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ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR COUPLED INTEGRO-DIFFERENTIAL SYSTEMS WITH TWO VARIABLE DELAYS

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Abstract. The purpose of this paper is to investigate the existence of periodic solutions of a system of coupled nonlinear integro-differential equations with variable delays. By applying the Krasnoselskii fixed point theorem and under sufficient conditions we establish the existence of such solutions. An example is given to illustrate our results.

1. Introduction

Existence of periodic solutions of differential equations have been investigated extensively in recent times, we refer the interested reader to the references [**1**]-[**8**], [**10**], [**12**], [**14**]-[**16**], [**19**]-[**21**] in this article and references therein for a wealth of information on this subject. In 2019, in the paper [**11**] Gabsi, Ardjouni and Djoudi use the Mawhin coincidence degree theory and the Krasnoselskii fixed point theorem, to obtained the existence of positive periodic solutions of the neutral nonlinear differential system

$$\begin{cases} x'(t) = \beta x'\left(t-\tau\left(t\right)\right) + f\left(x\left(t-\tau\left(t\right)\right)\right) + g\left(u\left(t-\sigma\left(t\right)\right)\right) + p\left(t\right), \\ u'(t) = -a\left(t\right)u\left(t\right) + \frac{d}{dt}F\left(t, u\left(t-\sigma\left(t\right)\right)\right) \\ \qquad +c\left(t\right)G\left(t, x\left(t-\tau\left(t\right)\right), u\left(t-\sigma\left(t\right)\right)\right). \end{cases}$$

In the paper [**17**] Raffoul investigate the existence of asymptotically periodic and periodic solutions for the following coupled nonlinear Volterra integro-differential

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system with infinite delay

$$\begin{cases} x'(t) = h_1(t)x(t) + h_2(t)y(t) + \int_{-\infty}^{t} a(t,s) f(x(s), y(s)) ds, \\ y'(t) = p_1(t)y(t) + p_2(t)x(t) + \int_{-\infty}^{t} b(t,s) g(x(s), y(s)) ds. \end{cases}$$

The author use the Schauder fixed point theorem to obtained here results. Deham [**9**] consider the second order nonlinear integro-differential equation

$$x''(t) + p(t)x'(t) + q(t)h(x(t)) = \int_{-\infty}^{t} Q(t,s) f(s, x(s - g(s))) ds,$$

and by the Krasnoselskii-Burton fixed point theorem show that the existence of periodic solutions is concluded. In this paper and motivated by the papers [**9**], [**11**], [**17**] and the references therein and by using fixed point technique, we study the existence of periodic solutions for the following coupled nonlinear integro-differential system

(1.1)
$$\begin{cases} x'(t) = h(t)x(t) + g_1(t, x(t), y(t), x(t-\tau_1(t)), y(t-\tau_2(t))) \\ +c_1(t)x'(t-\tau_1(t)) + \int_{-\infty}^{t} a_1(t,s) f_1(x(s), y(s)) ds, \\ y''(t) + p(t)y'(t) + q(t)y(t) = g_2(t, x(t), y(t), x(t-\tau_1(t)), y(t-\tau_2(t))) \\ +c_2(t)y'(t-\tau_2(t)) + \int_{-\infty}^{t} a_2(t,s) f_2(x(s), y(s)) ds, \end{cases}$$

where p and q are positive continuous real-valued functions and the functions h, c_i, a_i, $i = 1, 2$ are continuous functions. The functions $g_i(t, x, y, z, w)$, $i = 1, 2$ are continuous, periodic in t and Lipschitz continuous in x, y, z and w, $f_i(x, y)$, $i = 1, 2$ are continuous and Lipschitz continuous in x and y, and for some positive constants η_j, μ_j, $j = \overline{1,4}$ we have

$$|g_1(t, y_1, y_2, y_3, y_4) - g_1(t, x_1, x_2, x_3, x_4)| \leqslant \sum_{j=1}^{4} \eta_j |y_j - x_j|,$$

$$|g_2(t, y_1, y_2, y_3, y_4) - g_2(t, x_1, x_2, x_3, x_4)| \leqslant \sum_{j=1}^{4} \mu_j |y_j - x_j|,$$

and for some positive constants ρ_j, ξ_j, $j = 1, 2$ we have

$$|f_1(y_1, y_2) - f_1(x_1, x_2)| \leqslant \sum_{j=1}^{2} \rho_j |y_j - x_j|,$$

and

$$|f_2(y_1, y_2) - f_2(x_1, x_2)| \leqslant \sum_{j=1}^{2} \xi_j |y_j - x_j|.$$

We assume that $g_1(t, 0, 0, 0, 0) = g_2(t, 0, 0, 0, 0) = f_1(0, 0) = f_2(0, 0) = 0$.

We also suppose that there exists a positive real number T such that

$$\begin{cases} h(t+T) = h(t), \ \int_0^T h(s)ds \neq 0, \ a_i(t+T, s+T) = a_i(t,s), \\ c_i(t+T) = c_i(t), \ \tau_i(t+T) = \tau_i(t), \ i = 1, 2, \end{cases} \tag{1.2}$$

for $t \in \mathbb{R}$, with τ_i being scalar functions, continuous and $\tau_i(t) \geqslant \tau_i^* > 0$, $\tau_i'(t) \neq 1$.

To have a well behaved mapping we must assume that

$$p\left(t+T\right)=p\left(t\right),\ q\left(t+T\right)=q\left(t\right),\ \int_0^T p(s)ds>0,\ \int_0^T q(s)ds>0. \tag{1.3}$$

Define

$$P_T=\left\{(\varphi,\psi):(\varphi,\psi)\left(t+T\right)=(\varphi,\psi)\left(t\right)\right\},$$

where φ and ψ are continuous functions on $\mathbb{R}$. Then, P_T is a Banach space when endowed with the maximum norm

$$\|(x,y)\|=\max\left\{\max_{t\in[0,T]}\left|x\left(t\right)\right|,\ \max_{t\in[0,T]}\left|y\left(t\right)\right|\right\}.$$

Lemma 1.1 ([**13**]). *Assume that (1.2) and (1.3) hold and*

$$\frac{R}{QT}\left(e^{\int_0^T p(u)du}-1\right)\geqslant 1, \tag{1.4}$$

where

$$R=\max_{t\in[0,T]}\left|\int_t^{t+T}\frac{e^{\int_t^s p(u)du}}{e^{\int_0^T p(u)du}-1}q\left(s\right)ds\right|,\ Q=\left(1+e^{\int_0^T p(u)du}\right)^2R^2.$$

