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AN ALGORITHMIC OBSERVATION OF DIRECTED GRAPHS ON LATTICES

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Abstract. In this paper, we examine the directed graphs of lattices. We introduce new concepts such as upward directed graphs of lattices and downward directed graphs of lattices. For this aim, we put forward some relations on them by using isomorphism and anti-isomorphism concepts. Moreover, we support each of these steps with algorithmic construction.

1. Introduction

Many researchers have been studied the relationship between lattice theory and graph theory. Filipov approaches the comparability graphs of partially ordered sets by using the comparability relation, that is a, b are adjacent if either $a \leqslant b$ or $b \leqslant a$ for all a, b elements of a poset [**4**]. Duffus and Rival mention the covering graph of a poset [**3**]. Then, Gadeonova [**5**], Bollobás and Rival [**1**] investigate the properties of covering graphs obtained from lattices. Nimbhokar, Wasadikar, and DeMeyer introduce graphs on a lattice $\mathcal{L}$ with 0 such that x and y are adjacency if $x \wedge y = 0$ for all $x, y \in \mathcal{L}$ [**6**]. Moreover, Wasadikar and Survase determine the zero-divisor graph of a lattice and a meet-semilattice $\mathcal{L}$ with 0, by using the same defining way [**7, 10**]. This graph is shown by $\Gamma(\mathcal{L})$. Lately, Brešar et al. define the cover incomparability graphs of posets [**2**]. These graphs are called $C - I$ graphs of P. As a result, it can be defined as the graph in which the edge set is the union of the edge set of the corresponding covering graph and the corresponding incomparability graph. Moreover, Wasadikar and Survase introduce the incomparability graph of a lattice [**8, 9**].

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In this work, we handle new concepts such as upward directed graphs of lattices and downward directed graphs of lattices. We propound some relations on them by using isomorphism and anti-isomorphism concepts. We show a connection between each other of these upward and downward directed graphs. Besides, we support all of these notions with algorithmic construction and examples. In Section 2, we recall some fundamental concepts about lattices and graphs which are needed during this paper. In Section 3, we present the notions of upward directed graphs of lattices and downward directed graphs of lattices. We obtain some fundamental conclusions by using isomorphism and anti-isomorphism. Finally, we give some results about underlying graphs of them.

2. Preliminaries

In this section, we deal with some fundamental definitions, lemmas and propositions with reference to lattices and graphs that will be used in the following sections.

Definition 2.1. [**11**] A nonempty set L together with two binary operations $\vee$ and $\wedge$ on L is called lattice if it satisfies the following identities:
$(L1)$ $l_1 \vee l_2 = l_2 \vee l_1$ and $l_1 \wedge l_2 = l_2 \wedge l_1$,
$(L2)$ $l_1 \vee (l_2 \vee l_3) = (l_1 \vee l_2) \vee l_3$ and $l_1 \wedge (l_2 \wedge l_3) = (l_1 \wedge l_2) \wedge l_3$,
$(L3)$ $l_1 \vee l_1 = l_1$ and $l_1 \wedge l_1 = l_1$,
$(L4)$ $l_1 \vee (l_1 \wedge l_2) = l_1$ and $l_1 \wedge (l_1 \vee l_2) = l_1$.

Definition 2.2. [**11**] Two lattices L_1 and L_2 are isomorphic if there is a bijection f from L_1 to L_2 such that for every l_1, l_2 in L_1 the following two equations hold:
(i) $f(a \vee b) = f(a) \vee f(b)$,
(ii) $f(a \wedge b) = f(a) \wedge f(b)$.
Such an f is called an isomorphism.

Definition 2.3. [**12, 13**] A graph is indicated by $G = (V, E)$ where $V(G)$ and $E(G)$ represent the set of vertices and the set of edges in G, respectively.

Definition 2.4. [**12**] A directed graph (digraph) $D = (V, A)$ where $V(D)$ and $A(D)$ represent the set of vertices and the set of arcs (directed edges) which are ordered pairs of distinct vertices in D, respectively.

