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ON IDEALS OF THE NEARNESS RINGS

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Abstract. In this paper, we introduce a different approach for algebraic structures. Because, in the concept of ordinary algebraic structures, an algebraic structure consists of a nonempty set of abstract points with one or more binary operations, and it is required to satisfy certain axioms. But, we can use perceptual objects (non-abstract points) on weak nearness approximation space. This is like as a real life. The problem considered in this paper is to determine such features of ideals on weak nearness approximation space. We defined that upper nearness ideal generated by a subset of the ring on weak nearness approximation spaces and introduce some properties of these ideals. In addition to this, it is defined that nearness prime and nearness maximal ideals of the nearness rings and introduce some properties of these ideals.

1. Introduction

Set theory is very important tool especially for engineers and mathematicians. They use set theory as a base in their studies. Researchers defined new approaches when ordinary set theory is insufficient. Because, the real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [**31**]. In response to this situation Zadeh [**32**] introduced fuzzy set theory as an alternative to probability theory. Then, Rough set theory, proposed by Pawlak in 1982, focused on the uncertainty caused by indiscernible elements with different values in decision attributes ([**20**]). Worldwide, there has been a rapid growth in interest in rough set theory and its applications in recent years. In 1999, Molodtsov [**5**] suggested

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that these difficulties may be due to the inadequacy of the parametrization tool of the theory. Moreover, to overcome these difficulties, Molodtsov [**5**] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches.

In 2002, J. F. Peters introduced near set theory as a generalization of rough set theory. In this theory, Peters uses the features of objects to define the nearness of objects and consequently, the classification of our universal set with respect to the available information of the objects. The concept of near set theory was motivated by image analysis and inspired by a study of the perception of the nearness of familiar physical objects. Near set theory begins with the selection of probe functions that provide a basis for describing and discerning affinities between objects in distinct perceptual granules. A probe function is a real valued function representing a feature of physical objects such as images or behaviors of individual biological organisms. In [**23**], a indiscernibility relation that depends on the features of the objects in order to define the nearness of the objects was given. In more recent studies, it has been accepted as a generalized approach theory to investigate the nearness of similar non-empty sets (see [**21**], [**22**], [**24**], [**25**], and [**26**]).

We may be used near set theory to turn elements in algebraic structures into concrete elements. Because, in the concept of ordinary algebraic structures, such a structure that consists of a nonempty set of abstract points. But, this is not useful for real life problems. All researchers who study algebraic structures consider abstract elements. But, using them in our study some time is insufficient. We use perceptual objects (non-abstract points) in near set theory. Perceptual objects have some features such as colour, degree of maturation for an apple. The basic tool is consideration of upper approximations of the subsets of perceptual objects in the algebraic structures constructed on nearness approximation spaces or weak nearness approximation spaces. There are two important differences between ordinary algebraic structures and nearness algebraic structures. The first one is working with non-abstract points while the second one is considering of upper approximations of the subsets of perceptual objects for the closeness of binary operations.

The number sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are very useful in the field of engineering. But, we emphasize that the elements of the sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ have one and only one property. Having just one feature is not valuable to study for us. Because, many things has multiple features in real life. So, we must be take attention perceptible elements which has more than one property. Since the elements of the sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ has one and only one property, upper and lower approximation's and itself of these sets are equal to each other for $r = n$ $(n \in \mathbb{Z}^+)$. The sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are algebraic structures and also nearness algebraic structures.

Many researchers defined algebraic structures in different sets such as fuzzy set, rough set and soft set. For example, they defined fuzzy over a group as a fuzzy group. But, nearness algebraic structures defined on a set unlike them. Therefore, we think that nearness algebraic structure must be studied in which has property of $G \subset N_r(B)^*G$, where G is a nearness algebraic structure (see [**17**]).

In 2012, İnan and Öztürk analyzed the concept of nearness groups and investigated their basic properties ([**2, 3**]). After, in [**4**], the nearness semigroups and nearness rings were established and their basic properties were investigated (and other algebraic approaches of near sets in [**10**],[**11**], [**12**], [**13**] and [**30**]).

Recently, Öztürk [**9**] has established nearness semiring theory which is a generalization of semiring theory (see [**1**]) and has analyzed some properties of nearness semirings and ideals. Subsequently, researchers continued studies investigating the properties of various ideals of nearness semirings (see [**14**], [**15**], [**16**], [**19**], [**27**], [**28**], and [**29**]).

The problem considered in this paper is to determine features of ideals of the rings on weak nearness approximation space. We defined that upper nearness ideal generated by a subset of the ring on weak nearness approximation spaces and introduce some properties of these ideals. Besides, it is defined that nearness prime and nearness maximal ideals of the nearness rings and introduce some properties of these ideals.

2. Preliminaries

An object description is determined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in X$, which is a subset of an object space $\mathcal{O}$. Assume that $B \subseteq \mathcal{F}$ is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, where $\varphi_i : \mathcal{O} \to \mathbb{R}$ (set of reals). In combination, the functions representing object features provide a basis for an object description $\Phi : \mathcal{O} \to \mathbb{R}^L, \Phi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_L(x))$ a vector containing measurements (returned values) associated with each functional value $\varphi_i(x)$, where the description length $\mid \Phi \mid= L$ (See [**21**]).

The important thing to notice is the choice of functions $\varphi_i \in B$ used to describe an object of interest. Sample objects $X \subseteq \mathcal{O}$ are near each if and only if the objects have similar descriptions. Recall that each φ defines a description of an object. Then let $\triangle_{\varphi_i}$ denote $\triangle_{\varphi_i} =\mid \varphi_i(x^{'}) - \varphi_i(x) \mid$, where $x^{'}, x \in \mathcal{O}$. The difference φ leads to a description of the indiscernibility relation " $\sim_B$ " introduced by Peters in [**21**].

Definition 2.1. ([**21**]) Let $x, x^{'} \in \mathcal{O}, B \subseteq \mathcal{F}$.

$$\sim_B= \{(x, x^{'}) \in \mathcal{O} \times \mathcal{O} \mid \triangle_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on $\mathcal{O}$, where description length $i \leqslant\mid \Phi \mid$.

Comparing object descriptions is the basic idea in the near set approach to object recognition. Sets of object $X, X^{'}$ are called near each other if those sets contain the objects with at least partial matching descriptions.

Definition 2.2. ([**21**]) Let $X, X^{'} \subseteq \mathcal{O}, B \subseteq \mathcal{F}$. Then X is called near $X^{'}$ if there exists $x \in X, x^{'} \in X^{'}, \varphi_i \in B$ such that $x \sim_{\varphi_i} x^{'}$.

A weak nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$, where the approximation space is defined with respect to a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, $\sim_{B_r}$indiscernibility relation B_r defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, and collection of partitions (families of neighbourhoods) $N_r(B)$. This relation $\sim_{B_r}$defines a partition of $\mathcal{O}$ into non-empty, pairwise disjoint subsets that are equivalence classes denoted by $[x]_{B_r}$, where $[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$. These classes form a new set called the quotient set $\mathcal{O}/\sim_{B_r}$, where $\mathcal{O}/\sim_{B_r}= \left\{[x]_{B_r} \mid x \in \mathcal{O}\right\}$. In effect, each choice of probe functions B_r defines a partition $\xi_{\mathcal{O},B_r}$ on a set of objects $\mathcal{O}$, namely, $\xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r}$. Let we consider $X \subseteq \mathcal{O}$, then upper approximation of X defined by

$$N_r\left(B\right)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$$

and lower approximation of X defined by

$$N_r\left(B\right)_* X = \bigcup_{[x]_{B_r} \subseteq X} [x]_{B_r}$$

(See [**21**], [**11**]).

