implicative [**20**], associated [**18**], comparative [**19**], weak implicative [**28**], shift [**27**] and normal filters [**28**]. In [**19**], it is shown that every comparative filter of a quasi-ordered residuated system $\mathfrak{A}$ is an implicative filter of $\mathfrak{A}$ and the reverse it need not be valid. The concept of a strong quasi-ordered residuated system was introduced and discussed in [**21**]. In such systems, comparative and implicative filters coincide. The specificity of strong QRSs is that they allow us to determine the least upper bound for each their two elements. This allowed us to introduce the concept of prime and irreducible filters in such a relational system [**23**]. In addition to the previous one, in the strong quasi-ordered residuated system, three different types of prime filters [**26**] were introduced and analyzed, as well as weakly irreducible filters [**25**].

This article concisely presents the results obtained about the quasi-ordered residuated system and its substructures. This review paper is designed as follows:

In Section 2, which follows the introduction section, important definitions are given regarding the quasi-ordered relational system (by short, QRS) as well as some examples that should allow to a reader to get an impression of the specificity of this algebraic structure.

Section 3 describes the design of the concepts of pre-ideals and ideals in a quasi-ordered residuated system as well as some important features of these concepts. In addition to the above, a construction of a congruence relation on a quasi-ordered residuated system is described by relying on the concept of pre-ideals in this system. This allowed us to design a theorem that we can look at it as the First Isomorphism Theorem on these algebraic structures.

Section 4 presents the concept of filters in a quasi-ordered residuated system, which (according to first acquired impressions) differs somewhat from the usual determination of the notion of filters in implicit algebras as well as several different types of this substructure in QRS. Thus, the concepts of comparative, implicative, weak implicative, associated, shift and normal filters in a QRS are presented.

In section 5, a special class of quasi-ordered residual systems is presented - a strong quasi-ordered residuated system. In these QRSs, implicative and comparative filters are coincide. Second, in this QRS, it is possible to determine the least upper bound of any two elements. This allows the concepts of prime (three different types) and irreducible (and weak irreducible, also) filters in QRS to be determined.

2. Preliminaries: Concept of quasi-ordered residuated systems

In article [**4**], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([**4**], Definition 2.1). A *residuated relational system* is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:
(1) $(A, \cdot, 1)$ is a commutative monoid;
(2) $(\forall x \in A)((x, 1) \in R)$;
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.
We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The basic properties for residuated relational systems are subsumed in the following:

Theorem 2.1 ([**4**], Proposition 2.1). *Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ be a residuated relational system. Then*
(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$;
(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$;
(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$;
(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$;
(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a *quasi-order relation* $' \preccurlyeq '$ on a set A is a binary relation which is reflexive and transitive.

Definition 2.2 ([**4**]). A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$

Example 2.1. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	d	c	d
b	b	d	b	d	d
c	c	c	d	c	d
d	d	d	d	d	d

and

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$\preccurlyeq :=$
$\{(1,1),(a,1),(b,1),(c,1),(d,1),(b,b),(a,a),(c,c),(d,d),(c,a),(d,a),(d,b),(d,c)\}$.

EXAMPLE 2.2. For a commutative monoid A, let $\mathfrak{P}(A)$ denote the powerset of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A. Then $\langle \mathfrak{P}(A), \cdot, \rightarrow, A, \subseteq \rangle$ is a quasi-ordered residuated system in which the residuum are given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X := \{z \in A : Yz \subseteq X\}).$$

EXAMPLE 2.3. Let $\mathbb{R}$ be a field of real numbers. Define a binary operations '$\cdot$' and '$\rightarrow$' on $A = [0,1] \subset \mathbb{R}$ by

$$(\forall x, y \in [0,1])(x \cdot y := max\{0, x + y - 1\}) \text{ and } x \rightarrow y := min\{1, 1 - x + y\}).$$

Then, A is a commutative monoid with the identity 1 and $\langle A, \cdot, \rightarrow, <, 1 \rangle$ is a quasi-ordered residuated system.

EXAMPLE 2.4. Let $A = \langle -\infty, 1] \subset \mathbb{R}$ (the real numbers field). If we define '$\cdot$' and '$\rightarrow$' as follows, $(\forall u, v \in A)(u \cdot v := min\{u, v\})$ and $u \rightarrow v := 1$ if $u \leqslant v$ and $u \rightarrow v := v$ if $v < u$ for all $u, v \in A$, then $\mathfrak{A} := \langle A, \cdot, \rightarrow, 1, < \rangle$ is a quasi-ordered residuated system. □

EXAMPLE 2.5. ([**14**], Example 2.2) Any commutative residuated po-monoid (briefly, a CRPM) $(A, \cdot, \rightarrow, 1, \leqslant)$ is a quasi-ordered system.

EXAMPLE 2.6. Let $A = \{1, a, b, c, d, e\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$\cdot$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	b	a	c	d	e
b	b	b	a	c	d	e
c	c	b	c	c	d	e
d	d	d	d	d	e	e
e	e	e	e	e	e	d

and

$\rightarrow$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	c	1	1	1
b	1	c	1	1	1	1
c	1	c	c	1	1	1
d	1	d	d	d	1	e
e	1	e	e	e	d	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1 \rangle$ is a quasi-ordered residuated systems, where the relation '$\preccurlyeq$' is defined as follows

$\preccurlyeq := \{(1,1),\ (a,1)$,(b,1),(c,1),(d,1),(e,1),(a,a),(a,c),(a,d), $(a,e),\ (b,b),\ (b,c),\ (b,d),$ $(b,e),\ (c,c),\ (c,d),\ (c,e),\ (d,d),\ (e,e)\}$.

It should be noted that the elements a and b are not comparable.

REMARK 2.1. Quasi-ordered residauted system, generally speaking, differs from the commutative residuated lattice $\langle A, \cdot, \rightarrow, 0, 1, \sqcap, \sqcup, R \rangle$ where R is a lattice quasi-order. First, our observed system does not have to be limited from below. Second, the observed system does not have to be a lattice. However, the difference between a quasi-ordered relational system and a CRPM (Example 2.5) is only in order relations since a quasi-order relation does not have to be antisymmetric. Some more about this last-mentioned algebraic structure it can be found in [**13**].

The following proposition shows the basic properties of quasi-ordered residuated systems.

PROPOSITION 2.1 ([**4**], Proposition 3.1). *Let A be a quasi-ordered residuated system. Then*

(9) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$;
(10) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (y \to z \preccurlyeq x \to z \wedge z \to x \preccurlyeq z \to y))$;
(11) $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

3. Concept of ideals in QRS

Finding an appropriate substructure determination that should be recognized as an ideal in a quasi-ordered residuated system was challenging. Since the standard procedure could not be applied, this task was approached by a detour. Considering the properties of the right classes of the quasi-order relation in a quasi-ordered residuated system, some conditions were found that enabled the description of the notion of pre-ideal and the notion of ideal in this algebraic structure. The material presented in this section is mostly taken from article [**24**].

3.1. Right classes of quasi-order relations in a QRS. Let $\langle A, \cdot, \to, 1, \preccurlyeq\rangle$ be a quasi-ordered residuated system and let

$$L(a) = \{y \in A : a \preccurlyeq y\} \text{ and } R(b) = \{x \in A : x \preccurlyeq b\}$$

be the left class and the right class of the relation $\preccurlyeq$ generated by the elements a and b respectively. Then $R(1) = A$ by (2). In the following proposition, we give the basic properties of classes $R(b)$ $(b \in A)$:

PROPOSITION 3.1 ([**24**], Proposition 3.1). *Let a, b be elements in A. The following holds*

(12) $b \in R(b)$;
(13) *If $\preccurlyeq$ is an antisymmetric relation, then* $1 \in R(b) \Longleftrightarrow b = 1$.
(14) $(\forall u, v \in A)((v \in R(b) \wedge u \preccurlyeq v) \Longrightarrow u \in R(b))$;
(15) $(\forall u, v \in A)(u \in R(b) \vee v \in R(b)) \Longrightarrow u \cdot v \in R(b))$;
(16) $a \preccurlyeq b \Longrightarrow R(a) \subseteq R(b)$;
(17) $R(a \cdot b) \subseteq R(a) \cap R(b)$; *and*
(18) *Let us assume*
(O) $\neg(1 \in R(b))$.

Then the following holds

$$(\forall u, v \in A)(u \preccurlyeq v \Longrightarrow \neg(u \to v \in R(b))).$$

EXAMPLE 3.1. Let A be as in Example 2.1. Then, for example,

$$L(1) = \{1\},\ L(a) = \{1, a\},\ L(d) = A,$$

and

$$R(1) = A,\ R(a) = \{a, c, d\},\ R(b) = \{b.d\},\ R(c) = \{c, d\} \text{ and } R(d) = \{d\}.$$

It is obvious that the sets $R(a)$, $R(b)$, $R(c)$ and $R(d)$ satisfy the condition (O). Also, we have $a \cdot b = d$ and

$$\{d\} = R(d) = R(a \cdot b) \subseteq R(a) \cap R(b) = \{a, c, d\} \cap \{b, d\}$$

which illustrates the claim (17). □

Proposition 3.2 ([**24**], Proposition 3.2). *If we assuming that the right class $R(b)$ satisfies the condition* (O)*, than the condition*

(G) $(\forall u, v \in A)((\neg(u \to v \in R(b)) \wedge v \in R(b)) \implies u \in R(b))$

implies condition (14).

3.2. Analysis of conditions. The properties of the right classes of the relation $\preccurlyeq$ are a way for introducing the concept of ideals in a residuated relational system ordered under a quasi-order relation. Before that, we will analyze the interrelation of the following formulas:

(J1) $(\forall u, v \in A)(u \in J \vee v \in J \implies u \cdot v \in J)$;

(J2) $(\forall u, v \in A)((u \preccurlyeq v \wedge v \in J) \implies u \in J)$; and

(J3) $(\forall u, v \in A)((\neg(u \to v \in J) \wedge v \in J) \implies u \in J)$.

The condition (J1) implies that the subset J is an ideal in the monoid $(A, \cdot)$. Therefore, it provides a link between the idea of this substructure and the internal binary operation in A. The condition (J2) implies that the subset J is an ideal in the quasi-ordered ordered monoid $(A, \cdot)$. This condition will link the concept of ideals in a relational system $\mathfrak{A}$ with a quasi-order relation in A. The condition (J3) should involve the influence of the operation $\to$ into the substructure of the ideals in residuated relational systems ordered under quasi-order.