Then, there are continuous T-periodic functions a and b such that

$$b\left(t\right)>0,\ \int_0^T a(u)du>0,$$

and

$$a\left(t\right)+b\left(t\right)=p\left(t\right),\ b'\left(t\right)+a\left(t\right)b\left(t\right)=q\left(t\right),\ \textit{for } t\in\mathbb{R}.$$

Lemma 1.2 ([**22**]). *Assume the conditions of Lemma 1.1 hold and $\phi\in P_T$. Then, the equation*

$$x''\left(t\right)+p\left(t\right)x'\left(t\right)+q\left(t\right)x\left(t\right)=\phi\left(t\right),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x\left(t\right)=\int_t^{t+T}G\left(t,s\right)\phi\left(s\right)ds,$$

where

$$G\left(t,s\right)=\frac{\int_t^s e^{\int_t^u b(v)dv+\int_u^s a(v)dv}du+\int_s^{t+T}e^{\int_t^u b(v)dv+\int_u^{s+T} a(v)dv}du}{\left(e^{\int_0^T a(u)du}-1\right)\left(e^{\int_0^T b(u)du}-1\right)}.$$

Corollary 1.1 ([**22**]). *Green's function G satisfies the following properties*

$$G\left(t,t+T\right)=G\left(t,t\right),\ G\left(t+T,s+T\right)=G\left(t,s\right),$$
$$\frac{\partial}{\partial s}G\left(t,s\right)=a\left(s\right)G\left(t,s\right)-H\left(t,s\right),$$
$$\frac{\partial}{\partial t}G\left(t,s\right)=-b\left(t\right)G\left(t,s\right)+H^*\left(t,s\right),$$

where

$$H(t,s)=\frac{e^{\int_t^s b(v)dv}}{e^{\int_0^T b(v)dv}-1},\ H^*(t,s)=\frac{e^{\int_t^s a(v)dv}}{e^{\int_0^T a(v)dv}-1}.$$

LEMMA 1.3. *Assume (1.2)-(1.4). If $x,y\in P_T$, then x and y is a solution of (1.1) if and only if*

$$\begin{aligned}x(t)&=\frac{c_1(t)\,x(t-\tau_1(t))}{1-\tau_1'(t)}-\int_t^{t+T}\frac{e^{\int_u^{t+T}h(s)ds}}{1-e^{\int_0^T h(s)ds}}r_1(u)\,x(u-\tau_1(u))\,du\\&+\int_t^{t+T}\frac{e^{\int_u^{t+T}h(s)ds}}{1-e^{\int_0^T h(s)ds}}g_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))\,du\\&+\int_t^{t+T}\frac{e^{\int_u^{t+T}h(s)ds}}{1-e^{\int_0^T h(s)ds}}\int_{-\infty}^u a_1(u,s)\,f_1(x(s),y(s))\,dsdu,\end{aligned}\tag{1.5}$$

and

$$\begin{aligned}y(t)&=\int_t^{t+T}G(t,u)\,g_2(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))\,du\\&+\int_t^{t+T}G(t,u)\int_{-\infty}^u a_2(u,s)\,f_2(x(s),y(s))\,dsdu\\&+\int_t^{t+T}[h_2(u)\,H(t,u)-r_2(u)\,G(t,u)]\,y(u-\tau_2(u))\,du,\end{aligned}\tag{1.6}$$

where

$$r_1(u)=\frac{(c_1'(u)-c_1(u)\,h(u))\,(1-\tau_1'(u))+\tau_1''(u)\,c_1(u)}{(1-\tau_1'(u))^2},\tag{1.7}$$

$$h_2(u)=\frac{c_2(u)}{1-\tau_2'(u)},\tag{1.8}$$

$$r_2(u)=\frac{\left(a(u)\,c_2(u)+c_2'(u)\right)(1-\tau_2'(u))+\tau_2''(u)\,c_2(u)}{(1-\tau_2'(u))^2}.\tag{1.9}$$

PROOF. Let $x,y\in P_T$ be a solution of (1.1). Next, we multiply both sides of the first equation in (1.1) by $e^{-\int_0^t h(s)ds}$ and then integrate from t to $t+T$, to obtain

$$\begin{aligned}&\int_t^{t+T}\left[x(u)e^{-\int_0^u h(s)ds}\right]'du\\&=\int_t^{t+T}e^{-\int_0^u h(s)ds}g_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))\,du\\&+\int_t^{t+T}e^{-\int_0^u h(s)ds}c_1(u)\,x'(u-\tau_1(u))\,du\\&+\int_t^{t+T}e^{-\int_0^u h(s)ds}\int_{-\infty}^u a_1(u,s)\,f_1(x(s),y(s))\,dsdu.\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&x(t+T)e^{-\int_0^{t+T} h(s)ds} - x(t)e^{-\int_0^{t} h(s)ds} \\
&= \int_t^{t+T} e^{-\int_0^{u} h(s)ds} g_1\left(u, x\left(u\right), y\left(u\right), x\left(u-\tau_1\left(u\right)\right), y\left(u-\tau_2\left(u\right)\right)\right) du \\
&+ \int_t^{t+T} e^{-\int_0^{u} h(s)ds} c_1\left(u\right) x'\left(u-\tau_1\left(u\right)\right) du \\
&+ \int_t^{t+T} e^{-\int_0^{u} h(s)ds} \int_{-\infty}^{u} a_1\left(u,s\right) f_1\left(x\left(s\right), y\left(s\right)\right) ds du.
\end{aligned}$$

Multiply both sides with $e^{\int_0^{t+T} h(s)ds}$ and using the fact that $x(t+T) = x(t)$ and $e^{\int_t^{t+T} h(s)ds} = e^{\int_0^{T} h(s)ds}$, we obtain

$$\begin{aligned}
x(t) &= \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^{T} h(s)ds}} g_1\left(u, x\left(u\right), y\left(u\right), x\left(u-\tau_1\left(u\right)\right), y\left(u-\tau_2\left(u\right)\right)\right) du \\
&+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^{T} h(s)ds}} c_1\left(u\right) x'\left(u-\tau_1\left(u\right)\right) du \\
&+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^{T} h(s)ds}} \int_{-\infty}^{u} a_1\left(u,s\right) f_1\left(x\left(s\right), y\left(s\right)\right) ds du. \qquad (1.10)
\end{aligned}$$