The unique description of each object in the set characterize $A \subseteq \mathcal{O}$ which is a set of objects.

Definition 2.3. ([**10**]) Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and "+"and "·" binary operations defined on $\mathcal{O}$. A subset R of the set of perceptual objects $\mathcal{O}$ is called a nearness ring if the following properties are satisfied:

(NR_1) R is an abelian near group on $\mathcal{O}$ with binary operation "+",

(NR_2) R is a near semigroup on $\mathcal{O}$ with binary operation "·",

(NR_3) $x \cdot (y+z) = (x \cdot y) + (x \cdot z)$ and $(x+y) \cdot z = (x \cdot z) + (y \cdot z)$ properties hold in $N_r\left(B\right)^* R$ for all $x, y, z \in R$,

If in addition:

(NR_4) If $x \cdot y = y \cdot x$ for all $x, y \in R$, then R is said to be a commutative *nearness ring.*

(NR_5) If $N_r\left(B\right)^* R$ contains an element 1_R such that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$, then R is said to be a *nearness ring with identity.*

Definition 2.4. ([**10**]) Let R be a ring on nearness approximation space and S a nonempty subset of R. S is called subnearness ring of R, if S is a nearness ring with binary operations "+"and "·"on nearness ring R.

Lemma 2.1. ([**10**]) *Let R be a ring on nearness approximation space, $\{S_i \mid i \in \Delta\}$ a nonempty family of subnearness rings of R and $N_r\left(B\right)^* S_i$ groupoids. If*

$$\bigcap_{i \in \Delta} \left(N_r\left(B\right)^* S_i\right) = N_r\left(B\right)^* \left(\bigcap_{i \in \Delta} S_i\right),$$

then $\bigcap_{i \in \Delta} S_i$ is a subnearness ring of R.

Definition 2.5. ([**10**]) Let R be a ring on nearness approximation space and I be a nonempty subset of R. I is a left (right) nearness ideal of R provided for all $x, y \in I$ and for all $r \in R$, $x - y \in N_r\left(B\right)^* I$, $r \cdot x \in N_r\left(B\right)^* I$ ($x - y \in N_r\left(B\right)^* I$, $x \cdot r \in N_r\left(B\right)^* I$). A nonempty set I of a nearness ring R is called a nearness ideal of R if I is both a left and a right nearness ideal of R.

LEMMA 2.2. ([**10**]) *Let R be a ring on nearness approximation space, $\{I_i \mid i \in \Delta\}$ a nonempty family of nearness ideals of R and $N_r(B)^* I_i$ groupoids with the binary operations "$+$" and "$\cdot$". If*

$$\bigcap_{i\in\Delta} \left(N_r(B)^* I_i\right) = N_r(B)^* \Big(\bigcap_{i\in\Delta} I_i\Big),$$

then $\bigcap_{i\in\Delta} I_i$ is a nearness ideal of R.

In [**11**], $\nu_{N_r} : \wp(\mathcal{O})\times\wp(\mathcal{O}) \to [0,1]$ is not needed which is overlap function when algebraic structures are studied on the nearness approximation space $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$, the following definition was given.

DEFINITION 2.6. ([**11**]) Let $\mathcal{O}$ be a set of perceived objects, $\mathcal{F}$ a set of the probe functions, $\sim_{B_r}$an indiscernibility relation, and $N_r(B)$ a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is called a weak nearness approximation space.

THEOREM 2.1. ([**11**]) *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space and $X, Y \subset \mathcal{O}$, then the following statements hold;*

i) $N_r(B)_* X \subseteq X \subseteq N_r(B)^* X$,
ii) $N_r(B)^* (X \cup Y) = N_r(B)^* X \cup N_r(B)^* Y$,
iii) $N_r(B)_* (X \cap Y) = N_r(B)_* X \cap N_r(B)_* Y$,
iv) $X \subseteq Y$ *implies* $N_r(B)_* X \subseteq N_r(B)_* Y$,
v) $X \subseteq Y$ *implies* $N_r(B)^* X \subseteq N_r(B)^* Y$,
vi) $N_r(B)_* (X \cup Y) \supseteq N_r(B)_* X \cup N_r(B)_* Y$,
vii) $N_r(B)^* (X \cap Y) \subseteq N_r(B)^* X \cap N_r(B)^* Y$.

DEFINITION 2.7. ([**30**]) Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space, $G \subseteq \mathcal{O}$ and " $\cdot$ " be a operation by $\cdot : G \times G \to N_r(B)^* G$. G is called a group on $\mathcal{O}$, or shortly, nearness group if the following properties are satisfied:

(NG_1) $x \cdot y \in N_r(B)^* G$ for all $x, y \in G$,
(NG_2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^* G$ for all $x, y, z \in G$,
(NG_3) There exists $e \in N_r(B)^* G$ such that $x \cdot e = x = e \cdot x$ for all $x \in G$,
(NG_4) There exists $y \in G$ such that $x \cdot y = e = y \cdot x$ for all $x \in G$.

DEFINITION 2.8. ([**30**]) Let G be a nearness group such that $N_r(B)^*\big(N_r(B)^*G\big) = N_r(B)^* G$ and H be a non-empty subset of $N_r(B)^* G$. If the following properties are hold, then H is called upper nearness subgroup of G and it is denoted by $H \preccurlyeq G$.

i) $x \cdot y \in N_r(B)^* H$ for all $x, y \in H$,
ii) $x^{-1} \in H$ for all $x \in H$.

3. Nearness Ideal Generated by a Subset of The Nearness Rings

When Definition 2.6 is considered, Definition 2.3, Definition 2.4, Lemma 2.1, Definition 2.5, Lemma 2.2 can be restated as follow for weak nearness approximation space.

DEFINITION 3.1. Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space, $R \subseteq \mathcal{O}$ and " $+$ " and " $\cdot$ " operations by $+ : R \times R \to N_r(B)^* R$, and

$\cdot : R \times R \to N_r(B)^* R$, respectively. R is called a ring on $\mathcal{O}$, or shortly, nearness ring if the following properties are satisfied:

(NR_1) R is an abelian group on $\mathcal{O}$ with nearness binary operation "+",
(NR_2) R is a semigroup on $\mathcal{O}$ with nearness binary operation "$\cdot$",
(NR_3) For all $x, y, z \in R$,
$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$
properties hold in $N_r(B)^* R$.

If in addition:

(NR_4) If $x \cdot y = y \cdot x$ for all $x, y \in R$, then R is said to be a commutative *nearness ring*.
(NR_5) If $N_r(B)^* R$ contains an element 1_R such that $1_R \cdot x = x = x \cdot 1_R$ for all $x \in R$, then R is said to be a *nearness ring with identity*.

Definition 3.2. Let R be a ring on weak nearness approximation space and S a nonempty subset of R. S is called subnearness ring of R, if S is a nearness ring with binary operations "+" and "$\cdot$"on nearness ring R.

We will give the following lemma, which is the same proof as the proof of Lemma 2.1.