Proposition 3.3 ([**24**], Proposition 4.1). $(J2) \implies (J1)$.

Proposition 3.4 ([**24**], Proposition 4.2). *If for a proper subset J of a quasi-ordered residuated system holds (J2), then the following is valid also*

$$(\forall u, v \in A)(u \preccurlyeq v \implies \neg(u \to v \in J)).$$

Proposition 3.5 ([**24**], Proposition 4.3). *For a proper subset J of a quasi-ordered residuated system $\mathfrak{A}$ holds*

$$J(3) \Longrightarrow (J2).$$

Proposition 3.6 ([**24**], Proposition 4.4). *The condition* $(J2)$ *is equivalent to the condition*

(J4) $(\forall u, v, z \in A)((u \preccurlyeq v \to z \wedge z \in J) \implies u \cdot v \in J)$.

Proposition 3.7 ([**24**], Proposition 4.5). *Let J be a consistent subset of the monoid $(A, \cdot)$ in a quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$. Then*

$$J(2) \Longrightarrow J(3).$$

Remark 3.1. Therefore, the conditions (J2) and (J3) are not equivalent in the general case. If we choose to determine the concept of ideals by condition (J3), then the defined substructure in a quasi-ordered residuated system will satisfy the conditions (J1) and (J2). In this case, the right classes, in the general case, will not be ideals. If we choose to define the concept of ideals by condition (2), then we will have at least two types of ideals in these systems: ideals of type 1 - the class of ideals defined by condition (J2), and ideals of type 2 - the class of ideals determined by condition (J3). The second type of ideal, determined by the condition (J3), will be at the same time the first kind of ideal, because $(J3) \Longrightarrow (J2)$.

PROPOSITION 3.8 ([**24**], Proposition 4.6). *The condition (J3) is equivalent to the condition*

(J5) $(\forall u, v \in A)((\neg(u \in J) \wedge v \in J) \Longrightarrow u \rightarrow v \in J)$.

COROLLARY 3.1. *Let J be an ideal of q quasi-ordered residuated system $\mathfrak{A}$ such that $1 \notin J$. Then the following holds*

$$(\forall v \in A)(v \in J \Longrightarrow 1 \rightarrow v \in J).$$

3.3. Concept of pre-ideals. As is usual in any algebraic structure $\mathfrak{A}$, A is one of the ideals of structure $\mathfrak{A}$. By basing on our orientation about the determination of the notion of ideals in a residuated relational system ordered $\mathfrak{A}$ under a quasi-order by the properties of the right classes $R(a)$ $(a \in A)$ of the relation $\preccurlyeq$ generated by elements of the monoid $(A, \cdot)$, the concept of the pre-ideal in the system $\mathfrak{A}$ we introduce on the following way.

DEFINITION 3.1. ([**24**], Definition 4.1) Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$ be a quasi-ordered residuated system. For a subset J of the set A we say that it is a *pre-ideal* in $\mathfrak{A}$ if the condition (J2) is valid formula.

The class $R(b)$ $(b \in A)$ is a pre-ideal in $\mathfrak{A}$. Furthermore, the sets $\emptyset$ and A are trivial pre-ideals in $\mathfrak{A}$. Therefore, the family $\mathfrak{pJ}(A)$ of all pre-ideals in a quasi-ordered resuduated system $\mathfrak{A}$ is not empty.

THEOREM 3.1. *The family $\mathfrak{pJ}(A)$ of all pre-ideals in a quasi-ordered residuated system $\mathfrak{A}$ forms a complete lattice.*

COROLLARY 3.2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system and T be a non-empty subset of A. Then $T^U = \bigcup_{b \in T} R(b)$ is a pre-ideal in $\mathfrak{A}$.*

This type of pre-ideal is called *normal pre-ideals*. Without difficulty it can be the following shown:

THEOREM 3.2. *If $\{J_k\}_{k \in K}$ is a family of normal pre-ideals in $\mathfrak{A}$, then $\bigcup_{k \in K} J_k$ is also a normal pre-ideal in $\mathfrak{A}$.*

THEOREM 3.3. *Every pre-ideal J in a quasi-ordered reiduated system $\mathfrak{A}$ is a normal pre-ideal.*

REMARK 3.2. On the other hand, the intersection of two normal pre-ideals does not have to be a pre-ideal. If J_1 and J_2 are pre-ideals in $\mathfrak{A}$, then there are subsets $T1$ and $T2$ in A, such that $J_1 = T_1^U$ and $J_2 = T_2^U$. Thus $J_1 \cap J_2 = T_1^U \cap T_2^U = (\bigcup_{t \in T_1} R(t)) \cap \bigcup_{s \in T_2} R(s)) = \bigcup_{t \in T_1} \bigcup_{s \in T_2} (R(t) \cap R(s)) \supseteq \bigcup_{t \in T_1} \bigcup_{s \in T_2} R(t \cdot s)$ by (17).

EXAMPLE 3.2. Let A be as in Example 2.2. Then the sunsets $R(1) = A$, $R(a) = \{a, c, d, e\}$, $R(b) = \{b\}$, $R(c) = \{a, b, c\}$, $R(d) = \{a, b, c, d\}$ and $R(e) = \{a, b, c, e\}$ are pre-ideals in the quasi-ordered residuated system $\langle A, \cdot, \rightarrow, 1\rangle$. For example, for incomparable elements a and b, we have $\{a, b\}^U = R(a) \cup R(b) = A$ which illustrates the Corollary 3.2. $\square$

3.4. Concept of ideals. In this subsection we introduce and analyze the notion of ideals in a quasi-ordered residuated system.

DEFINITION 3.2. ([**24**], Definition 4.2) Let $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$ be a quasi-ordered residuated system. For a subset J of the set A we say that it is an *ideal* in $\mathfrak{A}$ if $J = A$ or the condition (J3) is valid formula.

PROPOSITION 3.9. *Every ideal J in a quasi-ordered reiduated system $\mathfrak{A}$ is a pre-ideal in $\mathfrak{A}$.*

The class $R(b)$ satisfying the conditions (O) and (G) is an ideal in $\mathfrak{A}$. Furthermore, the sets $\emptyset$ and A are trivial ideals in $\mathfrak{A}$. Therefore, the family $\mathfrak{J}(A)$ of all ideals in a quasi-ordered resuduated system $\mathfrak{A}$ is not empty.

THEOREM 3.4. *The family $\mathfrak{J}(A)$ of all ideals in a quasi-ordered residuated system $\mathfrak{A}$ forms a completely lattice.*

COROLLARY 3.3. *For each subset B in a quasi-ordered residauted system $\mathfrak{A}$ there is the maximal ideal in $\mathfrak{A}$ included in B.*

COROLLARY 3.4. *For any element a in a quasi-ordered residuated system $\mathfrak{A}$ there exists the maximal ideal M_a such that $\neg(a \in M_a)$.*

EXAMPLE 3.3. Let A be as in Example 2.1. Then, the subset $R(a) = \{a, c, d\}$ is an ideal in $\mathfrak{A}$. Indeed, for the elements $1, b \in A$ the following $1 \notin R(a)$ and $b \notin R(a)$ holds. By direct verification, it can be verified that the following to be valid:

$a \rightarrow a = 1 \notin R(a) \wedge a \in R(a) \implies a \in R(a)$,
$c \rightarrow a = 1 \notin R(a)\, a \in R(a) \implies c \in R(a)$,
$c \rightarrow c = 1 \notin R(a)\, c \in R(a) \implies c \in R(a)$,
$d \rightarrow a \notin R(a)\, a \in R(a) \implies d \in R(a)$,
$d \rightarrow c \notin R(a) \wedge c \in R(a) \implies d \in R(a)$,
$d \rightarrow d = 1 \notin R(a) \wedge d \in R(a) \implies d \in R(a)$ and
$c \rightarrow d = b \notin R(a) \wedge d \in R(a) \implies c \in R(a)$.

Analogously, it can be verified that the set $R(b)$ is an ideal in $\mathfrak{A}$. □

3.5. Congruence on QRS. We recall that any quasi-order relation on a set A generates an equivalence relation as follows

($C_{\preccurlyeq}$) $(\forall x, y \in A)((x, y) \in \theta_{\preccurlyeq} \iff (x \preccurlyeq y \wedge y \preccurlyeq x))$.

The internal binary operations in A are compatible with the relation $\theta_{\preccurlyeq}$ according to (9) and (10).

The equivalence relation $\theta_{\preccurlyeq}$, described by ($C_{\preccurlyeq}$), suggests that we can form the concept of congruences on a quasi-ordered residuated system analogously.

DEFINITION 3.3. ([**24**], Definition 5.1) An equivalence relation θ on a quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$ is a *congruence* on $\mathfrak{A}$ if the the following conditions hold

(C1) $(\forall x, y, z \in A)((x, y) \in \theta \implies ((x \cdot z, y \cdot z) \in \theta \wedge (z \cdot x, z \cdot y) \in \theta))$,
(C2) $(\forall x, y, z \in A)((x, y) \in \theta \implies ((x \rightarrow z, y \rightarrow z) \in \theta \wedge (z \rightarrow x, z \rightarrow y) \in \theta))$.

NOTE 3.1. Due to the commutativity of the multiplication operation in A, in implication (C1) it is sufficient to search for one of two requirements.

The relation of the congruence $C_{\preccurlyeq}$, in what follows we will write as follows $\equiv_{\preccurlyeq}$.