Letting

$$\begin{aligned}
&\int_t^{t+T} e^{\int_u^{t+T} h(s)ds} c_1\left(u\right) x'\left(u-\tau_1\left(u\right)\right) du \\
&= \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds} c_1\left(u\right)}{1-\tau_1'\left(u\right)} \left(1-\tau_1'\left(u\right)\right) x'\left(u-\tau_1\left(u\right)\right) du,
\end{aligned}$$

performing an integration by parts, we get

$$\begin{aligned}
&\int_t^{t+T} e^{\int_u^{t+T} h(s)ds} c_1\left(u\right) x'\left(u-\tau_1\left(u\right)\right) du \\
&= \left[\frac{c_1\left(u\right) x\left(u-\tau_1\left(u\right)\right)}{1-\tau_1'\left(u\right)} e^{\int_u^{t+T} h(s)ds}\right]_t^{t+T} - \int_t^{t+T} e^{\int_u^{t+T} h(s)ds} r_1\left(u\right) x\left(u-\tau_1\left(u\right)\right) du \\
&= \frac{c_1\left(t\right) x\left(t-\tau_1\left(t\right)\right)}{1-\tau_1'\left(t\right)} \left(1-e^{\int_0^{T} h(s)ds}\right) - \int_t^{t+T} e^{\int_u^{t+T} h(s)ds} r_1\left(u\right) x\left(u-\tau_1\left(u\right)\right) du, \qquad (1.11)
\end{aligned}$$

where $r_1(u)$ is given by (1.7). Substituting (1.11) into (1.10), we obtain

$$\begin{aligned} x(t) &= \frac{c_1(t)\,x(t-\tau_1(t))}{1-\tau_1'(t)} - \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} r_1(u)\,x(u-\tau_1(u))\,du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} g_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\,du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \int_{-\infty}^{u} a_1(u,s)\,f_1(x(s), y(s))\,dsdu. \end{aligned}$$

For the second equation in (1.1). From Lemma 1.2, we get

$$\begin{aligned} y(t) &= \int_t^{t+T} G(t,u)\,g_2(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\,du \\ &+ \int_t^{t+T} G(t,u)\,c_2(t)\,y'(t-\tau_2(t))\,du \\ &+ \int_t^{t+T} G(t,u) \int_{-\infty}^{u} a_2(u,s)\,f_2(x(s), y(s))\,dsdu, \end{aligned} \tag{1.12}$$

Letting

$$\begin{aligned} &\int_t^{t+T} G(t,u)\,c_2(u)\,y'(u-\tau_2(u))\,du \\ &= \int_t^{t+T} \frac{G(t,u)\,c_2(u)}{1-\tau_2'(u)} (1-\tau_2'(u))\,y'(u-\tau_2(u))\,du, \end{aligned}$$

performing an integration by parts, we get

$$\begin{aligned} \int_t^{t+T} G(t,u)\,c_2(u)\,y'(u-\tau_2(u))\,du &= \left[\frac{G(t,u)\,c_2(u)\,y(u-\tau_2(u))}{1-\tau_2'(u)}\right]_t^{t+T} \\ - \int_t^{t+T} [r_2(u)\,G(t,u) - h_2(u)\,H(t,u)]\,y(u-\tau_2(u))\,du. \end{aligned}$$

Since

$$\left[\frac{G(t,u)\,c_2(u)\,y(u-\tau_2(u))}{1-\tau_2'(u)}\right]_t^{t+T} = 0,$$

we obtain

$$\begin{aligned} &\int_t^{t+T} G(t,u)\,c_2(u)\,y'(u-\tau_2(u))\,du \\ &= \int_t^{t+T} [h_2(u)\,H(t,u) - r_2(u)\,G(t,u)]\,y(u-\tau_2(u))\,du. \end{aligned} \tag{1.13}$$

where $h_2(u)$ and $r_2(u)$ are given by (1.8) and (1.9). Substituting (1.13) into (1.12), we obtain

$$\begin{aligned} y(t) &= \int_t^{t+T} G(t,u)\, g_2(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\, du \\ &+ \int_t^{t+T} G(t,u) \int_{-\infty}^{u} a_2(u,s)\, f_2(x(s), y(s))\, ds du \\ &+ \int_t^{t+T} [h_2(u) H(t,u) - r_2(u) G(t,u)]\, y(u-\tau_2(u))\, du. \end{aligned}$$

□

Lemma 1.4 ([**22**]). *Let* $\Gamma = \int_0^T p(u)\,du$, $\Lambda = T^2 e^{\frac{1}{T}\int_0^T \ln(q(u))du}$. *If* $\Gamma^2 \geqslant 4\Lambda$, *then we have*

$$\min\left\{\int_0^T a(u)\,du,\ \int_0^T b(u)\,du\right\} \geqslant \frac{1}{2}\left(\Gamma - \sqrt{\Gamma^2 - 4\Lambda}\right) := l,$$

$$\max\left\{\int_0^T a(u)\,du,\ \int_0^T b(u)\,du\right\} \leqslant \frac{1}{2}\left(\Gamma + \sqrt{\Gamma^2 - 4\Lambda}\right) := m.$$

Corollary 1.2 ([**22**]). *Functions* G *and* H *satisfy*

$$\frac{T}{(e^m - 1)^2} \leqslant G(t,s) \leqslant \frac{T e^{\int_0^T p(v)dv}}{(e^l - 1)^2},\ |H(t,s)| \leqslant \frac{e^m}{e^l - 1}.$$

To simplify notation, we introduce the constants

$$\alpha = \frac{T e^{\int_0^T p(v)dv}}{(e^l - 1)^2},\ \gamma = \frac{e^m}{e^l - 1},\ \theta = \max_{t\in[0,T]} |h_2(t)|,$$
$$\beta = \max_{t\in[0,T]} |r_2(t)|,\ \lambda = \max_{t\in[0,T]} |a(t)|,\ \delta = \max_{t\in[0,T]} |b(t)|.$$

2. Periodic solutions

Lemma 2.1 ([**18**]). *Let* $\mathbb{M}$ *be a bounded closed convex nonempty subset of a Banach space* $(S, \|\cdot\|)$. *Assume that* A *and* B *map* $\mathbb{M}$ *into* S *such that*

(i) $x, y \in \mathbb{M}$, *implies* $Ax + By \in \mathbb{M}$,

(ii) A *is continuous and compact,*

(iii) B *is a contraction mapping.*

Then, there exists $z \in \mathbb{M}$ *with* $z = Az + Bz$.