Lemma 3.1. *Let R be a ring on weak nearness approximation space, $\{S_i \mid i \in I\}$ a nonempty family of subnearness rings of R. If*

$$\bigcap_{i \in \Delta} \left(N_r(B)^* S_i\right) = N_r(B)^* \left(\bigcap_{i \in \Delta} S_i\right),$$

then $\bigcap_{i \in \Delta} S_i$ is a subnearness ring of R.

Definition 3.3. Let R be a ring on weak nearness approximation space and I be a nonempty subset of R. I is called a left (right) nearness ideal of R provided $x - y \in N_r(B)^* I$, $r \cdot x \in N_r(B)^* I$ $(x - y \in N_r(B)^* I$, $x \cdot r \in N_r(B)^* I)$ for all $x, y \in I$ and for all $r \in R$, respectively. A nonempty set I of a nearness ring R is called a nearness ideal of R if I is both a left and a right nearness ideal of R.

We will give the following Lemma, which is the same proof as the proof of Lemma 2.2.

Lemma 3.2. *Let R be a ring on weak nearness approximation space, $\{I_i \mid i \in \Delta\}$ a nonempty family of nearness ideals of R. If*

$$\bigcap_{i \in \Delta} \left(N_r(B)^* I_i\right) = N_r(B)^* \left(\bigcap_{i \in \Delta} I_i\right),$$

then $\bigcap_{i \in \Delta} I_i$ is a nearness ideal of R.

Let's take the sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. Since each elements of these sets has one and only one property; these sets, upper and lower approximation's of these sets are equal to each other for $r = 1$. In other words;

$$N_1(B)_*\mathbb{N} = \mathbb{N} = N_1(B)^*\mathbb{N}$$
$$N_1(B)_*\mathbb{Z} = \mathbb{Z} = N_1(B)^*\mathbb{Z}$$
$$N_1(B)_*\mathbb{Q} = \mathbb{Q} = N_1(B)^*\mathbb{Q}$$
$$N_1(B)_*\mathbb{R} = \mathbb{R} = N_1(B)^*\mathbb{R}$$
$$N_1(B)_*\mathbb{C} = \mathbb{C} = N_1(B)^*\mathbb{C}$$

Then, the sets of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are algebraic structures so that also nearness algebraic structures. For example, $\mathbb{Z}$ is a nearness ring. In this paper, we study nearness ring in which has property of $R \subsetneqq N_r(B)^*R$.

Definition 3.4. Let R be a nearness ring, $B_r \subseteq \mathcal{F}$ where $r \leqslant |B|$ and $B \subseteq \mathcal{F}$, $\sim_{B_r}$ be a indiscernibility relation on $\mathcal{O}$. Then, $\sim_{B_r}$ is called a congruence indiscernibility relation on R, if $x \sim_{B_r} y$, where $x, y \in R$ implies $x + a \sim_{B_r} y + a$, $a + x \sim_{B_r} a + y$, $xa \sim_{B_r} ya$, and $ax \sim_{B_r} ay$, for all $a \in R$.

Lemma 3.3. *Let R be a nearness ring. If $\sim_{B_r}$ is a congruence indiscernibility relation on R, then $[x]_{B_r}+[y]_{B_r} \subseteq [x+y]_{B_r}$ and $[x]_{B_r}[y]_{B_r} \subseteq [xy]_{B_r}$ for all $x, y \in R$.*

Proof. Let $z \in [x]_{B_r} + [y]_{B_r}$. In his case, $z = a + b; a \in [x]_{B_r}, b \in [y]_{B_r}$. From here $x \sim_{B_r} a$,and $y \sim_{B_r} b$, and so, we have $x + y \sim_{B_r} a + y$, and $a + y \sim_{B_r} a + b$ by hypothesis. Thus, $x + y \sim_{B_r} a + b \Rightarrow z = a + b \in [x + y]_{B_r}$. Similarly, $[x]_{B_r}[y]_{B_r} \subseteq [xy]_{B_r}$ is obtained. □

Let R be a nearness ring. Let $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$ and $XY = \{\sum\limits_{finite} x_i y_i \mid x_i \in X \text{ and } y_i \in Y\}$, where subsets X and Y of R.

Lemma 3.4. *Let R be a nearness ring and $\sim_{B_r}$ be a congruence indiscernibility relation on R. The following properties hold:*
i) If $X, Y \subseteq R$, then $(N_r(B)^* X) + (N_r(B)^* Y) \subseteq N_r(B)^* (X + Y)$,
ii) If $X, Y \subseteq R$, then $(N_r(B)^* X)(N_r(B)^* Y) \subseteq N_r(B)^* (XY)$,

Proof. *i)* Let $x \in N_r(B)^* X + N_r(B)^* Y$. We have $x = a+b; a \in N_r(B)^* X, b \in N_r(B)^* Y$. $a \in N_r(B)^* X \Rightarrow [a]_{B_r} \cap X \neq \varnothing \Rightarrow \exists y \in [a]_{B_r} \cap X \Rightarrow y \in [a]_{B_r}$ and $y \in X$. Likewise, $b \in N_r(B)^* Y \Rightarrow [b]_{B_r} \cap Y \neq \varnothing \Rightarrow \exists z \in [b]_{B_r} \cap Y \Rightarrow z \in [b]_{B_r}$ and $z \in Y$. Since $w = y + z \in [a]_{B_r} + [b]_{B_r} \subseteq [a + b]_{B_r}$, we get $w \in [a + b]_{B_r}$ and $w \in X + Y$. Thus, $w \in [a + b]_{B_r} \cap (X + Y) \Rightarrow [a + b]_{B_r} \cap (X + Y) \neq \varnothing$, and so $a + b = x \in N_r(B)^* (X + Y)$.

ii) Let $x \in (N_r(B)^* X)(N_r(B)^* Y)$. Then $x = \sum\limits_{i=1} a_i b_i$, where $a_i \in N_r(B)^* X$ and $b_i \in N_r(B)^* Y$, $1 \leqslant i \leqslant n$. Thus, $[a_i]_{B_r} \cap X \neq \varnothing$ and $[b_i]_{B_r} \cap Y \neq \varnothing$. So, there exists elements $x_i \in [a_i]_{B_r}$, $x_i \in X$ and $y_i \in [b_i]_{B_r}$, $y_i \in Y$, $1 \leqslant i \leqslant n$. Therefore, $x_i y_i \in [a_i]_{B_r}[b_i]_{B_r} \subseteq [a_i b_i]_{B_r}$, $1 \leqslant i \leqslant n$ by Lemma 3.3. Hence, we get $\sum\limits_{i=1} x_i y_i \in [\sum\limits_{i=1} a_i b_i]_{B_r} = [x]_{B_r}$ and $\sum\limits_{i=1} x_i y_i \in XY$. In this case, $[x]_{B_r} \cap (XY) \neq \varnothing$, which implies that $x \in N_r(B)^* (XY)$. □

Definition 3.5. Let R be a nearness ring, $B_r \subseteq \mathcal{F}$, where $r \leqslant \mid B \mid$and $B \subseteq \mathcal{F}$, $\sim_{B_r}$be a indiscernibility relation on $\mathcal{O}$. Then, $\sim_{B_r}$ is called a complete congruence indiscernibility relation on R, if $[x]_{B_r} + [y]_{B_r} = [x+y]_{B_r}$ and $[x]_{B_r}[y]_{B_r} = [xy]_{B_r}$ for all $x, y \in R$.