Let $\langle A, \cdot, \rightarrow, \preccurlyeq, 1\rangle$ and $\langle B, \cdot', \rightarrow', \preccurlyeq', 1'\rangle$ be two quasi-ordered residuated systems. A mapping $f : A \longrightarrow B$ is a homomorphism between quasi-ordered residuated systems if the following

$$(\forall u, v \in A)(f(1) \equiv_{\preccurlyeq'} 1' \wedge f(u \cdot v) \equiv_{\preccurlyeq'} f(u) \cdot' f(v) \wedge f(u \rightarrow v) \equiv_{\preccurlyeq'} f(u) \rightarrow' f(v))$$

holds. A homomorphism f is a monotone if holds

$$(\forall u, v \in A)(u \preccurlyeq v \implies f(u) \preccurlyeq' f(v)).$$

EXAMPLE 3.4. Without much difficulties, one can check that the relation $Kerf := \{u, v) \in A \times A : f(u) \equiv_{\preccurlyeq'} f(v)\}$, where f is a monotone homomorphism, is a congruence on A. Indeed:

Let $u, v, z \in A$ be elements such that $(u, v) \in Kerf$. Then $f(u) \equiv_{\preccurlyeq'} f(v)$. Thus

$f(x \cdot z) \equiv_{\preccurlyeq'} f(u) \cdot' f(z) \preccurlyeq' f(v) \cdot' f(z) \preccurlyeq' f(v \cdot z)$ by (9) and
$f(v \cdot z) \equiv_{\preccurlyeq'} f(v) \cdot' f(z) \preccurlyeq' f(u) \cdot' f(z) \equiv_{\preccurlyeq'} f(u \cdot z)$ by (9).
Hence $(u \cdot z, v \cdot z) \in Kerf$.

Let $u, v, z \in A$ be elements such that $(u, v) \in Kerf$. Then $f(u) \equiv_{\preccurlyeq'} f(v)$. This means $f(u) \preccurlyeq' f(v)$ and $f(v) \preccurlyeq' f(u)$. Thus

$f(u \rightarrow z) \equiv_{\preccurlyeq'} f(u) \rightarrow' f(z) \preccurlyeq' f(v) \rightarrow' f(z) \preccurlyeq' f(v \rightarrow z)$ by (10) and
$f(v \rightarrow z) \equiv_{\preccurlyeq'} f(v) \rightarrow' f(z) \preccurlyeq' f(u) \rightarrow' f(z) \equiv_{\preccurlyeq'} f(u \rightarrow z)$ by (10).
Hence $(u \rightarrow z, v \rightarrow z) \in Kerf$. The second implication can be proved analogously to the previous one. □

THEOREM 3.5. *The family* $\mathfrak{C}(A)$ *of all congruences on a quasi-ordered residuated system* $\mathfrak{A}$ *forms a complete lattice.*

In the following theorem, we show a construction of congruence on a quasi-ordered residuatesd system A for a given pre-ideal J of A.

THEOREM 3.6. *Let* J *be a pre-ideal of a quasi-ordered residuated system* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$. *then the relation* θ_J, *defined by*

(C_J) $(\forall x, y \in A)((x, y) \in \theta_J \iff (\exists a \in J)(a \cdot x \preccurlyeq y \wedge a \cdot y \preccurlyeq x))$,

is a convex congruence on A.

NOTE 3.2. It is obvious that the following holds $\theta_{\preccurlyeq} \cap (J \times J) \subseteq \theta_J$. Indeed, for example we can take $a = y$. Then we have $a \cdot x = x \cdot y \preccurlyeq y$ and $a \cdot y = y \cdot y \preccurlyeq y \preccurlyeq x$ according to (11).

REMARK 3.3. The condition (C_J) is equivalent to the condition

$$(x, y) \in \theta_J \iff (\exists a \in J)(x \rightarrow y \in L(a) \wedge y \rightarrow x \in L(a)))$$

where $L(a) = \{u \in A : a \preccurlyeq u\}$ is the left class of the relation $\preccurlyeq$ generated by the element a.

Remark 3.4. Let us note that in the previous theorem for given elements $x, y \in A$ such that $(x, y) \in \theta_J$ element $a \in J$, which appears in the definition (C_J), is not unique. Indeed, for every $a' \preccurlyeq a$, the following are valid $a' \in J$ (according to (J2)) and $a' \cdot x \preccurlyeq a \cdot x \preccurlyeq y$ and $a' \cdot y \preccurlyeq a \cdot y \preccurlyeq x$ by (9).

The importance of a congruence relation θ on a quasi-ordered residuated system $\mathfrak{A}$ is justified by the fact that the quotient A/θ turns naturally into a ordered set. It is commonly known that if $(A, \preccurlyeq)$ is a quasi-ordered set and θ an equivalence relation on A, then the relation $\leqq$, defined by $(\forall x, y \in A)([x]_\theta \leqq [y]_\theta \iff x \preccurlyeq y)$ is a order on A/θ. Let us define operations '$\odot$ and '$\rightrightarrows$' as

$$(\forall x, y \in A)([x]_\theta \odot [y]_\theta = [x \cdot y]_\theta) \text{ and}$$

$$(\forall x, y \in A)([x]_\theta \rightrightarrows [y]_\theta = [x \to y]_\theta).$$

Theorem 3.7. *Let $\mathfrak{A} = \langle A, \cdot, \to, 1, \preccurlyeq \rangle$ be a quasi-ordered relational system and let θ be a congruence on $\mathfrak{A}$. Then*

$$\langle A/\theta, \odot, \rightrightarrows, [1]_\theta, \leqq \rangle$$

is a quasi-ordered residuated system.

According to the Theorem 3.6 and Theorem 3.7, we have:

Theorem 3.8. *Let J be a pre-ideal of a quasi-ordered residuated system $\langle A, \cdot, \to, 1, \preccurlyeq \rangle$. Then the quasi-ordered residuated system $\langle A/\theta_J, \odot, \rightrightarrows, [1]_{\theta_J}, \leqq \rangle$ can be constructed. In addition, natural mapping $\pi : A \longrightarrow A/\theta_J$ is an monotone surjective homomorphism of quasi-ordered residuated systems.*

As a conclusion of this section, we can express the following theorem:

Theorem 3.9. *Let $f : A \longrightarrow A'$ be a monotone homomorphism between quasi-ordered residuated systems. Then there exists the injective homomorphism*

$$g : \langle A/Kerf, \odot, \rightrightarrows, [1]_{Kerf}, \leqq \rangle \longrightarrow \langle A', \cdot', \to', \preccurlyeq', 1' \rangle$$

such that $f = g \circ \pi$.

4. Concept of filters in QRS

The content presented in this section is taken from the papers [**17, 18, 19, 20, 22, 27, 28**]. In order to design the filter concept in a quasi-ordered residuated system, we first look at the properties of the left class of the quasi-order relation.

Let $L(a) = \{y \in A : a \preccurlyeq y\}$ be the left class and $R(b) = \{x \in A : x \preccurlyeq b\}$ be the right class of the relation $\preccurlyeq$ generated by the elements a and b respectively. Then $R(1) = A$. Some authors use the notation $U(a)$ instead of $L(a)$ (see, for example [**4**]).

Lemma 4.1. *If $\preccurlyeq$ is an antisymmetric relation, then $L(1) = \{1\}$.*

In the following propositions, we give the basic properties of classes $L(a)$:

Proposition 4.1. *The following holds*
(19) $a \in L(a) \wedge 1 \in L(a)$;
(20) $(\forall u, v \in A)((u \in L(a) \wedge u \preccurlyeq v) \implies v \in L(a))$;
(21) $(\forall u, v \in A)(u \cdot v \in L(a) \implies (u \in L(a) \wedge v \in L(a)))$;
(22) $L(a) \cap L(b) \subseteq L(a \cdot b)$;
(23) $(\forall u, v \in A)(v \in L(a) \implies u \to v \in L(a))$;
(24) $(\forall u, v \in A)(u \preccurlyeq v \implies u \to v \in L(a))$.

Corollary 4.1. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. If the implication*
(H) $(\forall u, v \in A)((u \in L(a) \wedge u \to v \in L(a)) \implies v \in L(a))$
is valid, the the formula (20) *is valid too.*

4.1. Filters on QRS. The properties of class $L(a)$ are the motivation for introducing the concept of filters in a quasi-ordered residuated system. Before that, let us analyze the interrelations of the following formulas
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$;
(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$; and
(F3) $(\forall u, v \in A)((u \in F \wedge u \to v \in F) \implies v \in F)$.

The recognizing of the essence of the interrelation between the conditions (F1), (F2) and (F3) are exhibited successively.

Proposition 4.2. $(F2) \Longrightarrow (F1)$.

The formula (F1) says that F is a prime subset of the monoid $(A, \cdot)$.

Proposition 4.3. *Let F be a subset of a quasi-ordered residuated system $\mathfrak{A}$. If $u, v \in A$ are elements such that* $(F2)$ *holds, then $u \to v \in F$.*

Proposition 4.4. *Let F be a nonempty subset of a quasi-ordered residuated system A. Then* $(F2)$ *implies* $1 \in F$.

Remark 4.1. Assuming $1 \in F$ and (F2), we can reinforce the claim in Proposition 4.3 and prove $u \preccurlyeq v \implies u \to v \in F$ in the following way:

Proposition 4.5. *Let F be a subset of a quasi-ordered residuated system $\mathfrak{A}$. Then the condition* $(F2)$ *is equivalent to the condition*
(F4) $(\forall u, v, z \in A)((u \cdot v \in F \wedge u \preccurlyeq v \to z) \implies z \in F)$.

A interrelation between conditions (F2) and (F3) we describes in the following proposition.

Proposition 4.6. *Let F be a submonoid of the monoid $(A, \cdot)$ in a quasi-ordered residuated system $\mathfrak{A} = \langle (A, \cdot, \to, 1, \preccurlyeq \rangle$. Then* $F(2) \implies F(3)$.

Based on our previous analysis of the interrelationship between conditions (F1), F(2), (F3) and (F4) in a quasi-ordered residual system, we introduce the concept of filters in the following definition.

Definition 4.1. ([**17**], Definition 3.1) For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* in A if it satisfies conditions (F2) and (F3).

EXAMPLE 4.1. Let $A = \{1, 2, 3, 4\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$$\begin{array}{c|cccc} \cdot & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 4 \\ 3 & 3 & 2 & 2 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{array} \quad \text{and} \quad \begin{array}{c|cccc} \rightarrow & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 2 & 1 & 4 \\ 4 & 1 & 2 & 3 & 1 \end{array}$$

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 1), (3, 1), (2, 1), (2, 3), (2, 4)\}.$$

Then the subsets $\{1\}$, $\{1, 3\}$ and $\{1, 3, 4\}$ are filters in $\mathfrak{A}$.