Let $\alpha_1(t) = \frac{c_1(t)}{1-\tau_1'(t)}$, we assume that $\sup_{t\geqslant u} |\alpha_1(t)| = \beta_1 < 1$. Let V_i, $i = 1, 2$ be a positive constants such that $0 < V_1 + \beta_1 < 1$. Moreover, assume the existence of positive constants M_i, K_i, L_i, $i = 1, 2$ and σ such that

$$|f_1(x,y)| \leqslant M_1, \tag{2.1}$$

$$|f_2(x,y)| \leqslant M_2, \tag{2.2}$$

$$|g_1(t,x,y,z,w)| \leqslant K_1,\ |g_2(t,x,y,z,w)| \leqslant K_2, \tag{2.3}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \right| \int_{-\infty}^u |a_1(u,s)| \, ds du \leqslant L_1, \tag{2.4}$$

$$\int_t^{t+T} \int_{-\infty}^u |a_2(u,s)| \, ds du \leqslant L_2, \tag{2.5}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \right| |r_1(u)| \, du \leqslant V_1, \tag{2.6}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \right| du \leqslant V_2, \tag{2.7}$$

and

$$\int_{-\infty}^u |a_1(u,s)| \, ds \leqslant \sigma. \tag{2.8}$$

Set

$$M = \max \left\{ \frac{V_2 K_1 + L_1 M_1}{1 - V_1 - \beta_1}, \frac{(T K_2 + L_2 M_2)\alpha}{1 - T(\theta\gamma + \beta\alpha)} \right\}, \tag{2.9}$$

with $0 < T(\theta\gamma + \beta\alpha) < 1$.

We define subset $\Omega_{x,y}$ of P_T as follows

$$\Omega_{x,y} = \{(x,y) : (x,y) \in P_T \text{ with } \|(x,y)\| \leqslant M\}.$$

Then $\Omega_{x,y}$ is a bounded, closed and convex subset of P_T. Now for $(x,y) \in \Omega_{x,y}$ we can define an operator $E : \Omega_{x,y} \to P_T$ by

$$E(x,y)(t) = (E_1(x,y)(t), \ E_2(x,y)(t)),$$

where

$$\begin{aligned} E_1(x,y)(t) &= \frac{c_1(t)\, x(t-\tau_1(t))}{1-\tau_1'(t)} - \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} r_1(u)\, x(u-\tau_1(u))\, du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} g_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\, du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \int_{-\infty}^u a_1(u,s) f_1(x(s), y(s))\, ds du, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} E_2(x,y)(t) &= \int_t^{t+T} G(t,u)\, g_2(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\, du \\ &+ \int_t^{t+T} G(t,u) \int_{-\infty}^u a_2(u,s) f_2(x(s), y(s))\, ds du \\ &+ \int_t^{t+T} [h_2(u) H(t,u) - r_2(u) G(t,u)]\, y(u-\tau_2(u))\, du. \end{aligned} \tag{2.11}$$

To apply Lemma 2.1, we need to construct two mappings, one is a contraction and the other is compact. Therefore, we state (2.10) as

$$E_1(x,y)(t) = B_1(x,y)(t) + A_1(x,y)(t),$$

where $B_1, A_1 : \Omega_{x,y} \to P_T$ are given by

$$B_1(x,y)(t) = \frac{c_1(t)\,x(t-\tau_1(t))}{1-\tau_1'(t)},$$

and

$$\begin{aligned} A_1(x,y)(t) &= -\int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} r_1(u)\,x(u-\tau_1(u))\,du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} g_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\,du \\ &+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \int_{-\infty}^{u} a_1(u,s)\,f_1(x(s), y(s))\,ds du. \end{aligned}$$

And we state (2.11) as

$$E_2(x,y)(t) = B_2(x,y)(t) + A_2(x,y)(t),$$

where $B_2, A_2 : \Omega_{x,y} \to P_T$ are given by

$$B_2(x,y)(t) = \int_t^{t+T} G(t,u)\,g_2(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u)))\,du,$$

and

$$\begin{aligned} A_2(x,y)(t) &= \int_t^{t+T} G(t,u) \int_{-\infty}^{u} a_2(u,s)\,f_2(x(s), y(s))\,ds du \\ &+ \int_t^{t+T} [h_2(u)\,H(t,u) - r_2(u)\,G(t,u)]\,y(u-\tau_2(u))\,du. \end{aligned}$$

Now for $(x,y) \in \Omega_{x,y}$ we can define the operators $B, A : \Omega_{x,y} \to P_T$ by

$$B(x,y)(t) = (B_1(x,y)(t),\ B_2(x,y)(t)),$$

$$A(x,y)(t) = (A_1(x,y)(t),\ A_2(x,y)(t)).$$

Observe that, since the functions $g_i(t, x_1, x_2, x_3, x_4),\ i = 1,2$ is Lipschitz continuous in x_1, x_2, x_3, x_4 and $f_i(x_1, x_2),\ i = 1,2$ is Lipschitz continuous in x_1, x_2 we have

$$\begin{aligned} &|g_1(t, x_1, x_2, x_3, x_4)| \\ &= |g_1(t, x_1, x_2, x_3, x_4) - g_1(t,0,0,0,0) + g_1(t,0,0,0,0)| \\ &\leqslant |g_1(t, x_1, x_2, x_3, x_4) - g_1(t,0,0,0,0)| + |g_1(t,0,0,0,0)| \\ &\leqslant \sum_{j=1}^{4} \eta_j\,|x_j|, \end{aligned}$$

$$\begin{aligned}
&|g_2(t,x_1,x_2,x_3,x_4)| \\
&= |g_2(t,x_1,x_2,x_3,x_4) - g_2(t,0,0,0,0) + g_2(t,0,0,0,0)| \\
&\leqslant |g_2(t,x_1,x_2,x_3,x_4) - g_2(t,0,0,0,0)| + |g_2(t,0,0,0,0)| \\
&\leqslant \sum_{j=1}^{4} \mu_j |x_j|,
\end{aligned}$$

$$\begin{aligned}
&|f_1(x_1,x_2)| \\
&= |f_1(x_1,x_2) - f_1(0,0) + f_1(0,0)| \\
&\leqslant |f_1(x_1,x_2) - f_1(0,0)| + |f_1(0,0)| \leqslant \sum_{j=1}^{2} \rho_j |x_j|,
\end{aligned}$$

and

$$\begin{aligned}
&|f_2(x_1,x_2)| \\
&= |f_2(x_1,x_2) - f_2(0,0) + f_2(0,0)| \\
&\leqslant |f_2(x_1,x_2) - f_2(0,0)| + |f_2(0,0)| \leqslant \sum_{j=1}^{2} \xi_j |x_j|.
\end{aligned}$$

THEOREM 2.1. *Suppose (1.2)-(1.4) and (2.1)-(2.8) hold. Suppose that*

$$V_1 + V_2 \sum_{j=1}^{4} \eta_j + L_1 \sum_{j=1}^{2} \rho_j \leqslant 1 \text{ and } L_2 \alpha \sum_{j=1}^{2} \xi_j + T(\theta\gamma + \beta\alpha) \leqslant 1,$$

and $T\alpha V < 1$, *where* $V = \max(\mu_1 + \mu_3, \mu_2 + \mu_4)$. *Then (1.1) has a* T*-periodic solution.*