We will give the following Theorem, which is the same proof as the proof of Lemma 3.4.

Theorem 3.1. *Let R be a nearness ring, $\sim_{B_r}$be a complete congruence indiscernibility relation on R, and X, Y be two non-empty subsets of R. The following properties hold:*
i) $N_r(B)^* X + N_r(B)^* Y = N_r(B)^* (X+Y)$,
ii) $(N_r(B)^* X)(N_r(B)^* Y) = N_r(B)^* (XY)$.

Definition 3.6. Let R be a nearness ring such that $N_r(B)^* \left(N_r(B)^* R\right) = N_r(B)^* R$ and S be a non-empty subset of $N_r(B)^* R$. If the following properties are hold, then S is called upper subnearness ring of R.
i) $x - y \in N_r(B)^* S$ for all $x, y \in S$,
ii) $x \cdot y \in N_r(B)^* S$ for all $x, y \in S$.

Definition 3.7. Let R be a nearness ring such that $N_r(B)^* \left(N_r(B)^* R\right) = N_r(B)^* R$ and A be a nonempty subset of $N_r(B)^* R$. If the following properties are hold, then A is called left (right) upper nearness ideal of R.
i) $x - y \in N_r(B)^* A$ for all $x, y \in A$,
ii) $r \cdot x \in N_r(B)^* A$ ($x \cdot r \in N_r(B)^* A$) for all $x \in A$ and for all $r \in N_r(B)^* R$.
A nonempty set A of $N_r(B)^* R$ is called upper nearness ideal of R if A is both left upper and right upper nearness ideal of R.

Theorem 3.2. *Let R be a nearness ring such that $N_r(B)^* \left(N_r(B)^* R\right) = N_r(B)^* R$ and $\{A_i \mid i \in I\}$ be a family of all upper nearness ideals of R. If*

$$\bigcap_{i \in I} \left(N_r(B)^* A_i\right) = N_r(B)^* (\bigcap_{i \in I} A_i),$$

then $\bigcap_{i \in I} A_i$ is upper nearness ideal of R.

Proof. It is similar to the proof of Lemma 2.2. □

Definition 3.8. Let R be a nearness ring such that $N_r(B)^* \left(N_r(B)^* R\right) = N_r(B)^* R$ and $\{A_i \mid i \in I\}$ be a family of all upper nearness ideals of R that contain X. If

$$\bigcap_{i \in I} \left(N_r(B)^* A_i\right) = N_r(B)^* (\bigcap_{i \in I} A_i),$$

then $\bigcap_{i \in I} A_i$ is called upper nearness ideal generated by the set X and it is denoted by $\langle X \rangle$. The elements of X is called the generators of upper nearness ideal $\langle X \rangle$.

If $X = \{x_1, x_2, \cdots, x_n\}$, then $\langle X \rangle = \langle x_1, x_2, \cdots, x_n \rangle$. In this case, it is called finitely generated by X. If $\langle X \rangle = \{a\}$, then $\langle X \rangle = \langle a \rangle$ is called the principal upper nearness ideal generated by a.

THEOREM 3.3. *Let R be a nearness ring such that $N_r(B)^*(N_r(B)^* R) = N_r(B)^* R$, $a \in N_r(B)^* R$, and $N_r(B)^* R$ satisfies associative and distributive property.*

i) If $N_r(B)^*(N_r(B)^* \langle a\rangle) = N_r(B)^* \langle a\rangle$, then

$$\langle a\rangle \quad = \{na + ra + as + \textstyle\sum_{i=1}^{m} r_i\alpha_i a s_i \ \mid r, s, r_i, s_i \in N_r(B)^* R; n \in \mathbb{Z}, m \in \mathbb{Z}^+\},$$

ii) If $N_r(B)^*(N_r(B)^* \langle a\rangle) = N_r(B)^* \langle a\rangle$ and R is a nearness ring with identity, then

$$\langle a\rangle = \left\{\sum_{i=1}^{m} r_i a s_i \ \mid r_i, s_i \in N_r(B)^* R, m \in \mathbb{Z}^+\right\}.$$

PROOF. *i)* Let

$$T \quad = \{na + ra + as + \textstyle\sum_{i=1}^{m} r_i a s_i \ \mid r, s, r_i, s_i \in N_r(B)^* R; n \in \mathbb{Z}, m \in \mathbb{Z}^+\}.$$

Firstly, we prove that T is upper nearness ideal of R.

$$0_R = 0_{\mathbb{Z}} a + 0_M a + a 0_M + \sum_{i=1}^{m} r_i a 0_R.$$

Then, $T \neq \varnothing$. Let $x, y \in T$.

$$\begin{aligned} x \quad &= na + ra + as + \textstyle\sum_{i=1}^{m} r_i a s_i;\ r, s, r_i, s_i \in N_r(B)^* R,\ n \in \mathbb{Z}, m \in \mathbb{Z}^+, \\ y \quad &= n'a + r'a + as + \textstyle\sum_{i=1}^{t} r_i' a s_i';\ r', s', r_i', s_i' \in N_r(B)^* R,\ n' \in \mathbb{Z},\ t \in \mathbb{Z}^+. \end{aligned}$$

Now, show that $x - y \in N_r(B)^* T$.

$$\begin{aligned} x - y &= na + ra + as + \sum_{i=1}^{m} r_i a s_i - (n'a + r'a + as' + \sum_{i=1}^{t} r_i' a s_i') \\ &= na + ra + as + \sum_{i=1}^{m} r_i a s_i - n'a - r'a - as' - \sum_{i=1}^{t} r_i' a s_i' \\ &= na + ra + as - n'a - r'a - as' + \sum_{i=1}^{m} r_i a s_i - \sum_{i=1}^{t} r_i' a s_i' \\ &= (n - n')a + (r - r')a + a(s - s') + \sum_{i=1}^{m+t} r_i'' a s_i'' \\ &= n''a + r''a + as'' + \sum_{i=1}^{m+t} r_i'' a s_i''. \end{aligned}$$

Hence, $x-y \in T$. From Theorem 2.1.(i), $x-y \in N_r(B)^* T$. Next, we prove that $kx\ (xk) \in N_r(B)^* T$ for all $x \in K$ and $k \in N_r(B)^* R$.

$$\begin{aligned} kx &= k(na + ra + as + \sum_{i=1}^{m} r_i a s_i) \\ &= k(na) + k(ra) + k(as) + k(\sum_{i=1}^{m} r_i a s_i) \\ &= k(a + a + \cdots + a) + (kr)a + kas + \sum_{i=1}^{m} k(r_i a s_i) \\ &= ka + ka + \cdots + ka + (kr)a + kas + \sum_{i=1}^{m} (kr_i) a s_i \\ &= ka + ka + \cdots + ka + r'a + kas + \sum_{i=1}^{m} r_i' a s_i \\ &= r''a + \sum_{i=1}^{m+1} r_i'' a s_i'' \\ &= 0_{\mathbb{Z}} a + r''a + a0_R + \sum_{i=1}^{m+1} r_i'' a s_i''. \end{aligned}$$

Then, $kx \in T$. Afterwards, $kx \in N_r(B)^* T$ by Theorem 2.1.(i). Similarly, $xk \in N_r(B)^* T$.