REMARK 4.2. In implicative algebras, the term 'implicative filter' is used instead of the term 'filter' we use (see, for example [**5, 15**]) because in the structure we study the concept of filter is determined more complexly than requirements (F3). It is obvious that our filter concept is also a filter in the sense of [**5, 6, 15**]. The term 'special implicative filter' is also used in the aforementioned sources if the implicative filter in the sense of [**15**] satisfies some additional condition.

The sets $\emptyset$ and A are trivial filters in $\mathfrak{A}$. Therefore, the family $\mathfrak{F}(A)$ of all filters in a quasi-ordered resuduated system $\mathfrak{A}$ is not empty.

THEOREM 4.1. *The family $\mathfrak{F}(A)$ of all filters in a quasi-ordered residuated system forms a completely lattice.*

COROLLARY 4.2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system and let B be a subset in A. Then there exists the minimal filter in $\mathfrak{A}$ which contains B.*

COROLLARY 4.3. *For any element a in A there exists the minimal filter F_a in $\mathfrak{A}$ such that $a \in F_a$.*

In the following definition we introduce the concept of 2-filters, which is somewhat more complex than the concept of filters.

DEFINITION 4.2. For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that the *2-filter* in $\mathfrak{A}$ if (F2) and the following

(F5) $(\forall u, v, z \in A)(((u \rightarrow v) \rightarrow z \in F \wedge u \rightarrow z \in F) \implies v \cdot z \in F)$

are valid.

It is immediately seen that $1 \in F$ if F is not an empty set and that, besides, the 2-filter satisfies condition (F1). Since a non-empty 2-filter F satisfies condition (F2), the claim of the Proposition 4.3 is valid for F also. Let us show that every 2-filter in $\mathfrak{A}$ is a filer in $\mathfrak{A}$.

THEOREM 4.2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then every 2-filter in $\mathfrak{A}$ is a filter u $\mathfrak{A}$.*

The previously presented material is contained in our text [**17**].

4.2. Implicative filters in QRS. In this subsection, we introduce the concept of implicative filters in a quasi-ordered residuared systems and analyze it ([**20**]).

DEFINITION 4.3. ([**20**], Definition 3.1) For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that the *implicative filter* in $\mathfrak{A}$ if (F2) is valid and the following condition
(IF) $(\forall u, v, z \in A)((u \to (v \to z) \in F \wedge u \to v \in F) \implies u \to z \in F)$
holds.

EXAMPLE 4.2. In Example 4.1, the filter $\{1, 3\}$ is an implicative filter in $\mathfrak{A}$.

It is immediately seen that $1 \in F$ and F satisfies condition (F1) because F satisfies the condition (F2) and F is a non-empty subset.

PROPOSITION 4.7. *Let F be an implicative filter of a quasi-ordered residuated system $\mathfrak{A}$. Then the following holds*
(F6) $(\forall u, v \in A)(u \to (u \to v) \in F \implies u \to v \in F)$.

The concept of implicative filters is a generalization of the concept of filters, that is, every implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ is a filter in $\mathfrak{A}$.

THEOREM 4.3. *Every implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ is a filter of $\mathfrak{A}$.*

We intend to more accurately describe this class of filters in quasi-ordered residuated systems. In what follows, we need the following two lemmas

LEMMA 4.2. *Let F be a subset of a quasi-ordered residuated system $\mathfrak{A}$. Then the condition $(F2)$ is equivalent to the condition*
(F7) $(\forall u, v, z \in A)((u \cdot v \in F \wedge u \preccurlyeq v \to z) \implies z \in F)$.

LEMMA 4.3. *Let $a \in A$ be an element of a quasi-ordered residuated system $\mathfrak{A}$. Then $L(a) = \{y \in A : a \preccurlyeq y\}$ is a filter of $\mathfrak{A}$ if and only if the following holds*
(25) $(\forall u, v \in A)((a \preccurlyeq u \wedge au \preccurlyeq v) \implies a \preccurlyeq v)$.

Another important result in this report is the following theorem:

THEOREM 4.4. *In a quasi-ordered residuated system $\mathfrak{A}$, the set $L(1)$ is an implicative filter if and only if $L(a)$ is a filter of $\mathfrak{A}$ for any $a \in A$.*

The following theorem gives another condition for a filter of a quasi-ordered residuates system $\mathfrak{A}$ to be an implicative filter in $\mathfrak{A}$.

THEOREM 4.5. *Let F be a filter of a quasi-ordered sesiduated system $\mathfrak{A}$. Then F is an implicative filter in $\mathfrak{A}$ if and only if the set $F_a = \{x \in A : a \to x \in F\}$ is a filter of $\mathfrak{A}$ for any $a \in A$.*

We can design another condition equivalent to a condition (IF):

THEOREM 4.6. *Let F be a non-empty subset of a quasi-ordered residuated system $\mathfrak{A}$ satisfying* (F2). *If F satisfies the additional condition*

(F8) $(\forall u, v, z \in A)((u \to (v \to (v \to z)) \in F \wedge u \in F) \implies v \to z \in F)$

then F is an implicative filter in $\mathfrak{A}$.

We conclude this report with the following theorem

THEOREM 4.7. *If a non-empty subset F of a quasi-ordered residuates system $\mathfrak{A}$ satisfies conditions* (F2), (F3) *and* (F6), *then F is an implicative filter in $\mathfrak{A}$.*

4.3. Comparative filters in QRS. In this subsection we introduce the concept of comparative filter in quasi-ordered residuated system ([**19**]). We then relate it to the concept of filters and the concept of implicative filters in this algebraic system.

DEFINITION 4.4. For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that a *comparative filter* in $\mathfrak{A}$ if (F2) is valid and the following condition

(CF) $(\forall u, v, z \in A)((u \to ((v \to z) \to v) \in F \wedge u \in F) \implies v \in F)$

holds.

EXAMPLE 4.3. Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 2.1. Then the set $F := \{1, a, b\}$ is a comparative filter in $\mathfrak{A}$.

Since any comparative filter F of $\mathfrak{A}$ satisfies the condition (F2), F also satisfies the condition (F0): $1 \in F$.

It can be seen immediately that the following proposition is valid:

PROPOSITION 4.8. *For any comparative filter F in a quasi-ordered residuated system $\mathfrak{A}$ holds*

(26) $(\forall v, z \in A)((v \to z) \to v \in F \implies v \in F)$.

LEMMA 4.4 ([**20**], Lemma 3.4). *Let a subset F of a quasi-ordered residuated system $\mathfrak{A}$ satisfy the condition* (F2). *Then the following holds*

(27) $(\forall u, v, z \in A)(u \to (v \to z) \in F \iff v \to (u \to z) \in F)$.

REMARK 4.3. Considering equivalence (27), the condition (CF) is equivalent to the condition

(FC (27)) $(\forall u, v, z \in A)(((v \to z) \to (u \to v) \in F \wedge u \in F) \implies v \in F)$.

Let us show that every comparative filter in $\mathfrak{A}$ is a filter of $\mathfrak{A}$.

THEOREM 4.8. *Every comparative filter of a quasi-ordered residuated system $\mathfrak{A}$ is a filter of $\mathfrak{A}$.*

The converse of Theorem 4.8 is not true in general as seen in the following example.

EXAMPLE 4.4. Let $\mathfrak{A}$ be a quasu-ordered residuated system as in Example 2.1. Then $F := \{1, b\}$ is a filter of $\mathfrak{A}$, but it is not a comparative filter of $\mathfrak{A}$, since for $u = b$, $v = a$ and $z = d$, we have $b \to ((a \to d) \to a) = 1 \in F$ and $b \in F$, but $a \notin F$.

Let us show that the condition (26) is sufficient for a filter of a quasi-ordered residuated system $\mathfrak{A}$ to be a comparative filter of $\mathfrak{A}$.

THEOREM 4.9. *Let F be a filter of a quasi-ordered residuated system $\mathfrak{A}$. Then F is a comparative filter of $\mathfrak{A}$ if and only if the condition* (26) *is valid.*

In what follows, we assume that a quasi-ordered residuated system $\mathfrak{A}$ satisfies the condition

(28) $(\forall u, v \in A)(u \rightarrow v = ((u \rightarrow v) \rightarrow v) \rightarrow v)$.

PROPOSITION 4.9. *Let $\mathfrak{A}$ be a quasi-ordered residuated system that satisfies the condition (28). Then any comparative filter F in $\mathfrak{A}$ satisfies the the condition*

(29) $(\forall u, v \in A)((u \rightarrow v) \rightarrow v \in F \implies (v \rightarrow u) \rightarrow u \in F)$.

THEOREM 4.10. *Let F be an implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ satisfying (29). Then F is a comparative filter of $\mathfrak{A}$.*

THEOREM 4.11. *Any comparative filter of a quasi-ordered residuated system $\mathfrak{A}$ is an implicative filter of $\mathfrak{A}$.*

EXAMPLE 4.5. Let $\mathfrak{A}$ be a quasi-ordered residuated system as in Example 4.4. Then the subset $F := \{1, b\}$ is an implicative filter but it is not a comparative filter.

We end this section with the following theorem.

THEOREM 4.12. *The family $\mathfrak{F}_c(A)$ of all comparative filters of a quasi-ordered residuated system $\mathfrak{A}$ forms a complete lattice.*

COROLLARY 4.4. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any subset T of A, there is the unique minimum comparative filter in $\mathfrak{A}$ that contains T.*

COROLLARY 4.5. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any element x of A, there is the unique minimum comparative filter in $\mathfrak{A}$ that contains x.*

4.4. Associated filters of QRS. In this subsection, we introduce the concept of associated filters in quasi-ordered residuared systems and analyze it ([**18**]).

DEFINITION 4.5. ([**18**], Definition 3.1) Let $\mathfrak{A}$ be a QRS and x be a fixed element of A. A non-empty subset F of A is called an *associated* filter of $\mathfrak{A}$ *with respect to* x (briefly, *x-associated filter* of $\mathfrak{A}$) if it satisfies the condition (F-2) and

(AF) $(\forall y, z \in A)((x \rightarrow (y \rightarrow z) \in F \wedge x \rightarrow y \in F) \implies z \in F)$.

By an *associated filter* of $\mathfrak{A}$, we mean an x-associated filter of $\mathfrak{A}$ for all x in A.

It is immediately seen that $1 \in F$ and F satisfies condition (F1) because F satisfies the condition (F2) and F is a non-empty subset.