PROOF. In order to prove that (1.1) has a T-periodic solution, we shall make sure that A and B satisfy the conditions of Lemma 2.1. For all $(x,y) \in \Omega_{x,y}$, we have $(x,y)(t+T) = (x,y)(t)$ and $\|(x,y)\| \leqslant M$. It is easy to show that $E(x,y)(t+T) = E(x,y)(t)$. For any $(x,y) \in \Omega_{x,y}$, we will show that $|E(x,y)(t)| \leqslant M$. In view of the above estimates, we have

$$|B_1(x,y)(t)| \leqslant \left| \frac{c_1(t)}{1-\tau_1'(t)} \right| |x(t-\tau_1(t))| \leqslant \beta_1 M,$$

and

$$|A_1(x,y)(t)| \leqslant \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \right| |r_1(u)| \, |x(u-\tau_1(u))| \, du$$
$$+ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \right| |g_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))| \, du$$
$$+ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \right| \int_{-\infty}^{u} |a_1(u,s)| \, |f_1(x(s),y(s))| \, dsdu$$
$$\leqslant V_1 M + V_2 K_1 + L_1 M_1.$$

As a consequence of (2.9),

$$\frac{V_2K_1+L_1M_1}{1-V_1-\beta_1} \leqslant M, \text{ we have } V_2K_1+L_1M_1 \leqslant (1-V_1-\beta_1)\, M.$$

This implies that

$$|E_1(x,y)(t)| \leqslant \beta_1 M + V_1 M + V_2 K_1 + L_1 M_1$$
$$\leqslant \beta_1 M + V_1 M + (1-V_1-\beta_1)\, M = M.$$

On the other hand

$$|B_2(x,y)(t)|$$
$$\leqslant \int_t^{t+T} G(t,u)\, |g_2(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))| \, du$$
$$\leqslant TK_2\alpha,$$

and

$$|A_2(x,y)(t)| \leqslant \int_t^{t+T} G(t,u) \int_{-\infty}^{u} |a_2(u,s)| \, |f_2(x(s),y(s))| \, dsdu$$
$$+ \int_t^{t+T} \left[|h_2(u)| \, |H(t,u)| + |r_2(u)|\, G(t,u) \right] |y(u-\tau_2(u))| \, du$$
$$\leqslant L_2 M_2 \alpha + T(\theta\gamma+\beta\alpha)\, M.$$

As a consequence of (2.9),

$$\frac{(TK_2+L_2M_2)\,\alpha}{1-T(\theta\gamma+\beta\alpha)} \leqslant M, \text{ we have } (TK_2+L_2M_2)\,\alpha \leqslant (1-T(\theta\gamma+\beta\alpha))\, M.$$

This implies that

$$|E_2(x,y)(t)| \leqslant TK_2\alpha + L_2M_2\alpha + T(\theta\gamma+\beta\alpha)\, M$$
$$\leqslant (1-T(\theta\gamma+\beta\alpha))\, M + T(\theta\gamma+\beta\alpha)\, M = M.$$

Thus, E maps $\Omega_{x,y}$ into itself, i.e. $E(\Omega_{x,y}) \subseteq \Omega_{x,y}$. We will now show that A is continuous. Let $\{(x_n,y_n)\}$ be a sequence in $\Omega_{x,y}$ such that

$$\lim_{n\to\infty} \|(x_n,y_n)-(x,y)\| = 0.$$

Since $\Omega_{x,y}$ is closed, we get $(x,y) \in \Omega_{x,y}$. Then by the definition of A, we obtain

$$\begin{aligned}\|A(x_n,y_n)-A(x,y)\| = \max\Big\{&\max_{t\in[0,T]}|A_1(x_n,y_n)(t)-A_1(x,y)(t)|\,,\\ &\max_{t\in[0,T]}|A_2(x_n,y_n)(t)-A_2(x,y)(t)|\Big\},\end{aligned}$$

in which

$$\begin{aligned}&|A_1(x_n,y_n)(t)-A_1(x,y)(t)|\\ &\leqslant \int_t^{t+T}\left|\frac{e^{\int_u^{t+T}h(s)ds}}{1-e^{\int_0^T h(s)ds}}\right||r_1(u)|\,|x_n(u-\tau_1(u))-x(u-\tau_1(u))|\,du\\ &+\int_t^{t+T}\left|\frac{e^{\int_u^{t+T}h(s)ds}}{1-e^{\int_0^T h(s)ds}}\right||g_1(u,x_n(u),y_n(u),x_n(u-\tau_1(u)),y_n(u-\tau_2(u)))\\ &-g_1(u,x(u),y(u),x(u-\tau_1(u)),y(u-\tau_2(u)))|\,du\\ &+\int_t^{t+T}\left|\frac{e^{\int_u^{t+T}h(s)ds}}{1-e^{\int_0^T h(s)ds}}\right|\int_{-\infty}^{u}|a_1(u,s)|\,|f_1(x_n(s),y_n(s))-f_1(x(s),y(s))|\,dsdu,\end{aligned}$$

the continuity of g_1 and f_1 along with the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\max_{t\in[0,T]}|A_1(x_n,y_n)(t)-A_1(x,y)(t)|=0.$$

And on the other hand

$$\begin{aligned}&|A_2(x_n,y_n)(t)-A_2(x,y)(t)|\\ &\leqslant \int_t^{t+T}G(t,u)\int_{-\infty}^{u}|a_2(u,s)|\,|f_2(x_n(s),y_n(s))-f_2(x(s),y(s))|\,dsdu\\ &+\int_t^{t+T}[|h_2(u)|\,|H(t,u)|+|r_2(u)|\,G(t,u)]\,|y_n(u-\tau_2(u))-y(u-\tau_2(u))|\,du.\end{aligned}$$

The continuity of f_2 along with the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\max_{t\in[0,T]}|A_2(x_n,y_n)(t)-A_2(x,y)(t)|=0.$$

Thus

$$\lim_{n\to\infty}\|A(x_n,y_n)-A(x,y)\|=0.$$

This result proves that A is continuous.