Now, we show that $T = \langle a \rangle$. By definition of principal ideal,

$$\langle a \rangle = \bigcap_{i \in I} A_i;\ a \in A_i,\ A_i \text{ is upper nearness ideal of } R,\ \forall i \in I. \tag{3.1}$$

Let $x \in \langle a \rangle$. From here, $x \in \bigcap_{i \in I} A_i$. So, $\forall i \in I$, $x \in A_i$. On the other hand,

$$a = 1_{\mathbb{Z}} a + 0_R a + a0_R + \textstyle\sum_{i=1}^{m} r_i a 0_R;\ 0_R, r_i \in N_r(B)^* R, 1_{\mathbb{Z}} \in \mathbb{Z}, m \in \mathbb{Z}^+.$$

From here, $a \in T$. By 3.1, T is one of A_i's that contains a. Hence, $x \in T$. In this case,

$$\langle a \rangle \subseteq T. \tag{3.2}$$

On the other hand, let $x \in T$. Therefore,

$$x = na + ra + as + \textstyle\sum_{i=1}^{m} r_i a s_i; r, s, r_i, s_i \in N_r(B)^* R, n \in \mathbb{Z}, m \in \mathbb{Z}^+.$$

By 3.1, we have $a \in \langle a \rangle$. Then, we get $x \in \langle a \rangle$ by the hypothesis. Hence,

$$T \subseteq \langle a \rangle. \tag{3.3}$$

Thus, from 3.2 and 3.3, $\langle a \rangle = T$.

ii) Let $D = \left\{\sum_{i=1}^{m} r_i a s_i \mid r_i, s_i \in N_r(B)^* R, m \in \mathbb{Z}^+\right\}$. By using ($i$), we have

$$\langle a \rangle = \{na + ra + as + \textstyle\sum_{i=1}^{m} r_i a s_i \mid r, s, r_i, s_i \in N_r(B)^* R; n \in \mathbb{Z}; m \in \mathbb{Z}^+\}.$$

For $x \in \langle a \rangle$ and since R is nearness ring with identity;

$$\begin{aligned}
x &= na + ra + as + \sum_{i=1}^{m} r_i a s_i \\
&= a + a + \cdots + a + ra + as + \sum_{i=1}^{m} r_i a s_i \\
&= 1_R a 1_R + 1_R a 1_R + \cdots + 1_R a 1_R + r a 1_R + 1_R a s + \sum_{i=1}^{m} r_i a s_i \\
&= 1_R a 1_R + 1_R a 1_R + \cdots + 1_R a 1_R + r a 1_R + 1_R a s + r_1 a s_1 + \cdots + r_m a s_m \\
&= r_1 a s_1 + r_2 a s_2 + \cdots + r_n a s_n + r_{n+1} a s_{n+1} + r_{n+2} a s_{n+2} + \cdots + r_{n+m+2} a s_{n+m+2} \\
&= \sum_{i=1}^{n+m+2} r_i a s_i .
\end{aligned}$$

Therefore, $x \in T$, and so we get $\langle a \rangle = T$. □

Definition 3.9. Let R be a nearness ring. Let $C(R) := \{c \in R \mid cr = rc$ for all $r \in R\}$. $C(R)$ is called nearness center of R.

Definition 3.10. Let R be a nearness ring such that $N_r(B)^* (N_r(B)^* R) = N_r(B)^* R$. Let $C(N_r(B)^* R) := \{c \in N_r(B)^* R \mid cr = rc$ for all $r \in N_r(B)^* R\}$. $C(N_r(B)^* R)$ is called upper nearness center of R.

$C(R)$ (resp. $C(N_r(B)^* R)$) is easily seen to be a subnearness ring (resp. upper subnearness ring) of R, but may not be a nearness ideal (resp. upper nearness ideal).

Theorem 3.4. *Let R be a nearness ring such that $N_r(B)^* (N_r(B)^* R) = N_r(B)^* R$, $C(N_r(B)^* R)$ be upper nearness center of R, X be a non-empty subset of $N_r(B)^* R$, $a \in N_r(B)^* R$, and $N_r(B)^* R$ satisfies associative and distributive property.*

i) If $N_r(B)^* (N_r(B)^* \langle a \rangle) = N_r(B)^* \langle a \rangle$ and $a \in C(N_r(B)^* R)$, then

$$\langle a \rangle = \left\{ na + ra \ \mid r \in N_r(B)^* R, n \in \mathbb{Z} \right\},$$

ii) $(N_r(B)^* R)a = \left\{ ra \mid r \in N_r(B)^* R \right\}$ $(a(N_r(B)^* R) = \left\{ ar \mid r \in N_r(B)^* R \right\})$ is a left (right) upper nearness ideal of R. If R is a nearness ring with identity, $N_r(B)^* (N_r(B)^* \langle a \rangle) = N_r(B)^* \langle a \rangle$, and $a \in C(N_r(B)^* R)$, then $(N_r(B)^* R)a = \langle a \rangle = a(N_r(B)^* R)$,

iii) If R is a nearness ring with identity, $N_r(B)^* (N_r(B)^* \langle X \rangle) = N_r(B)^* \langle X \rangle$, and $X \subset C(N_r(B)^* R)$, then

$$\langle X \rangle = \left\{ \sum_{i=1}^{n} r_i a_i \ \mid r_i \in N_r(B)^* R,\ a_i \in X,\ n \in \mathbb{Z}^+ \right\}.$$

PROOF. *i)* Let $I = \left\{na + ra \mid r \in N_r(B)^* R,\, n \in \mathbb{Z}\right\}$. From Theorem 3.3.(*i*),

$$\langle a\rangle = \left\{na + ra + as + \sum_{i=1}^{m} r_i a s_i \mid r, s, r_i, s_i \in N_r(B)^* R,\, n \in \mathbb{Z},\, m \in \mathbb{Z}^+\right\}.$$

If $a \in C(N_r(B)^* R)$, then $ar = ra$ for all $r \in N_r(B)^* R$. So,

$$\begin{aligned} x &= na + ra + as + \sum_{i=1}^{m} r_i a s_i \\ &= na + ra + sa + \sum_{i=1}^{m} r_i s_i a \\ &= na + (r + s + \sum_{i=1}^{m} r_i s_i) a \\ &= na + r^{'} a, r^{'} \end{aligned}$$

and $x \in I$. Hence, we get that $I = \langle a\rangle$.

ii) We prove that $(N_r(B)^* R)a = \left\{ra \in N_r(B)^* R \mid r \in N_r(B)^* R\right\}$ is left upper nearness ideal of R. Let $x, y \in (N_r(B)^* R)a$. Then, there exists elements $r_1, r_2 \in N_r(B)^* R$ such that $x = r_1 a$, $y = r_2 a$. In this case, $x - y = r_1 a - r_2 a = (r_1 - r_2)a \in (N_r(B)^* R)a$ by the hypothesis. From Theorem 2.1.(*i*), $x - y \in N_r(B)^* ((N_r(B)^* R)a)$.

Let $x \in (N_r(B)^* R)a$ and $r \in N_r(B)^* R$. we have $x = r_1 a$ such that $r_1 \in N_r(B)^* R$. Hence, $rx = r(r_1 a) = (rr_1)a$ is obtained. From the hypothesis, since $rr_1 \in N_r(B)^* R$, we get $rx \in (N_r(B)^* R)a$. From Theorem 2.1.(*i*), $rx \in N_r(B)^* ((N_r(B)^* R)a)$. Thus, $(N_r(B)^* R)a$ is left upper nearness ideal of R.