EXAMPLE 4.6. Let $A = \{1, 2, 3, 4\}$ and operations '$\cdot$' and '$\rightarrow$' defined on A as follows:

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

and

$\rightarrow$	1	2	3	4
1	1	2	3	4
2	1	1	2	3
3	1	1	1	4
4	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1,1), (2,2), (3,3), (4,4), (4,1), (3,1), (2,1), (3,2), (4,2), (4,3)\}.$$

Then $\{1\}$ is a 1-associated filter of $\mathfrak{A}$. The subset $\{1, 2, 3\}$ are associated in $\mathfrak{A}$ with respect to 1, 2 and 3, but it is a 4-associated filter of $\mathfrak{A}$ because $4 \rightarrow (4 \rightarrow 4) = 4 \rightarrow 1 = 1 \in \{1, 2, 3\}$ and $4 \rightarrow 4 = 1 \in \{1, 2, 3\}$ but $\neg(4 \in \{1, 2, 3\})$.

THEOREM 4.13. *For any $x \in A$, every x-associated filter of a QRS $\mathfrak{A}$ contains x itself.*

NOTE 4.1. The previous theorem suggests that there are no proper associated filters in quasi-ordered residuated systems. So, the only associated filter in a QRS $\mathfrak{A}$ is A.

THEOREM 4.14. *Let $\mathfrak{A}$ be a QRS. For any $x \in A$, if F is a filter of $\mathfrak{A}$ which contains x, then it is an x-associated filter of $\mathfrak{A}$.*

THEOREM 4.15. *Let $\mathfrak{A}$ be a QRS. Every 1-associated filter of $\mathfrak{A}$ is a filter of $\mathfrak{A}$, and vice versa.*

THEOREM 4.16. *Let F be a filter of a QRS $\mathfrak{A}$ and let $x, x' \in A$ be such that $x \preccurlyeq x'$. If F is an x-associated filter of $\mathfrak{A}$, then it is a x'-associated filter of $\mathfrak{A}$.*

In the following two theorems we show some conditions for the filter F of QRS $\mathfrak{A}$ to be a x-associated filter of $\mathfrak{A}$ for a given $x \in A$.

THEOREM 4.17. *Let F be a filter of a QRS $\mathfrak{A}$. If F satisfies the additional condition*

(30) $(\forall y, z \in A)(x \rightarrow (y \rightarrow z) \in F \implies (x \rightarrow y) \rightarrow z \in F)$,

then F is an x-associated filter of A.

THEOREM 4.18. *Let F be a filter of a QRS $\mathfrak{A}$. If F satisfies the additional condition*

(31) $(\forall z \in A)(x \rightarrow (x \rightarrow z) \in F \implies z \in F)$,

then F is an x-associated filter of A.

REMARK 4.4. The reverse of the previous theorem is obvious since the implication (31) we can get from (FA) if we choose $y = z$ and $z = y$.

4.5. Weak implicative filters in QRS. It is known ([**19**], Theorem 3.1) that any implicative filter in a quasi-ordered residuated system is a filter in that system. As, in the general case, the opposite does not have to be true, our intention in this subsection is to analyze the possibility of the existence of some kind of filter between the previous two and which would be different from both of them. In this subsection, we describe the concept of weak implicit filters in quasi-ordered residuared systems introduced and analyzed in paper [**22**].

DEFINITION 4.6. ([**22**], Definition 3.1) For a non-empty subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that the *weak implicative filter* in $\mathfrak{A}$ if (F2) and the following condition

(WIF) $(\forall u, v, z \in A)((u \to (v \to z) \in F \wedge u \to v \in F) \implies u \to (u \to z) \in F)$ are valid.

Let us show that every weak implicative filter in $\mathfrak{A}$ is a filter of $\mathfrak{A}$.

THEOREM 4.19. *Any weak implicative filter in a quasi-ordered residuated system $\mathfrak{A}$ is a filter of $\mathfrak{A}$.*

THEOREM 4.20. *Every implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ is a weak implicative filter of $\mathfrak{A}$.*

In the following theorem, we demonstrate one equivalent condition when a weak implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ to become an implicative filter of $\mathfrak{A}$.

THEOREM 4.21. *Let F be a weak implicative filter of a quasi-ordered residuated system $\mathfrak{A}$. Then F is an implicative filter of $\mathfrak{A}$ if and only if the following condition holds*

(32) $(\forall u, z \in A)(u \to (u \to z) \iff u \to z \in F)$.

EXAMPLE 4.7. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	b
b	b	a	b	b	d
c	c	a	b	c	d
d	d	b	d	d	d

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1,1), (a,1), (b,1), (c,1), (d,1), (a,a), (b,b), (c,c), (d,d), (d,a), (d,b), (c,b)\}.$$

Then the subsets $\{1, a\}$ is a weak implicative filter in $\mathfrak{A}$.

THEOREM 4.22. *Let $\mathfrak{A}$ be a quasi-ordered residuated system and let F be a filter of $\mathfrak{A}$. Then F is a weak implicative filter of $\mathfrak{A}$ if it satisfies the following condition:*

(32) $(\forall y, v, z \in A)(u \to (v \to z) \in F \implies u \to ((u \to v) \to (u \to z)) \in F)$.

THEOREM 4.23. *Let the quasi-ordered residuated system $\mathfrak{A}$ satisfy the additional condition*

(33) $(\forall u, v, z \in A)(v \to z \preccurlyeq (u \to v) \to (u \to z))$.

Then every filter of $\mathfrak{A}$ is a weak implicative filter of $\mathfrak{A}$.

4.6. Normal filter in QRS. Normal filters in BL-algebra were defined in paper [**30**]. Borzooei and Paad studied the normal filter in BL-algebras ([**2**]) by comparing it with other types of filters in residuated lattices. Ahadpanah and Torkzadeh are studied normal filters in residuated lattices ([**1**]). Wang er al. also dealt with normal filters in some logical algebras ([**34**]).

In this subsection, the concept of normal filters in a quasi-ordered residuated system is presented and some of its important features are shown ([**28**]).

DEFINITION 4.7. ([**28**], Definition 7) A filter F of a quasi-ordered residuated system $\mathfrak{A}$ is called *normal* if the following holds

(NF) $(\forall x, y, z \in A)((z \to ((y \to x) \to x) \in F \wedge z \in F) \implies (x \to y) \to y \in F)$.

In the general case, the filter in a quasi-ordered residuated system does not have to be a normal filter.

THEOREM 4.24. *If $\mathfrak{A}$ is a strong quasi-ordered residuated system, i.e., if it satisfies the condition (37) in the Definition 5.1, then any filter of $\mathfrak{A}$ is a normal filter of $\mathfrak{A}$.*

EXAMPLE 4.8. Let $\mathfrak{A}$ be as in Example 2.4. We will show that $\mathfrak{A}$ does not have any proper normal filter. Let $u, v \in A$ be such that $u < v < x$ and let $F := \langle x, 1]$ be a filter in $\mathfrak{A}$ where $x < 1$. Then $(v \to u) \to u = 1 \in F$ and $(u \to v) \to v = u \to v = u \notin F$. So, F is not a filter in $\mathfrak{A}$. Thus, F is not a normal filter in A according to Theorem 4.24.

THEOREM 4.25. *Let F be a filter of a quasi-ordered residuated system $\mathfrak{A}$. F is a normal filter of $\mathfrak{A}$ if and only if the followings holds*

(34) $(\forall x, y, z \in A)((y \to x) \to x \in F \implies (x \to y) \to y \in F)$.

THEOREM 4.26. *Any comparative filter in a quasi-ordered residuated sustem $\mathfrak{A}$ is a normal filter in $\mathfrak{A}$.*

EXAMPLE 4.9. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	a	a	1

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	b	1	b
c	1	b	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (a,a), (a,b), (a,c), (b,b), (c,b), (c,c)\}.$$

Subset $\{1\}$ is a normal filter of $\mathfrak{A}$ but it is not a comparative filter in $\mathfrak{A}$ because it does not satisfy the condition (34) since, for example, for $v = b$ and $u = a$, we have $(b \to a) \to b = 1 \in F$ while $b \notin F$.

EXAMPLE 4.10. Let $\mathfrak{A}$ be as in Example 4.9. $F = \{1\}$ is a normal filter in $\mathfrak{A}$ but it is not an implicative filter in $\mathfrak{A}$ since it does not satisfy the condition (14) in Lemma 3 in the paper [**28**]. For example, for $u = b$ and $v = c$ we have $u \to (u \to v) = b \to (b \to c) = b \to b = 1 \in F$ but $u \to v = b \to c = b \notin F$.

Now, we express the following theorems:

THEOREM 4.27. *Every comparative filter in a quasi-ordered residuated system is an implicative and a normal filter in it.*

THEOREM 4.28. *If F is an implicative and normal filter in a quasi-ordered residuated system $\mathfrak{A}$, then F is a comparative filter in $\mathfrak{A}$.*

The notion of MV-filters in residuated lattices was introduced in [**1**] as follows: A subset F of a residuated lattice L is called an MV-filter of L if it is a filter of L that satisfies in the condition

(MVF) $(\forall u, v \in L)(((u \to v) \to v) \to ((v \to u) \to u) \in F)$.

Also, in [**1**] it is shown (Theorem 3.10) that every MV-filter of L is a normal filter of L. It has been shown there that the reverse need not be true (Example 3.11). In our case, the relationships between conditions (MVF) and (NF) are similar.

THEOREM 4.29. *Any filter in a quasi-ordered residuated system $\mathfrak{A}$ which satisfies the condition (MVF), is a normal filter in $\mathfrak{A}$.*

EXAMPLE 4.11. Consider the quasi-ordered residuated system $\mathfrak{A}$ as in Example 4.9. The filter $F = \{1\}$ in $\mathfrak{A}$ is a normal filter in $\mathfrak{A}$ while it does not satisfy the condition (MVF), since $((c \to a) \to a) \to ((a \to c) \to x) = a \notin F$ holds.

4.7. Shift filters in QRS. In this subsection, we introduce and analyze the concept of shift filters of quasi-ordered residuated system. The material presented here is taken from the text [**27**].