We now have to show that A is compact. For $n \in \mathbb{N}$, let $(x_n, y_n) \in \Omega_{x,y}$, we have

$$\begin{aligned}
&\left|A_1\left(x_n, y_n\right)(t)\right| \\
&\leqslant \int_t^{t+T}\left|\frac{e^{\int_u^{t+T} h(s) ds}}{1-e^{\int_0^T h(s) ds}}\right|\left|r_1(u)\right|\left|x_n\left(u-\tau_1(u)\right)\right| du \\
&+\int_t^{t+T}\left|\frac{e^{\int_u^{t+T} h(s) ds}}{1-e^{\int_0^T h(s) ds}}\right|\left|g_1\left(u, x_n(u), y_n(u), x_n\left(u-\tau_1(u)\right), y_n\left(u-\tau_2(u)\right)\right)\right| du \\
&+\int_t^{t+T}\left|\frac{e^{\int_u^{t+T} h(s) ds}}{1-e^{\int_0^T h(s) ds}}\right| \int_{-\infty}^u\left|a_1(u, s)\right|\left|f_1\left(x_n(s), y_n(s)\right)\right| ds du \\
&\leqslant\left(V_1+V_2 \sum_{j=1}^4 \eta_j+L_1 \sum_{j=1}^2 \rho_j\right) M \leqslant M .
\end{aligned}$$

And

$$\begin{aligned}
&\left|A_2\left(x_n, y_n\right)(t)\right| \\
&\leqslant \int_t^{t+T} G(t, u) \int_{-\infty}^u\left|a_2(u, s)\right|\left|f_2\left(x_n(s), y_n(s)\right)\right| ds du \\
&+\int_t^{t+T}\left[\left|h_2(u)\right||H(t, u)|+\left|r_2(u)\right| G(t, u)\right]\left|y_n\left(u-\tau_2(u)\right)\right| du \\
&\leqslant\left(L_2 \alpha \sum_{j=1}^2 \xi_j+T(\theta \gamma+\beta \alpha)\right) M \leqslant M .
\end{aligned}$$

Thus

$$\left\|A\left(x_n, y_n\right)\right\| \leqslant M .$$

If we calculate $(A(x_n, y_n))'(t)$, then

$$
\begin{aligned}
&(A_1(x_n, y_n))'(t) \\
&= g_1(t, x_n(t), y_n(t), x_n(t-\tau_1(t)), y_n(t-\tau_2(t))) - r_1(t) x_n(t-\tau_1(t)) \\
&+ \int_{-\infty}^{t} a_1(t,s) f_1(x_n(s), y_n(s)) ds + h(t) \\
&\times \left[\int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} g_1(u, x_n(u), y_n(u), x_n(u-\tau_1(u)), y_n(u-\tau_2(u))) du \right. \\
&- \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} r_1(u) x_n(u-\tau_1(u)) du \\
&\left. + \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}} \int_{-\infty}^{u} a_1(u,s) f_1(x(s), y(s)) dsdu \right] \\
&= g_1(t, x_n(t), y_n(t), x_n(t-\tau_1(t)), y_n(t-\tau_2(t))) - r_1(t) x_n(t-\tau_1(t)) \\
&+ \int_{-\infty}^{t} a_1(t,s) f_1(x_n(s), y_n(s)) ds + h(t) A_1(x_n, y_n)(t),
\end{aligned}
$$

hence, for some positive constant D_1, we obtain

$$
\begin{aligned}
&\left|(A_1(x_n, y_n))'(t)\right| \\
&\leqslant |g_1(t, x_n(t), y_n(t), x_n(t-\tau_1(t)), y_n(t-\tau_2(t)))| + |r_1(t)| |x_n(t-\tau_1(t))| \\
&+ \int_{-\infty}^{t} |a_1(t,s)| |f_1(x_n(s), y_n(s))| ds + |h(t)| |A_1(x_n, y_n)(t)| \\
&\leqslant \left[\sum_{j=1}^{4} \eta_j + \sigma \sum_{j=1}^{2} \rho_j + \theta_1 + \theta_2 \right] M \leqslant D_1,
\end{aligned}
$$

where $\sup_{t \geqslant u} |r_1(t)| = \theta_1$, $\sup_{t \geqslant u} |h(t)| = \theta_2$. On the other hand

$$
\begin{aligned}
&(A_2(x_n, y_n))'(t) \\
&= \int_t^{t+T} [-b(t) G(t,u) + H^*(t,u)] \int_{-\infty}^{u} a_2(u,s) f_2(x_n(s), y_n(s)) dsdu \\
&+ h_2(t) y_n(t-\tau_2(t)) - \int_t^{t+T} [b(t)(h_2(u) H(t,u) - r_2(u) G(t,u)) \\
&+ r_2(u) H^*(t,u)] y_n(u-\tau_2(u)) du,
\end{aligned}
$$

hence, for some positive constant D_2, we obtain

$$\begin{aligned}
&\left|\left(A_2\left(x_n, y_n\right)\right)'(t)\right| \\
&\leqslant \int_t^{t+T}\left(|b(t)| G(t, u)+\left|H^*(t, u)\right|\right) \int_{-\infty}^u\left|a_2(u, s)\right|\left|f_2\left(x_n(s), y_n(s)\right)\right| d s d u \\
&+\left|h_2(t)\right|\left|y_n\left(t-\tau_2(t)\right)\right|+\int_t^{t+T}\left[|b(t)|\left(\left|h_2(u)\right||H(t, u)|+\left|r_2(u)\right| G(t, u)\right)\right. \\
&\left.+\left|r_2(u)\right|\left|H^*(t, u)\right|\right]\left|y_n\left(u-\tau_2(u)\right)\right| d u \\
&\leqslant(\delta \alpha+\gamma) L_2 M \sum_{j=1}^2 \xi_j+\theta M+T[\delta(\theta \gamma+\beta \alpha)+\beta \gamma] M \leqslant D_2 .
\end{aligned}$$

Thus

$$\left\|\left(A\left(x_n, y_n\right)\right)'\right\| \leqslant D,$$

where $D=\max \left(D_1, D_2\right)$. Thus, the sequence $\left(A\left(x_n, y_n\right)\right)$ is uniformly bounded and equi-continuous. The Arzela-Ascoli theorem implies that there exists a subsequence $\left(A\left(x_{n_k}, y_{n_k}\right)\right)$ of $\left(A\left(x_n, y_n\right)\right)$ converges uniformly to a continuous T-periodic function. Thus, A is compact.