Now, since $N_r(B)^* (N_r(B)^* \langle a\rangle) = N_r(B)^* \langle a\rangle$ and $a \in C(N_r(B)^* R)$, we get $\langle a\rangle = \{na + ra \mid n \in \mathbb{Z}, r \in R\}$ from (*i*). From here, $x \in (N_r(B)^* R)a$ and $x = ra, r \in N_r(B)^* R$. Thus, we get $x = 0_{\mathbb{Z}} a + ra \in \langle a\rangle$, and so

$$(N_r(B)^* R)a \subseteq \langle a\rangle \tag{3.4}$$

is obtained. Let $x \in \langle a\rangle$. Then, $x = na + ra; n \in \mathbb{Z}, r \in N_r(B)^* R$. In this case, since R is with identity and $N_r(B)^* (N_r(B)^* R) = N_r(B)^* R$,

$$\begin{aligned} x &= a + a + \cdots + a + ra \\ &= 1_R a + 1_R a + \cdots + 1_R a + ra \\ &= (1_R + 1_R + \cdots + 1_R + r)\, a \\ &= r' a. \end{aligned}$$

Thus, we obtain $x = r'a \in (N_r(B)^* R)a$, and so

$$\langle a\rangle \subseteq (N_r(B)^* R)a. \tag{3.5}$$

Hence, from 3.4 and 3.5, $\langle a\rangle = (N_r(B)^* R)a$. Similarly, it can be shown that $a(N_r(B)^* R) = \langle a\rangle$. Thus, $(N_r(B)^* R)a = \langle a\rangle = a(N_r(B)^* R)$.

iii) Let R has an identity and $X \subseteq C(N_r\,(B)^*\,R)$. Then, we take

$$T = \left\{ \sum_{i=1}^{n} r_i a_i \mid r_i \in N_r\,(B)^*\,R, a_i \in X, n \in \mathbb{Z}^+ \right\}.$$

Firstly, we show that T is upper nearness ideal of R. For all $x, y \in T$ $x = \sum_{i=1}^{n} r_i a_i$, $y = \sum_{i=1}^{n'} r_i' a_i'$; $r_i, r_i' \in N_r\,(B)^*\,R$; $a_i, a_i' \in X$; $n, n' \in \mathbb{Z}^+$. Thus,

$$\begin{aligned} x - y &= \sum_{i=1}^{n} r_i a_i - \sum_{i=1}^{n'} r_i' a_i' \\ &= (r_1 a_1 + r_2 a_2 + \cdots + r_n a_n) - \left(r_1' a_1' + r_2' a_2' + \cdots + r_{n'}' a_{n'}' \right) \\ &= r_1 a_1 + r_2 a_2 + \cdots + r_n a_n - r_1' a_1' - r_2' a_2' - \cdots - r_{n'}' a_{n'}' \\ &= r_1 a_1 + r_2 a_2 + \cdots + r_n a_n + r_{n+1} a_{n+1} + r_{n+2} a_{n+2} + \cdots + r_{n+n'} a_{n+n'} \\ &= \sum_{i=1}^{n+n'} r_i a_i. \end{aligned}$$

Then, $x - y \in T$. From Theorem 2.1.(i), we have $x - y \in N_r(B)^* T$, i.e. $(T, +) \preccurlyeq (R, +)$.

Let $x \in T$ and $s \in N_r\,(B)^*\,R$. $x = \sum_{i=1}^{n} r_i a_i$; $r_i \in N_r\,(B)^*\,R$, $a_i \in X$, $n \in \mathbb{Z}^+$.

$$xs = \left(\sum_{i=1}^{n} r_i a_i \right) s = \sum_{i=1}^{n} r_i a_i s = \sum_{i=1}^{n} (r_i s) a_i;\ r_i s \in N_r\,(B)^*\,R,\ a_i \in X, n \in \mathbb{Z}^+.$$

$$sx = s \left(\sum_{i=1}^{n} r_i a_i \right) = \sum_{i=1}^{n} (s r_i) a_i;\ s r_i \in N_r\,(B)^*\,R,\ , a_i \in X, n \in \mathbb{Z}^+.$$

From Theorem 2.1.(i), $xs, sx \in N_r(B)^* T$. Hence, T is upper nearness ideal of R.

Now, we show that $T = \langle X \rangle$.

$$\langle X \rangle = \bigcap_{i \in I} A_i, X \subseteq A_i, A_i\text{'s are upper nearness ideals of } R.$$

Let $x \in T$. Then, $x = \sum_{i=1}^{n} r_i a_i$; $r_i \in R$, $a_i \in X \subset A_i$, $n \in \mathbb{Z}^+$. Since A_i's are upper nearness ideals of R, $x \in \bigcap_{i \in I} A_i = \langle X \rangle$ and so,

$$T \subseteq \langle X \rangle\,. \tag{3.6}$$

$y \in X \Rightarrow y = 1_R y \Rightarrow y \in T \Rightarrow X \subseteq T$. Since $\langle X \rangle = \bigcap_{i \in I} A_i$, $X \subseteq A_i$ and A_i's are upper nearness ideals of R, T is one of the nearness ideal of R that contains X. Because of $\bigcap_{i \in I} A_i \subseteq A_i$, $\langle X \rangle = \bigcap_{i \in I} A_i \subseteq T$. Hence,

$$\langle X \rangle \subseteq T. \tag{3.7}$$

Hence, from 3.6 and 3.7, we get that $\langle X \rangle = T$. □

Definition 3.11. Let R be nearness ring and A, B and P are upper nearness ideals of R. P is called a prime upper nearness ideal of R if $A \cdot B \subseteq N_r(B)^*P$ implies that either $A \subseteq N_r(B)^*P$ or $B \subseteq N_r(B)^*P$.

In other words, let R be nearness ring and P be upper nearness ideal of R. P is called a prime upper nearness ideal of R if $ab \in N_r(B)^*P$ implies that either $a \in N_r(B)^*P$ or $b \in N_r(B)^*P$ for $a, b \in N_r(B)^*R$.

Theorem 3.5. *Let R be a nearness ring such that $N_r(B)^*(N_r(B)^*R) = N_r(B)^*R$, $\sim_{B_r}$ be a conqruence indiscernibility relation on $N_r(B)^*R$, P be upper nearness ideal of R such that $N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$, and $N_r(B)^*R$ satisfies associative and distributive property. Then, the following conditions are equivalent to each other.*

i) P is a prime upper nearness ideal.

ii) If $a, b \in N_r(B)^*R$ such that $a(N_r(B)^*R)b \subseteq N_r(B)^*P$, then $a \in N_r(B)^*P$ or $b \in N_r(B)^*P$.

iii) If $\langle a\rangle$ and $\langle b\rangle$ are principal upper nearness ideals of R such that $\langle a\rangle\langle b\rangle \subseteq N_r(B)^*P$, $N_r(B)^*(N_r(B)^*\langle a\rangle) = N_r(B)^*\langle a\rangle$ and $N_r(B)^*(N_r(B)^*\langle b\rangle) = N_r(B)^*\langle b\rangle$, then $a \in N_r(B)^*P$ or $b \in N_r(B)^*P$.

iv) If U and V be upper nearness ideals of R such that $UV \subseteq N_r(B)^*P$, then $U \subseteq N_r(B)^*P$ or $V \subseteq N_r(B)^*P$.