DEFINITION 4.8. ([**27**], Definition 3.1) Let $\mathfrak{A}$ be a quasi-ordered residuated system. A non empty subset F of A is a *shift filter* of $\mathfrak{A}$ if it satisfies the conditions (F2) and the following condition

(SF) $(\forall u, v, z \in A)((u \to (v \to z) \in F \wedge u \in F) \implies ((z \to v) \to v) \to z \in F)$.

REMARK 4.5. In some other algebraic systems, request (SF) is recognized as a fantastic filter.

EXAMPLE 4.12. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	c
b	b	a	b	c
c	c	a	c	c

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	a	1	c
c	1	a	b	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1, 1), (a, a), (b, b), (c, c), (a, 1), (b, 1), (c, 1), (a, b)\}.$$

Then the subsets $\{1, b\}$ is a shift filter of $\mathfrak{A}$.

EXAMPLE 4.13. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	a	b	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(a,b),(a,c)\}.$$

Then the subsets $\{1,b\}$ is a filter of $\mathfrak{A}$ but it is not a shift filter of $\mathfrak{A}$. For example, for $u = b$, $v = a$ and $z = c$, we have $b \to (a \to c) = b \to 1 = 1 \in \{1,b\}$ and $b \in \{1,b\}$, but $((c \to a) \to a) \to c = (a \to a) \to c = 1 \to c = c \notin \{1,b\}$.

It can be verified that a shift filter of a quasi-ordered residuated system $\mathfrak{A}$ has the following property

Proposition 4.10. *Let F be a shift filter of a quasi-ordered residuated system $\mathfrak{A}$. Then*

(35) $(\forall u,v \in A)(u \to v \in F \implies ((v \to u) \to u) \to v \in F)$.

Let us show now that the condition (35) is sufficient for a filter F of a quasi-ordered system $\mathfrak{A}$ satisfying the condition (35) to be a shift filter of $\mathfrak{A}$.

Theorem 4.30. *Let F be a filter of a quasi-ordered residuated system $\mathfrak{A}$ and suppose that F satisfies the condition (35). Then F is a shift filter of $\mathfrak{A}$.*

Our second theorem on this class of filters of quasi-ordered residuated systems is the following

Theorem 4.31. *Every comparative filter of a quasi-ordered residuared system $\mathfrak{A}$ is a shift filter of $\mathfrak{A}$.*

The following example shows that any shift filter of a quasi-ordered residuated system $\mathfrak{A}$ does not have to be a comparative filter of $\mathfrak{A}$.

Example 4.14. Let $A = \{1,a,b,c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	a	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	a	b
b	1	a	1	b
c	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1,1),(a,a),(b,b),(c,c),(a,1),(b,1),(c,1),(b,a),(c,a),(c,b)\}.$$

Then the subsets $\{1\}$ is a shift filter of $\mathfrak{A}$ but it is not a comparative filter of $\mathfrak{A}$. For example, for $u = 1$, $v = a$ and $z = b$, we have

$$1 \to ((a \to b) \to a) = 1 \to (a \to a) = 1 \to 1 = 1 \in \{1\}$$

and $1 \in \{1\}$, but $a \notin \{1\}$.

THEOREM 4.32. *Let F be an implicative filter of a quasi-ordered residuated system $\mathfrak{A}$ satisfying*

(36) $(\forall u, v \in A)((u \to v) \to v \in F \implies (v \to u) \to u \in F)$.

Then F is a shift of $\mathfrak{A}$.

The following example shows that any shift filter of a quasi-ordered residuated system $\mathfrak{A}$ does not have to be an implicative filter of $\mathfrak{A}$.

EXAMPLE 4.15. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	a	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	a	b
b	1	1	1	b
c	1	1	1	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq = \{(1, 1), (a, a), (b, b), (c, c), (a, 1), (b, 1), (c, 1), (b, a), (c, a), (c, b)\}.$$

Then the subsets $\{1\}$ is a shift filter of $\mathfrak{A}$ but it is not an implicative filter of $\mathfrak{A}$. For example, for $u = b$, $v = b$ and $z = c$, we have $b \to (b \to c) = 1 \in \{1\}$ and $b \to b = 1 \in \{1\}$, but $b \to c = b \notin \{1\}$.

We end this subsection with the following theorem.

THEOREM 4.33. *The family $\mathfrak{F}_s(A)$ of all shift filters of a quasi-ordered residuated system $\mathfrak{A}$ forms a complete lattice.*

Let $\mathfrak{A}$ be a quasi-ordered residuated system. Before embarking on further conclusions, let us recall the terms 'minimum fulter' and 'maximum filter' in a quasi-ordered residuated system: We shall say that a filter A is a minimal filter of $\mathfrak{A}$ if there is no a filter B of $\mathfrak{A}$ such that $B \subset A$. Also, dually, we shall say that a filter A is a maximal filter of $\mathfrak{A}$ if there is no a filter B of $\mathfrak{A}$ such that $A \subset B$. It is easy to conclude that if A and B are two minimum interiors filters of a quasi-ordered residuated system $\mathfrak{A}$, then $A \cap B = \emptyset$, because, otherwise, according to the previous theorem, $A \cap B$ would be a filter of $\mathfrak{A}$ contained in A and contained in B, which is impossible.

COROLLARY 4.6. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any subset T of A, there is the unique minimum shift filter of $\mathfrak{A}$ that contains T.*

COROLLARY 4.7. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any element x of A, there is the unique minimum shift filter of $\mathfrak{A}$ that contains x.*

5. Strong quasi-ordered residuated systems

In this section we introduce the concept of strong quasi-ordered residuated systems. This concept was introduced and analyzed in articles [**21, 23**] by this author.

Considering the fact that the quasi-order relation '$\preccurlyeq$', which appears in the determination of this algebraic system, does not have to be antisymmetric, the following definition gets a clearer meaning.

Definition 5.1. ([**21**], Definition 6) For a quasi-ordered residuated system $\mathfrak{A}$ it is said to be a *strong quasi-ordered residuated system* if the following holds

(37) $(\forall u, v \in A)((u \to v) \to v \preccurlyeq (v \to u) \to u \wedge (v \to u) \to u \preccurlyeq (u \to v) \to v)$.

The following is an example of a non strong QRS:

Example 5.1. Let $A = \{1, a, b\}$ and the operations '$\cdot$' and '$\to$' be defined on A as follows:

$\cdot$	1	a	b
1	1	a	b
a	a	a	a
b	b	a	b

and

$\to$	1	a	b
1	1	a	b
a	1	1	1
b	1	a	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,a), (b,b), (a,1), (b,1), (a,b)\}.$$

It can be easily checked that $\mathfrak{A}$ is a quasi-ordered residuated system. Since

$$(a \to b) \to b = 1 \to b = b \text{ and } (b \to a) \to a = a \to a = 1,$$

we have $(a \to b) \to b \preccurlyeq (b \to a) \to a$ but $\neg((b \to a) \to a \preccurlyeq (a \to b) \to b)$. Thus, $\mathfrak{A}$ is not a strong quasi-ordered residuated system.

Now, we give an example of strong quasi-ordered residuated system.

Example 5.2. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	b	b	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (a,a), (b,b), (c,c), (a,b), (a,c)\}.$$

Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system.

REMARK 5.1. It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates an equivalence relation $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (9) and (10), this equivalence is compatible with the operations in $\mathfrak{A}$. Thus, $\equiv_{\preccurlyeq}$ is a congruence on $\mathfrak{A}$. In light of this remark, the condition (37) can be written in the form

$$(\forall u, v \in A)((u \to v) \to v \equiv_{\preccurlyeq} (v \to u) \to u).$$

In this section, we shall present the structure of strong quasi-ordered residuated systems.

THEOREM 5.1. *In a strong quasi-ordered residuated system* $\mathfrak{A}$ *the following holds*

(38) $(\forall u, v \in A)(u \preccurlyeq v \implies v \equiv_{\preccurlyeq} (v \to u) \to u)$.

As a significant consequences of this theorem we can prove some important properties of strong quasi-ordered residuared systems.

THEOREM 5.2. *Let* $\mathfrak{A}$ *be a strong quasi-ordered residuated system. Then the comparative and implicative filters of* $\mathfrak{A}$ *coincide.*

The notion "least upper bound" is well-defined for partial order relations. Although it is not common to use this definition for quasi-order relations because, in the general case, for quasi-order relations the "least upper bound" is not unique, when it comes to a strong quasi-ordered relational system, this concept can be determined as shown in the following theorem.

THEOREM 5.3. *Let* $\mathfrak{A}$ *be a strong quasi-ordered residuated system. For any* $u, v \in A$*, the element*

$$u \sqcup v := (v \to u) \to u \equiv_{\preccurlyeq} (u \to v) \to v$$

is the least upper bound of u *and* v.

The following example shows that condition (37) is crucial for the defining of the least upper bound.

EXAMPLE 5.3. Let $A = \{1, a, b, c, d, e\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	a	a	a	a	a
b	b	a	b	a	b	a
c	c	a	a	a	a	c
d	d	a	b	a	v	c
e	e	a	a	c	c	e

and

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	1	1	1
b	1	e	1	e	1	e
c	1	b	b	1	1	1
d	1	a	b	e	1	e
e	1	b	b	d	b	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (d,1), (e,1), (a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (b,d), (c,c), (c,d), (c,e), (d,d)\}$. Direct verification it can prove that $\mathfrak{A}$ is a quasi-ordered residuated system but is is not a strong system. For example, for $u = b$ and $v = c$, we have $(b \to c) \to c = e \to c = d$ and

$(c \to b) \to b = b \to b = 1$. Since $d \neq 1$, we conclude that $\mathfrak{A}$ is not a strong system. However, on the other hand, in this case we still have $b \preccurlyeq d$ and $c \preccurlyeq d$. Obviously, 1 is an upper bound of b and c also, but we cannot define the least upper bound $b \sqcup c$ as in the previous theorem.

We end this part of Section 5 by the following statement:

THEOREM 5.4. *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system. Then $(\mathfrak{A}, \sqcup)$ is a distributive upper semi-lattice in the following sense*

$$(\forall x, y, z \in A)((x \sqcup y) \sqcup z \equiv_{\preccurlyeq} (x \sqcup z) \sqcup (y \sqcup z)).$$

REMARK 5.2. The term 'distributive upper semi-lattice' used here differs from the concept of 'distributive join semi-lattice' which appears for example in the book [**7**] (pp. 99).