Definition 2.5. [**12**] A directed graph D_1 is isomorphic to a directed graph D_2, written $D_1 \cong D_2$, if there exists a bijective function $\psi : V(D_1) \to V(D_2)$ such that $(u, v) \in A(D_1)$ if and only if $(\psi(u), \psi(v)) \in A(D_2)$. The function ψ is called an isomorphism from D_1 to D_2.

Definition 2.6. [**12**] The underlying graph of a directed graph D is that graph obtained by replacing each arc (u, v) or symmetric pair (u, v), (v, u) of arcs by the edge uv.

3. Directed graphs of lattices and their algorithmic construction

In this section, we present the notions of upward directed graphs of lattices and downward directed graphs of lattices. Firstly, we attain the upward directed graph and the downward directed graph are uniquely determined for each $\mathcal{L}$. We give anti-isomorphic directed graph concept. Then, we show that the upward directed graph and the downward directed graph for the same lattice are anti-isomorphic. We prove that the upward (downward) directed graphs produced from two isomorphic lattices are isomorphic. We also obtain some fundamental results related these concepts. Moreover, we support all of these notions with algorithmic construction and examples.

Definition 3.1. Let $\mathcal{L}$ be a lattice. The upward directed graph of $\mathcal{L}$ is shown by $D(\mathcal{L}) = (V, A)$ such that the set of vertices and the set of arcs corresponding the following sets, respectively:

$$V(D(\mathcal{L})) = \{\ l \mid\ \ l \in L\}$$

and

$$A(D(\mathcal{L})) = \{(l_1, l_2) \mid\ \ l_1 \vee l_2 = l_2,\ \ \ l_1 \neq l_2\ \ \ and\ \ l_1, l_2 \in L\}.$$

In accordance with Definition 3.1, we can construct an upward directed graph of $\mathcal{L}$ by using Algorithm 1:

Algorithm 1: Constructing upward directed graph of lattice $\mathcal{L}$

Data: Lattice $\mathcal{L} = (L, \vee, \wedge)$
Result: Directed graph $D(\mathcal{L})$

ConstructUpwardDigraph($\mathcal{L}$)
$V(D(\mathcal{L})) = L$ `// initializing the vertex set to set` L
$A(D(\mathcal{L})) = \emptyset$ `// initializing the arc set to empty set`
foreach x **in** L **do**
  foreach y **in** L **do** where $x \neq y$
    if $x\ \vee\ y = y$ ***OR*** $y\ \vee\ x = y$ **then**
      $A(D(\mathcal{L})) = A(D(\mathcal{L})) \cup (x, y)$
$D(\mathcal{L}) = (V(D(\mathcal{L})), A(D(\mathcal{L})))$
Return $D(\mathcal{L})$

By the help of Definition 3.1 and Algorithm 1, we handle construction of upward directed graph on any lattice in Example 3.1.

Example 3.1. We take into account a structure $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ with the following Hasse diagram (Figure 1) such that the set $L = \{0, a, b, c, d, e, f, g, h, 1\}$.

The binary operations $\vee$ and $\wedge$ on $\mathcal{L}$ have Cayley tables (Table 1 and Table 2) as follows, respectively:

$\vee$	0	a	b	c	d	e	f	g	h	1
0	0	a	b	c	d	e	f	g	h	1
a	a	a	e	c	d	e	f	g	h	1
b	b	e	b	g	h	e	1	g	h	1
c	c	c	g	c	f	g	f	g	1	1
d	d	d	h	f	d	h	f	1	h	1
e	e	e	e	g	h	e	1	g	h	1
f	f	f	1	f	f	1	f	1	1	1
g	g	g	g	g	1	g	1	g	1	1
h	h	h	h	1	h	h	1	1	h	1
1	1	1	1	1	1	1	1	1	1	1