Proof. We first prove that $(i) \implies (ii)$. Let P be prime upper nearness ideal and $a, b \in N_r(B)^*R$ such that $a(N_r(B)^*R)b \subseteq N_r(B)^*P$. Then, since $N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$, we have $(N_r(B)^*R)a(N_r(B)^*R)b(N_r(B)^*R) \subseteq N_r(B)^*P$.

On the other hand, from Lemma 3.4.(ii), we have that $(N_r(B)^*R)(N_r(B)^*R) \subseteq N_r(B)^*(RR)$. Since R is a nearness ring such that $N_r(B)^*(N_r(B)^*R) = N_r(B)^*R$ and $RR \subseteq N_r(B)^*R$, we get $N_r(B)^*(RR) \subseteq N_r(B)^*(N_r(B)^*R)) = N_r(B)^*R$ by Theorem 2.1.(v). Therefore, we obtain that $(N_r(B)^*R)(N_r(B)^*R) \subseteq N_r(B)^*R$. In this case, we have

$$\begin{aligned} &((N_r(B)^*R)a(N_r(B)^*R))((N_r(B)^*R)b(N_r(B)^*R)) \\ &\qquad \subseteq (N_r(B)^*R)a(N_r(B)^*R)b(N_r(B)^*R) \\ &\qquad \subseteq N_r(B)^*P. \end{aligned}$$

Since $(N_r(B)^*R)a(N_r(B)^*R)$ and $(N_r(B)^*R)b(N_r(B)^*R)$ are upper nearness ideals of R, we get $(N_r(B)^*R)a(N_r(B)^*R) \subseteq N_r(B)^*P$ or $b(N_r(B)^*R) \subseteq N_r(B)^*P$ by the hypothesis.

Suppose that $(N_r(B)^*R)a(N_r(B)^*R) \subseteq N_r(B)^*P$. Let $A = \langle a\rangle$ such that $N_r(B)^*(N_r(B)^*\langle a\rangle) = N_r(B)^*\langle a\rangle$. Thus, $A^3 \subseteq (N_r(B)^*R)a(N_r(B)^*R) \subseteq N_r(B)^*P \Rightarrow A^2 \subseteq N_r(B)^*P$ or $A \subseteq N_r(B)^*P$. If $A^2 \subseteq N_r(B)^*P$, then $AA \subseteq N_r(B)^*P$, again using (i), we have $A \subseteq N_r(B)^*P$ and $a \in N_r(B)^*P$.

If $(N_r(B)^* R)b(N_r(B)^* R) \subseteq N_r(B)^*P$, then similarly, it follows that $b \in N_r(B)^*P$.

We show that $(ii) \Longrightarrow (iii)$. Let us next assume the truth of (ii) and $\langle a\rangle \langle b\rangle \subseteq N_r(B)^*P$, where $N_r(B)^*(N_r(B)^* \langle a\rangle) = N_r(B)^* \langle a\rangle$, and $N_r(B)^*(N_r(B)^* \langle b\rangle) = N_r(B)^* \langle b\rangle$ for $a, b \in N_r(B)^* R$. In this case, if $\langle a\rangle \langle b\rangle \subseteq N_r(B)^*P$, it prove easily that $a(N_r(B)^* R)b \subseteq \langle a\rangle \langle b\rangle \subseteq N_r(B)^*P$ and thus $a \in N_r(B)^*P$ or $b \in N_r(B)^*P$ from (ii).

We now proceed to show that $(iii) \Longrightarrow (iv)$. Suppose that U and V are upper nearness ideals of R such that $UV \subseteq N_r(B)^*P$. Let us assume that $U \nsubseteq N_r(B)^*P$, and $UV \subseteq N_r(B)^*P$. Then, there exists an element u such that $u \in U$ and $u \notin N_r(B)^*P$. Let $v \in V$. Since $\langle u\rangle \langle v\rangle \subseteq UV + (N_r(B)^* R)UV \subseteq N_r(B)^*P$ and $u \notin N_r(B)^*P$, property (iii) implies that $v \in N_r(B)^*P$ and so, $V \subseteq N_r(B)^*P$.

Lastly, we prove that $(iv) \Longrightarrow (i)$. Let U, V be upper nearness ideals of R and $UV \subseteq N_r(B)^*P$. Therefore, we have $U \subseteq N_r(B)^*P$ or $V \subseteq N_r(B)^*P$ from (iv). Thus, from Definition 3.11, P is prime upper nearness ideal. □

Definition 3.12. Let R be nearness ring and A, B and P are nearness ideals of R. P is called a prime nearness ideal of R if $A \cdot B \subseteq N_r(B)^*P$ implies that either $A \subseteq N_r(B)^*P$ or $B \subseteq N_r(B)^*P$.

In other words, let R be nearness ring and P be nearness ideal of R. P is called a prime nearness ideal of R if $ab \in N_r(B)^*P$ implies that either $a \in N_r(B)^*P$ or $b \in N_r(B)^*P$ for any $a, b \in R$.

Definition 3.13. Let R be a nearness ring and A be a nearness ideal of R. Then A is called a maximal nearness ideal of R if $R \neq A$ and there does not exist any nearness ideal B of R such that $A \subset B \subset R$.

Example 3.1. Let $\mathcal{O} = \{x, y, z, p, r, s, t\}$ be a set of perceptual objects for $r = 1$, $B = \{\varphi_1, \varphi_2\} \subseteq \mathcal{F}$ be a set of probe functions. Let $R = \{z, p, t\} \subset \mathcal{O}$. Probe functions' values

$$\varphi_1 : \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\},$$
$$\varphi_2 : \mathcal{O} \to V_2 = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7\},$$

are presented in *Table* 1.

	x	y	z	p	r	s	t
φ_1	α_1	α_2	α_3	α_3	α_2	α_4	α_5
φ_2	α_2	α_3	α_5	α_5	α_7	α_2	α_3

Table 1

Now, we find the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ of elements in $\mathcal{O}$:

$$\begin{aligned}
[x]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(x) = \alpha_1\} = \{x\} \\
[y]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(y) = \alpha_2\} = \{y, r\} \\
&= [r]_{\varphi_1}, \\
[z]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(z) = \alpha_3\} = \{z, p\} = [p]_{\varphi_1}, \\
[s]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(s) = \alpha_4\} = \{s\}, \\
[t]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(t) = \alpha_5\} = \{t\}.
\end{aligned}$$

Then, we have that $\xi_{\varphi_1} = \left\{[x]_{\varphi_1}, [y]_{\varphi_1}, [z]_{\varphi_1}, [s]_{\varphi_1}, [t]_{\varphi_1}\right\}$.

$$\begin{aligned}
[x]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(x) = \alpha_2\} = \{x, s\} \\
&= [s]_{\varphi_2}, \\
[y]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(y) = \alpha_3\} = \{y, t\} \\
&= [t]_{\varphi_2}, \\
[z]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(z) = \alpha_5\} = \{z, p\} \\
&= [p]_{\varphi_2}, \\
[r]_{\varphi_2} &= \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(r) = \alpha_7\} = \{r\}.
\end{aligned}$$

We attain that $\xi_{\varphi_2} = \left\{[x]_{\varphi_2}, [y]_{\varphi_2}, [z]_{\varphi_2}, [r]_{\varphi_2}\right\}$. Consequently, a set of partitions of $\mathcal{O}$ is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}\}$ for $r = 1$. Hence,

$$\begin{aligned}
N_1(B)^* R &= \bigcup_{[x]_{\varphi_i} \cap R \neq \varnothing} [x]_{\varphi_i} \\
&= [z]_{\varphi_1} \cup [t]_{\varphi_1} \cup [y]_{\varphi_2} \cup [z]_{\varphi_2} = \{y, z, p, t\}.
\end{aligned}$$

Taking operation tables for R in Table 2 and Table 3.