The following theorem lists some of the properties of a strong resudiated system in relation to the operation '$\sqcup$':

THEOREM 5.5 ([**20**], Proposition 2). *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system. Then:*
(a) $(\forall u, v \in A)(u \sqcup 1 = 1 \sqcup u = 1$ *and* $u \sqcup v = v \sqcup u)$,
(b) $(\forall x, y, z \in A)((z \cdot x) \sqcup (z \cdot y) \preccurlyeq x \sqcup y)$,
(c) $(\forall x, y, z \in A)((x \sqcup y) \to z \preccurlyeq (x \to z) \sqcup (y \to z))$,
(d) $(\forall x, y, z \in A)((z \to x) \sqcup (z \to y) \preccurlyeq z \to (x \sqcup y))$,
(e) $(\forall, y \in A)(x \sqcup y \preccurlyeq (y \to x) \sqcup (x \to y))$,
(f) $(\forall x, y \in A)((x \sqcup y) \sqcup x \equiv_{\preccurlyeq} x \sqcup y)$.

5.1. Three types of prime filters in QRS. The concepts of prime filters in a strong quasi-ordered residuated system were introduced and analyzed in articles [**23, 26**]. Thus, while in [**23**] the term 'prime filter of the first type' was introduced, in [**26**], the terms 'prime filter of the second type' and 'prime filter of the third type' were introduced.

The following definition gives the concept of prime filters of the first type in QRS's.

DEFINITION 5.2. ([**23, 26**]) Let F be a filter of a strong quasi-ordered residuated system $\mathfrak{A}$. Then F is said to be a *prime filter of the first type* in $\mathfrak{A}$ if the following holds

(PF1) $(\forall u, v \in A)(u \sqcup v \in F \implies (u \in F \vee v \in F))$.

EXAMPLE 5.4. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	a	a
c	c	b	a	a

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	1
c	1	b	c	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1),(a,1),(b,1),(c,1),(a,a),(b,b),(c,c),(a,b),(a,c),(b,c)\}.$$

Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system. The only proper filter in this system is the subset $F := \{1\}$. It is easily concluded by directly checking that F is a prime filter of the first type.

Example 5.5. Let $\mathfrak{A}$ be as in the Example 5.2. Then the subsets $\{1\}$, $\{1,b\}$ and $\{1,c\}$ are filters in $\mathfrak{A}$. It can be checked that $F_1 := \{1,b\}$ and $F_2 := \{1,c\}$ are prime filters of the first type in $\mathfrak{A}$ while the filter $\{1\}$ is not prime of the first type because we have $b \sqcup c = 1 \in \{1\}$ but $b \notin \{1\}$ and $c \notin \{1\}$.

We first show one important feature of prime filters in strong QRSs.

Proposition 5.1. *Let F be a prime filter of the first type in a strong quasi-ordered residuated system $\mathfrak{A}$. Then*

$$(\forall x, y \in A)(x \sqcup y \in F \implies (x \to y \in F \lor y \to x \in F)).$$

The result of the previous proposition is the motive for introducing the notion of prime filters of the second type in a strong quasi-ordered residuated system.

Definition 5.3. ([**26**]) A filter F of a strong quasi-ordered residuated system $\mathfrak{A}$ is a *prime filter of the second type* if the following holds

(PF2) $(\forall x, y \in A)(x \to y \in F \lor y \to x \in F)$.

Example 5.6. Let $\mathfrak{A}$ be as in the Example 2.5. Subsets $\{1,c\}$ is a prime UP-filter of the second type of $\mathfrak{A}$. The subset $F := \{1,b\}$ is a prime UP-filter of the first type but it is not a prime UP-filter of the second type because, for example, holds $a \to b = c \notin F$ and $b \to a = c \notin F$.

In the previous example it was shown that a filter of a quasi-ordered residuated system can be a prime filter of the first type but it does not have to be a prime filter of the second type. However, the following theorem shows that if F satisfies the condition (PF2), then it satisfies the condition (PF1) also, i.e. any prime filter of the second type of a strong quasi-ordered residuated system $\mathfrak{A}$ is a prime filter of the first type of $\mathfrak{A}$.

Theorem 5.6. *If F is s filter of the second type in a strong quasi-ordered residuated system $\mathfrak{A}$, then F is a prime filter of the first type in $\mathfrak{A}$.*

Theorem 5.7 (Extension property for prime filters of the second type). *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system and let F and G be filter of $\mathfrak{A}$ such that $F \subseteq G$. If F is a prime filter of the second type, then G is a prime filter of the second kind also.*

The following theorem gives one sufficient condition that a filter of first type in a strong quasi-ordered residuated system be a filter of the second type.

THEOREM 5.8. *Let a strong quasi-ordered residuated system* $\mathfrak{A}$ *satisfies the condition*

(U) $(\forall x, y \in A)((x \to y) \sqcup (y \to x) = 1)$.

Then any prime filter of the first type in $\mathfrak{A}$ *is a prime filter of the second type.*

Our next theorem gives one important property of prime filters of the second type in a strong QRS.

THEOREM 5.9. *If the relation* $\preccurlyeq$ *in a strong quasi-ordered residuated system* $\mathfrak{A}$ *is a linear relation in the following sense*

$$(\forall x, y \in A)(x \preccurlyeq y \ \vee \ y \preccurlyeq x),$$

then every filter in $\mathfrak{A}$ *is a prime filter of the second type.*

EXAMPLE 5.7. Let $\mathfrak{A}$ be as in the Example 5.4. The relation $\preccurlyeq$ is linear and the subset $F := \{1\}$ is a prime filter in $\mathfrak{A}$.

One connection between the linearity of the relation $\preccurlyeq$ and the requirement that the filter $\{1\}$ be a prime filter (of the first type) in a strong quasi-ordered residuated system is given by the following theorem.

THEOREM 5.10. *If* $\{1\}$ *is a prime filter of the first type in a strong quasi-ordered residuated system* $\mathfrak{A}$*, then holds*

$$(\forall x, y \in A)(x \sqcup y = 1 \implies (x \preccurlyeq y \ \vee \ y \preccurlyeq x)).$$

Obviously, if F is a prime filter of the second type of a strong quasi-ordered residuated system $\mathfrak{A}$, then it holds

$$(\forall x, y \in A)((x \to y) \sqcup (y \to x) \in F).$$

Indeed, from $x \to y \in F$ or $y \to x \in F$ it follows $(x \to y) \sqcup (y \to x) \in F$ because $x \to y \preccurlyeq (x \to y) \sqcup (y \to x)$ or $y \to x \preccurlyeq (x \to y) \sqcup (y \to x)$ with respect to (F-2).

The procedure exposed in the previous analysis is the motive for introducing the concept of prime filter of the third type in a strong quasi-ordered residuated system.

DEFINITION 5.4. ([**26**]) A *prime filter* F *of the third type* of a strong quasi-ordered residated system $\mathfrak{A}$ is a filter of $\mathfrak{A}$ satisfying

(PF3) $(\forall x, y \in A)((x \to y) \sqcup (y \to x) \in F)$.

From the definitions it is also clear that a prime filters of the third type has the following two properties. Therefore, we will state the following two theorems without proof.

THEOREM 5.11. *Any prime filter of the second type is a prime filter of the third type.*

THEOREM 5.12 (Extension property for prime filters of the third type). *Let* $\mathfrak{A}$ *be a strong quasi-ordered residuated system and let* F *and* G *be filter of* $\mathfrak{A}$ *such that* $F \subseteq G$*. If* F *is a prime filter of the third type, then* G *is a prime filter of the third type also.*

COROLLARY 5.1. *If* $\{1\}$ *is a prime filter of the third type of a strong quasi-ordered residuated system* $\mathfrak{A}$*, then every filter in* $\mathfrak{A}$ *is a prime filter of the third type in* $\mathfrak{A}$.

The following example shows that a prime filter of the first type does not have to be a prime filter of the third type.

EXAMPLE 5.8. Let $\mathfrak{A}$ be as in Example 5.4. The subset $F_2 := \{1, c\}$ is a prime filter of the second type of $\mathfrak{A}$. Thus, F_2 is a prime filter of the third type of $\mathfrak{A}$ also, by Theorem 5.11. On the other side, $F_1 := \{1, b\}$ is a prime filter of the first type but it is not a prime filter of the second type (see Example 5.4). Direct verification it can show that F_1 is not a prime filter of the first third type, too. Indeed, for example, for $x = a$ and $y = b$, we have $a \to b = c$ and $b \to a = c$ but $(x \to y) \sqcup (y \to x) = c \sqcup c = c \notin F_1$.

The following example shows that a prime filter of the third type of a strong quasi-ordered residuated system does not have to be a prime filter of the second type. Also, this example shows that a prime filter of the third type of a strong quasi-ordered residuated system does not have to be a prime filter of the first type.

EXAMPLE 5.9. Let $A = \{1, a, b, c, d\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	a	a	a	a
c	c	a	a	a	a
d	d	a	a	a	a

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	b	1	1	1
c	1	c	c	1	d
d	1	d	d	c	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows $\preccurlyeq := \{(a,1), (a,a), (a,b), (a,c), (a,d), (b,1), (b,b), (b,c), (b,d), (c,1), (c,c), (d,1), (d,d)\}$. Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system. Here it is $(c \to d) \to d = d \to d = 1$, and $(d \to c) \to c = c \to c = 1$. Subset $F := \{1\}$ is a filter of $\mathfrak{A}$. So $c \sqcup d = 1$. Obviously, this filter is not a prime filter of the first type because $c \sqcup d = 1 \in F$ but $c \notin F$ and $d \notin F$. It can be shown by direct verification that F is a prime filter of the third type of $\mathfrak{A}$. Also, this filter is not a prime filter of the second type, because for $x = c$ and $y = d$ we have $x \to y = c \to d = d \notin F$ and $y \to x = d \to c = c \notin F$.

Of course, at the end of the section in the following example we show that there is a filter in a strong quasi-ordered residuated system that it is not a prime filter of any kind listed in this article.

EXAMPLE 5.10. Let $\mathfrak{A}$ be as in Example 5.3. Subset $F := \{1\}$ is a filter in $\mathfrak{A}$. But:

(i) F is not a prime filter of the first type of $\mathfrak{A}$ because $c \sqcup d = 1 \in F$ but $c \notin F$ and $d \notin F$.