For all $\left(x_1, y_1\right),\left(x_2, y_2\right) \in \Omega_{x, y}$,

$$\begin{aligned}
&\left|B_1\left(x_1, y_1\right)(t)-B_1\left(x_2, y_2\right)(t)\right| \\
&=\left|\frac{c_1(t) x_1\left(t-\tau_1(t)\right)}{1-\tau_1'(t)}-\frac{c_1(t) x_2\left(t-\tau_1(t)\right)}{1-\tau_1'(t)}\right| \\
&=\left|\frac{c_1(t)}{1-\tau_1'(t)}\right|\left|x_1\left(t-\tau_1(t)\right)-x_2\left(t-\tau_1(t)\right)\right| \\
&\leqslant \beta_1\left|x_1\left(t-\tau_1(t)\right)-x_2\left(t-\tau_1(t)\right)\right|,
\end{aligned}$$

hence B_1 is contraction because $\beta_1 < 1$. And

$$\begin{aligned}
&\left|B_2(x_1,y_1)(t) - B_2(x_2,y_2)(t)\right| \\
&\leqslant \int_t^{t+T} G(t,u)\left|g_2(u,x_1(u),y_1(u),x_1(u-\tau_1(u)),y_1(u-\tau_2(u)))\right. \\
&\left.-g_2(u,x_2(u),y_2(u),x_2(u-\tau_1(u)),y_2(u-\tau_2(u)))\right| du \\
&\leqslant T\alpha\left(\mu_1\left|x_1(t)-x_2(t)\right| + \mu_3\left|x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))\right|\right. \\
&\left.+\mu_2\left|y_1(t)-y_2(t)\right| + \mu_4\left|y_1(t-\tau_2(t)) - y_2(t-\tau_2(t))\right|\right) \\
&\leqslant T\alpha\left(\mu_1 \max_{t\in[0,T]}\left|x_1(t)-x_2(t)\right| + \mu_3 \max_{t\in[0,T]}\left|x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))\right|\right. \\
&\left.+\mu_2 \max_{t\in[0,T]}\left|y_1(t)-y_2(t)\right| + \mu_4 \max_{t\in[0,T]}\left|y_1(t-\tau_2(t)) - y_2(t-\tau_2(t))\right|\right) \\
&\leqslant T\alpha\left((\mu_1+\mu_3) \max_{t\in[0,T]}\left|x_1(t)-x_2(t)\right| + (\mu_2+\mu_4) \max_{t\in[0,T]}\left|y_1(t)-y_2(t)\right|\right) \\
&\leqslant T\alpha V \max\left(\max_{t\in[0,T]}\left|x_1(t)-x_2(t)\right|, \max_{t\in[0,T]}\left|y_1(t)-y_2(t)\right|\right),
\end{aligned}$$

hence B_2 is contraction because $T\alpha V < 1$. Then

$$\begin{aligned}
&\left|B(x_1,y_1)(t) - B(x_2,y_2)(t)\right| \\
&= \max\left\{\left|B_1(x_1,y_1)(t) - B_1(x_2,y_2)(t)\right|, \left|B_2(x_1,y_1)(t) - B_2(x_2,y_2)(t)\right|\right\},
\end{aligned}$$

this implies that

$$\begin{aligned}
&\left\|B(x_1,y_1) - B(x_2,y_2)\right\| \\
&\leqslant \max(\beta_1, T\alpha V)\max\left(\max_{t\in[0,T]}\left|x_1(t)-x_2(t)\right|, \max_{t\in[0,T]}\left|y_1(t)-y_2(t)\right|\right).
\end{aligned}$$

Hence B is contraction. Thus, the conditions of Lemma 2.1 are satisfied and there is a $(x,y) \in \Omega_{x,y}$, such that $(x,y) = A(x,y) + B(x,y)$. □

In the next theorem, we relax condition (2.2).

THEOREM 2.2. *Assume (1.2)-(1.4), (2.1) and (2.3)-(2.8) hold. Suppose that*

$$V_1 + V_2\sum_{j=1}^{4}\eta_j + L_1\sum_{j=1}^{2}\rho_j \leqslant 1, \text{ and } L_2\alpha\sum_{j=1}^{2}\xi_j + T(\theta\gamma+\beta\alpha) \leqslant 1,$$

and $T\alpha V < 1$, where $V = \max(\mu_1+\mu_3, \mu_2+\mu_4)$. In addition, we assume the existence of continuous nondecreasing function W_2 such that

$$\left|f_2(x,y)\right| \leqslant f_2(\left|x\right|, y) \leqslant N_2 W_2(\left|x\right|),$$

for some positive constant N_2, and for $u > 0$ we ask that

$$\frac{W_2(u)}{u} \leqslant \frac{1 - T(\theta\gamma+\beta\alpha) - \frac{TK_2\alpha}{M}}{L_2N_2\alpha}. \tag{2.12}$$

Then, (1.1) has a T-periodic solution.

Proof. Set

$$M=\max\left\{\frac{V_2K_1+L_1M_1}{1-V_1-\beta_1},\frac{\left(TK_2+L_2N_2W_2\left(M\right)\right)\alpha}{1-T\left(\theta\gamma+\beta\alpha\right)}\right\}. \tag{2.13}$$

Note that due to (2.12) we get

$$M\geqslant\frac{\left(TK_2+L_2N_2W_2\left(M\right)\right)\alpha}{1-T\left(\theta\gamma+\beta\alpha\right)},$$

and hence (2.12) is well defined. For any $(x,y)\in\Omega_{x,y}$, we obtain by the proof of the previous theorem that

$$\left|E_1\left(x,y\right)\left(t\right)\right|\leqslant M.$$

Then

$$\begin{aligned}&\left|B_2\left(x,y\right)\left(t\right)\right|\\&\leqslant\int_t^{t+T}G\left(t,u\right)\left|g_2\left(u,x\left(u\right),y\left(u\right),x\left(u-\tau_1\left(u\right)\right),y\left(u-\tau_2\left(u\right)\right)\right)\right|du\\&\leqslant TK_2\alpha,\end{aligned}$$

and

$$\begin{aligned}\left|A_2\left(x,y\right)\left(t\right)\right|\leqslant&\int_t^{t+T}G\left(t,u\right)\int_{-\infty}^{u}\left|a_2\left(u,s\right)\right|f_2\left(\left|x\left(s\right)\right|,y\left(s\right)\right)dsdu\\&+\int_t^{t+T}\left[\left|h_2\left(u\right)\right|\left|H\left(t,u\right)\right|+\left|r_2\left(u\right)\right|G\left(t,u\right)\right]\left|y\left(u-\tau_2\left(u\right)\right)\right|du\\&\leqslant N_2W_2\left(M\right)\int_t^{t+T}G\left(t,u\right)\int_{-\infty}^{u}\left|a_2\left(u,s\right)\right|dsdu\\&+M\int_t^{t+T}\left[\left|h_2\left(u\right)\right|\left|H\left(t,u\right)\right|+\left|r_2\left(u\right)\right|G\left(t,u\right)\right]du\\&\leqslant L_2N_2\alpha W_2\left(M\right)+T\left(\theta\gamma+\beta\alpha\right)M.\end{aligned}$$

As a consequence of (2.13),

$$\frac{\left(TK_2+L_2N_2W_2\left(M\right)\right)\alpha}{1-T\left(\theta\gamma+\beta\alpha\right)}\leqslant M,$$

we have

$$\left(TK_2+L_2N_2W_2\left(M\right)\right)\alpha\leqslant\left(1-T\left(\theta\gamma+\beta\alpha\right)\right)M.$$

This implies that

$$\begin{aligned}|E_2(x,y)(t)|&\leqslant TK_2\alpha+L_2N_2\alpha W_2\left(M\right)+T\left(\theta\gamma+\beta\alpha\right)M\\&\leqslant\left(1-T\left(\theta\gamma+\beta\alpha\right)\right)M+T\left(\theta\gamma+\beta\alpha\right)M=M.\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.1. □

In the next theorem, we relax condition (2.1).