$+$	z	p	t
z	y	t	p
p	t	y	z
t	p	z	y

Table 2

$\cdot$	z	p	t
z	y	z	z
p	y	p	p
t	y	t	t

Table 3

In this case, $(R, +, \cdot)$ is a nearness ring. Let take $A = \{p, t\}$ is subset of R. A is nearness ideal of R and there does not exist any ideal B such that $A \subset B \subset R$. Hence, A is maximal ideal of R.

EXAMPLE 3.2. $\langle 3\rangle$ is a maximal nearness ideal, but $\langle 4\rangle$ is not a maximal nearness ideal in $\mathbb{Z}$. Since, $\langle 4\rangle \subsetneqq \langle 2\rangle \subsetneqq \mathbb{Z}$.

THEOREM 3.6. *Let R be commutative nearness ring with identity, $\sim_{B_r}$ be a conqruence indiscernibility relation on R and A be a nearness ideal of R such that $N_r(B)^*(N_r(B)^* A) = N_r(B)^* A$. If A is maximal nearness ideal, then A is prime nearness ideal.*

PROOF. We show that $a \in A$ or $b \in B$ for $a, b \in R$ and $r \in R$ when $arb \in N_r(B)^*A$. Since R is commutative, $aR = \{ax \mid x \in R\}$ is nearness ideal of R. If $a \notin A$, then $A + aR = \{m + ax \mid m \in A, x \in R\}$ is nearness ideal of R. Since $A \subsetneqq A + aR$ and A is maximal nearness ideal, $R = A + aR$. Since R has an identity, there are elements $m \in A$ and $x \in R$ so that $1 = m + ax$. $1 = m + ax \Rightarrow b = mb + axb \in N_r(B)^*A$. Hence, from Definition 3.12, A is prime nearness ideal. $\square$

References

[1] J. S. Golan, *Semirings and Their Applications*, 1st ed., Springer-Science+Business Media,B.V., 1999.

[2] E. İnan and M. A. Öztürk, Near groups on nearness approximation spaces. *Hacet. J. Math. Stat.*, **41**(4)(2012), 545–558.

[3] E. İnan and M. A. Öztürk, Erratum and notes for near groups on nearness approximation spaces. *Hacet. J. Math. Stat.*, **43**(2)(2014), 279–281.

[4] E. İnan and M. A. Öztürk, Near semigroups on nearness approximation spaces. *Ann. Fuzzy Math. Inform.*, **10**(2)(2015), 287–297.

[5] D. Molodtsov, Soft set theory - First results. *Comput. Math. Appl.*, **37**(1999), 19–31.

[6] M. A. Öztürk and E. İnan, Soft nearness approximation spaces. *Fund. Inform.*, **124**(1)(2013), 231–250.

[7] M. A. Öztürk, M. Uçkun, and E. İnan, Near groups of weak cosets on nearness approximation spaces. *Fund. Inform.*, **133**(2014), 433–448.

[8] M. A. Öztürk, İ. Çelik Siner, and Y. B. Jun, Nearness BCK-algebras. *Int. J. Open Probl. Comput. Sci. Math.*, **8**(4)(2015), 37–57.

[9] M. A. Öztürk, Semirings on weak nearness approximation spaces. *Ann. Fuzzy Math. Inform.*, **15**(3)(2018), 227–241.

[10] M. A. Öztürk and E. İnan, Nearness rings, *Ann. Fuzzy Math. Inform.*, **17**(2)(2019), 115–132.

[11] M. A. Öztürk, Y. B. Jun, and A. İz, Gamma semigroups on weak nearness approximation spaces. *J. Int. Math. Virtual Inst.*, **9**(2019), 53–72.

[12] M. A. Öztürk, Prime ideals of gamma semigroups on weak nearness approximation spaces. *Asian-Eur. J. Math.*, **12** (5)(2019), 1950080 (10 pages).

[13] M. A. Öztürk and Y. B. Jun, Nobusawa gamma nearness rings. *New Math. Nat. Comput.*, **15**(2)(2019), 373–394.

[14] M. A. Öztürk and İ. Temur, Prime ideals of nearness semirings. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **68**(2)(2019), 1867–1878.

[15] M. A. Öztürk and İ. H. Bekmezci, Gamma nearness semirings. *Southeast Asian Bull. Math.*, **44**(4)(2020), 567–586.

[16] M. A. Öztürk, A note on ideals of the nearness hemirings. *Ann. Fuzzy Math. Inform.*, **20**(3)(2020), 235–242.

[17] M. A. Öztürk, Nearness d-algebras. *J. Algebr. Hyperstruct. Log. Algebras*, **2**(3)(2021), 73–84.

[18] M. A. Öztürk and H. Yazarli, Nobusawa gamma nearness semirings. *New Math. Nat. Comput.*, **17**(3)(2021), 571–588.
[19] M. A. Öztürk and Ö. Tekin, Gamma nearness hemirings. *Afr. Math.*, **32**(7)(2021), 1491–1502.
[20] Z. Pawlak, Rough sets. *Int. J. Comput. Inform. Sci.*, **11**(5)(1982), 341–356.
[21] J. F. Peters, Near sets: General theory about nearness of objects. *Appl. Math. Sci.*, **1**(53-56)(2007), 2609–2629.
[22] J. F. Peters, Near sets: Special theory about nearness of objects. *Fund. Inform.*, **75**(1-4)(2007), 407–433.
[23] J. F. Peters, Classification of perceptual objects by means of features. *Int. J. Info. Technol. Intell. Comput.*, **3**(2)(2008), 1–35.
[24] J. F. Peters and P. Wasilewski, Foundations of near sets. *Inform. Sci.*, **179**(18)(2009), 3091–3109.
[25] J. F. Peters and S. A. Naimpally, Applications of near sets. *Notices Amer. Math. Soc.*, **59**(4)(2012), 536–542.
[26] J. F. Peters, Near sets: An introduction. *Math. Comput. Sci.*, **7**(1)(2013), 3–9.
[27] Ö. Tekin, Quasi ideals of nearness semirings. *Cumhuriyet Science Journal*, **42**(2)(2021), 333–338.
[28] Ö. Tekin, Bi-ideals of nearness semirings. *European Journal of Science and Technology*, Special Issue **28**(2021), 11–15.
[29] Ö. Tekin and M. A. Öztürk, Nobusawa gamma nearness hemirings. *Asian-Eur. J. Math.*, **15**(6)(2022), 2250112 (22 pages).
[30] Ö. Tekin and M. A. Öztürk, Nearness subgroups. *New Math. Nat. Comput.*, (In press) https://doi.org/10.1142/S1793005723500242.
[31] L. A. Zadeh, From circuit theory to system theory. *Proc. Inst. Radio Eng.*, **50**(1962), 856–865.
[32] L. A. Zadeh, Fuzzy sets. *Inform. and Control*, **8**(1965), 338–353.

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