(ii) F is not a prime filter of the second type of $\mathfrak{A}$ because for example, we have $b \to c = d \notin F$ and $c \to b = d \notin F$.

(iii) The filter F is not a prime filter of the third type of $\mathfrak{A}$ because for example, we have $(b \to c) \sqcup (c \to b) = d \notin F$.

5.2. Irreducible filter in QRS. The concept of irreducible filter in a strong quasi-ordered residuated system as well as its analysis has been the subject of study in the paper [**23**].

Definition 5.5. ([**23**], Definition 10) A filter F of a quasi-ordered residuated system $\mathfrak{A}$ is said to be an *irreducible filter* in $\mathfrak{A}$ if if for any filters S and T of $\mathfrak{A}$ the following implication holds

(IRF) $(F = S \cap T \implies (F = S \vee F = T))$.

Theorem 5.13. *Any prime filter of a strong quasi-ordered residuated system is an irreducible filter.*

Example 5.11. Let $A = \{1, a, b, c\}$ and operations '$\cdot$' and '$\to$' defined on A as follows:

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

and

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	b	b	1

Then $\mathfrak{A} = \langle A, \cdot, \to, 1\rangle$ is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows

$$\preccurlyeq := \{(1,1), (a,1), (b,1), (c,1), (a,a), (b,b), (c,c), (a,b), (a,c)\}.$$

Direct verification it can prove that $\mathfrak{A}$ is a strong quasi-ordered residuated system. Then the subsets $\{1\}$, $\{1, b\}$ and $\{1, c\}$ are filters in $\mathfrak{A}$ and $\{1\} = \{1, b\} \cap \{1, c\}$ holds because the filter $\{1\}$ is not a prime filter.

The concept of weakly irreducible filters in a strong quasi-ordered residuated system as well as its analysis has been the subject of study in the paper [**25**]. Now, we introduce the notion of 'weakly irreducible filters' in a strong quasi-ordered residuated system by weakening the condition that determine the concept of irreducible filters.

Definition 5.6. ([**25**], Definition 3.1) Let F be a proper filter of a strong quasi-ordered residuated system $\mathfrak{A}$. F is a *weakly irreducible filter* in $\mathfrak{A}$ if and only if for all filter S and T of A such that $S \cap T \subseteq F$ the following holds $S \subseteq F$ or $T \subseteq F$.

The following theorem gives one important characterization of weakly irreducible filters in strong quasi-ordered residuated systems.

Theorem 5.14. *Let F be a proper filter in a quasi-ordered residuated system $\mathfrak{A}$. Then F is weakly irreducible if and only if the following conditions hold*

(WIrF) $(\forall x, y \in A)((x \notin F \wedge y \notin F) \implies (\exists z \notin F)(x \leqslant z \wedge y \leqslant z))$.

THEOREM 5.15. *Every weakly irreducible filter in a strong quasi-ordered residuated system is a prime filter.*

THEOREM 5.16. *Every weakly irreducible filter in a strong quaso-ordered residuated system is an irreducible filter.*

REMARK 5.3. The proof of Theorem 5.16 can be derived without reference to Theorem 5.15. Indeed, let F be a weakly irreducible filter in $\mathfrak{A}$. Let S and T be filters in $\mathfrak{A}$ such that $F = S \cap T$. Then $F \subseteq S$ and $F \subseteq T$. If we assume that $F \neq S$ and $F \neq T$, then there are elements $x \in S$ and $y \in T$ such that $x \notin F$ and $y \notin F$. Since F is a weakly irreducible filter in $\mathfrak{A}$, by Theorem 5.14, there is an element $z \notin F$ with $x \leqslant z$ and $y \leqslant z$. On the other hand, we have $z \in S \cap T = F$ which is a contradiction. So, F is irreducible.

As shown in [**23**], any prime filter in a strong quasi-ordered residuated system is an irreducible filter. In this paper it is shown that any weakly irreducible filter is a prime filter and, therefore, an irreducible filter. It is quite natural to ask the question: When will the converse be valid? One of the possible answers can be recognized immediately:

THEOREM 5.17. *If the lattice $\mathfrak{F}(A)$ of a strong quasi-ordered residuated system $\mathfrak{A}$ is distributive, then any irreducible filter in $\mathfrak{A}$ is a weakly irreducible filter in $\mathfrak{A}$.*

The result presented in Theorem 5.17 opened more questions than it offered answers of interest for this research. For example:

- Since $(\mathfrak{A}, \sqcup)$ is an upper semi-lattice if $\mathfrak{A}$ is a strong quasi-ordered residuated system, can the general results on semi-lattices be applied in this case?
- Is it possible and how to design a modification of Stone's theorem from 1936 (M. H. Stone. The theory of representation for Boolean algebras. *Trans. Am. Math. Soc.*, 40 (1) (1936), 37-111)? See also [**7**], Part II, Chapter 1, Theorem 15 (pp. 63) and Chapter 5, Lemma 2 (pp. 100).
- If $\mathfrak{A}$ is a strong quasi-ordered residuard system, does every proper filter is the intersection of some irreducible filters?

5.3. Pre-linear strong quasi-ordered residuated systems. Here, we analyze a strong quasi-ordered residuated system that satisfies the pre-linearity condition

(U) $(\forall x, y \in A)((x \rightarrow y) \sqcup (y \rightarrow x) \equiv_{\preccurlyeq} 1)$.

The material presented in this subsection is taken from the paper [**29**].

EXAMPLE 5.12. Any Weisberg hoops is a pre-linear strong quasi-ordered residuated system. The opposite does not have to be true, because in the formula (6) the equality does not have to be present and, moreover, in the general case, it need not be valid $(\forall x, y \in A)(x \cdot (x \rightarrow y) \equiv_{\preccurlyeq} y \cdot (y \rightarrow x))$.

THEOREM 5.18. *Every non-empty filter in a pre-linear strong quasi-ordered residuated system $\mathfrak{A}$ is a filter of the third type.*

In addition, we also conclude:

THEOREM 5.19. *In any pre-linear strong quasi-ordered residuated system prime filters of the first type and prime filters of the second type are coincide.*

The following questions remains open:

QUESTION 1. *Is the algebraic structure designed in this way a MTL-algebra?*

This question seems to be justified in view of Remark 2.1. If the answer to the previous question is no, then:

QUESTION 2. *If prime filters of the first type and prime filters of the second type in a strong quasi-ordered residuated system coincide, is that system a pre-linear system?*

The treatment of this question in the case when $\mathfrak{A}$ is a commutative residuated lattice can be found in [**33, 10**]. In [**10**] it is proved (Theorem 2) that if the prime filters of the first type and prime filters of the second type of a residuated lattice $\mathfrak{L}$ coincide, then $\mathfrak{L}$ is a MTL-algebra and vice versa. As in [**33**], here it can be shown that if 1 is $\sqcup$-irreducible in a strong quasi-ordered residuated system $\mathfrak{A}$, then from the condition that the prime filters of the first type and prime filters of the second type in $\mathfrak{A}$ coincide, it follows that $\mathfrak{A}$ is pre-linear.

In what follows we need an explanation of the slogan 1 is $\sqcup$-irreducible: If $\mathfrak{A}$ is a strong quasi-ordered residuated system, then the greatest element 1 in $\mathfrak{A}$ is $\sqcup$-irreducible if the following holds

$$(\forall x, y \in A)(x \sqcup y \equiv_{\preccurlyeq} 1 \iff (x \equiv_{\preccurlyeq} 1 \vee y \equiv_{\preccurlyeq} 1)).$$

THEOREM 5.20. *Let $\mathfrak{A}$ be a strong quasi-ordered residuated system in which the class of all prime filters of the first type coincides with the class of all prime filters of the second type. If the greatest element* 1 *of* $\mathfrak{A}$ *is* $\sqcup$*-irreducible, then* $\mathfrak{A}$ *is pre-linear.*

6. Conclusion and further work

The concept of quasi-ordered residuated system was introduced in [**4**] by Bonzio and Chajda. The concept of filters in this algebraic structure as well as various types of filters were introduced by the author ([**17, 18, 19, 20, 22, 27, 28**]). The notion strong quasi-ordered residuated systems it is designed to form an environment in which implicative and comparative filters coincide ([**21**]). In such algebraic structure, the author designed the notions of prime and irreducible filters ([**23**]).

In paper [**26**], as a continuation of previous research, the author has dealt with the possibility of establishing three different concepts of prime filters in a strong quasi-ordered residuared system. The situation with prime filters in commutative residuated lattice $(A, \cdot, \rightarrow, \wedge, \vee, 0, 1, \preccurlyeq)$, where $\preccurlyeq$ is a quasi-ordered on A is different from the situation presented here. To this end, in order to gain insight into the types of prime filters in commutative residuated lattice, the reader can look at articles [**33, 10**].

In further research of these algebraic structures, one could, among other things, pay attention to the conditions that would lead to some of the types of prime filters coincide. Judging by the results obtained in this and some of the previous research, more attention should be paid to strong quasi-ordered residuated systems in which $\preccurlyeq$ is a linear relation.

It is possible to design an algebraic structure $(A, \cdot, \rightarrow, 1, \sqcup, \preccurlyeq)$ which has the following properties

(a) $(A, \cdot, 1)$ is a commutative monoid;
(b) $(\forall x, y, z \in A)(x \preccurlyeq z \iff x \preccurlyeq y \rightarrow z)$;
(c) $(A, \sqcup, 1)$ is a distributive upper semi-lattice; and
(d) $(\forall x, y \in A)((x \rightarrow y) \sqcup (y \rightarrow x) = 1)$.

Thus, it is an algebraic structure in which the last lower bound for a pair of elements does not have to be determined and it does not have to be bounded from below. The algebraic structure designed in this way is reminiscent of the determination of MTL-algebra in which the constraint requirements on the underside are omitted and, and moreover, it does not have to be a lower semi-lattice. Thus, an algebraic structure designed in this way would be an incomplete MTL-algebra. This reasoning could be accepted as a justification for studying such algebraic structures. Of course, it would be a generalization of MTL-algebra. In such an algebraic structure, any filter would be a prime filter of the third type. At the same time, each a prime filter of the first type would be a prime filter of the second type.

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