Theorem 2.3. *Assume (1.2)-(1.4) and (2.2)-(2.8) hold. Suppose that*

$$V_1 + V_2 \sum_{j=1}^{4} \eta_j + L_1 \sum_{j=1}^{2} \rho_j \leqslant 1 \text{ and } L_2 \alpha \sum_{j=1}^{2} \xi_j + T(\theta\gamma + \beta\alpha) \leqslant 1,$$

and $T\alpha V < 1$, *where* $V = \max(\mu_1 + \mu_3, \mu_2 + \mu_4)$. *In addition, we suppose the existence of continuous nondecreasing function* W_1 *such that*

$$|f_1(x, y)| \leqslant f_1(x, |y|) \leqslant N_1 W_1(|y|),$$

for some positive constant N_1, *and for* $u > 0$ *we ask that*

$$\frac{W_1(u)}{u} \leqslant \frac{1 - V_1 - \beta_1 - \frac{V_2 K_1}{M}}{L_1 N_1}. \tag{2.14}$$

Then, (1.1) has a T-periodic solution.

Proof. Set

$$M = \max\left\{ \frac{V_2 K_1 + L_1 N_1 W_1(M)}{1 - V_1 - \beta_1}, \frac{(T K_2 + L_2 M_2)\alpha}{1 - T(\theta\gamma + \beta\alpha)} \right\}. \tag{2.15}$$

Note that due to (2.14) we have

$$M \geqslant \frac{V_2 K_1 + L_1 N_1 W_1(M)}{1 - V_1 - \beta_1},$$

and hence (2.14) is well defined. For any $(x, y) \in \Omega_{x,y}$, we get by the proof of the previous theorem that

$$|E_2(x, y)(t)| \leqslant M.$$

Thus

$$|B_1(x, y)(t)| \leqslant \left| \frac{c_1(t)}{1 - \tau_1'(t)} \right| |x(t - \tau_1(t))| \leqslant \beta_1 M,$$

and

$$\begin{aligned}
&|A_1(x, y)(t)| \\
&\leqslant \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \right| |r_1(u)| \, |x(u - \tau_1(u))| \, du \\
&+ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \right| |g_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u)))| \, du \\
&+ \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \right| \int_{-\infty}^{t} |a_1(u, s)| \, f_1(x(s), |y(s)|) \, ds du \\
&\leqslant M \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \right| |r_1(u)| \, du + K_1 \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \right| du \\
&+ N_1 W_1(M) \int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \right| \int_{-\infty}^{t} |a_1(u, s)| \, ds du \\
&\leqslant V_1 M + V_2 K_1 + L_1 N_1 W_1(M).
\end{aligned}$$

As a consequence of (2.15),

$$\frac{V_2K_1 + L_1N_1W_1(M)}{1 - V_1 - \beta_1} \leqslant M, \text{ we have } V_2K_1 + L_1N_1W_1(M) \leqslant (1 - V_1 - \beta_1) M.$$

This implies that

$$\begin{aligned} |E_1(x, y)(t)| &\leqslant \beta_1 M + V_1 M + V_2 K_1 + L_1 N_1 W_1(M) \\ &\leqslant \beta_1 M + V_1 M + (1 - V_1 - \beta_1) M = M. \end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.1. □

Example 2.1. Consider the coupled nonlinear integro-differential system (2.16)

$$\begin{cases} x'(t) = h(t)x(t) + g_1(t, x(t), y(t), x(t - \tau_1(t)), y(t - \tau_2(t))) \\ +c_1(t)x'(t - \tau_1(t)) + \int_{-\infty}^{t} a_1(t, s) f_1(x(s), y(s)) ds, \\ y''(t) + p(t)y'(t) + q(t)y(t) = g_2(t, x(t), y(t), x(t - \tau_1(t)), y(t - \tau_2(t))) \\ +c_2(t)y'(t - \tau_2(t)) + \int_{-\infty}^{t} a_2(t, s) f_2(x(s), y(s)) ds, \end{cases}$$

where

$$\begin{gathered} p(t) = \frac{1}{\pi}, \ q(t) = \frac{1}{10^3}, \ c_1(t) = \frac{2}{10^8} \sin(t), \ c_2(t) = \frac{3}{10^8} \sin(t), \\ \tau_1(t) = 2\pi, \ \tau_2(t) = 4\pi, \ a_1(t, s) = \frac{1}{10^8} e^{-t+s}, \ a_2(t, s) = \frac{1}{10^9} e^{-t+s}, \end{gathered}$$

$$\begin{aligned} f_1(x_1, x_2) &= \frac{3}{10^2} \sin(x_1) + \frac{6}{10^3} \cos(x_2), \ h(t) = \frac{1}{10^2}, \\ f_2(x_1, x_2) &= \frac{7}{10^3} \cos(x_1) + \frac{5}{10^2} \sin(x_2), \ T = 2\pi, \end{aligned}$$

$$\begin{aligned} g_1(t, x_1, x_2, x_3, x_4) &= \frac{2}{10^5} \sin(x_1) + \frac{5}{10^5} \cos(x_2) + \frac{7}{10^5} \sin(x_3 + x_4), \\ g_2(t, x_1, x_2, x_3, x_4) &= \frac{3}{10^5} \sin(x_1 + x_2) + \frac{4}{10^5} \cos(x_3) + \frac{8}{10^5} \sin(x_4). \end{aligned}$$

The conditions of Theorem 2.1 are satisfied, then (2.16) has a 2π-periodic solution.

3. Conclusion

In the current paper, we have studied the existence of periodic solutions for a system of coupled nonlinear integro-differential equations with variable delays. We have presented the existence theorems for the problem (1.1) under some sufficient conditions due to the Krasnoselskii fixed point theorem. The main results have been well illustrated with the help of an example.

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