TABLE 1. Cayley table of $\vee$-operator in Example 3.1

$\wedge$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a	a	a
b	0	0	b	0	0	b	0	b	b	b
c	0	a	0	c	a	a	c	c	a	c
d	0	a	0	a	d	a	d	a	d	d
e	0	a	b	a	a	e	a	e	e	e
f	0	a	0	c	d	a	f	c	d	f
g	0	a	b	c	a	e	c	g	e	g
h	0	a	b	a	d	e	d	e	h	h
1	0	a	b	c	d	e	f	g	h	1

TABLE 2. Cayley table of $\wedge$-operator in Example 3.1

By the help of Cayley tables of Example 3.1, we obtain the lattice in Figure 1:

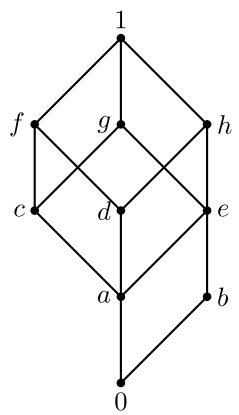

FIGURE 1. Hasse diagram of $\mathcal{L}$ in Example 3.1

As a result, we get the upward directed graph in Figure 2:

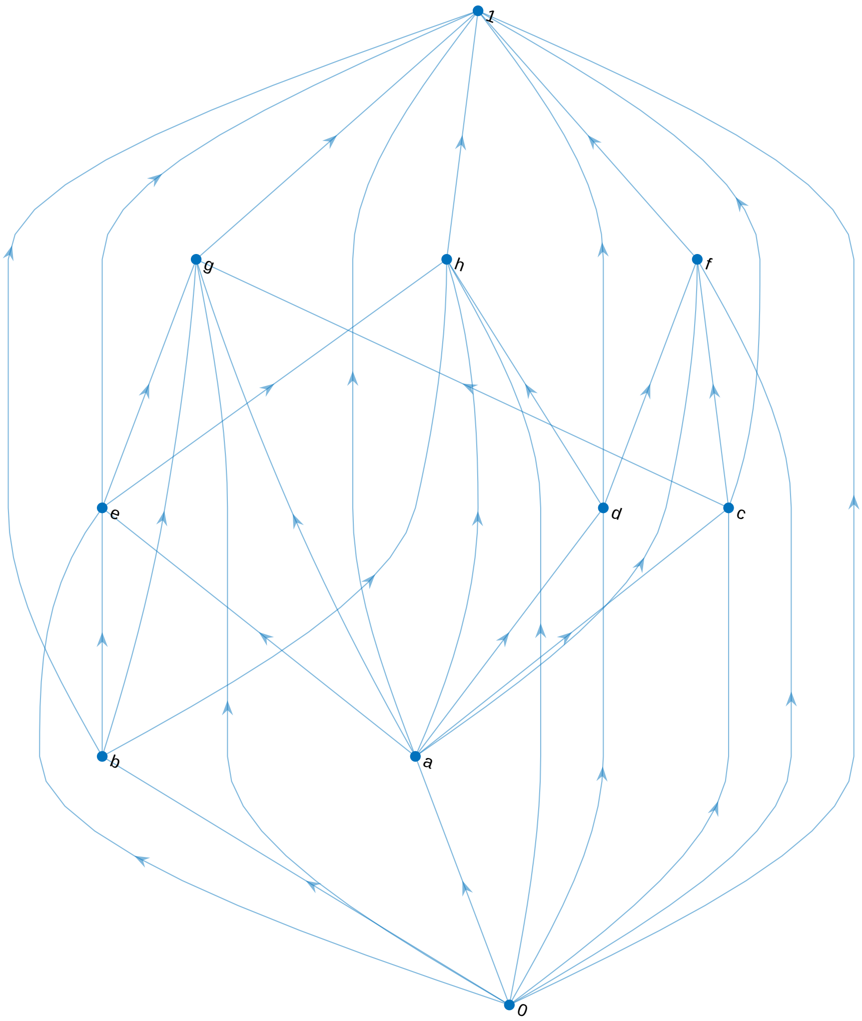

FIGURE 2. Upward directed graph $D(\mathcal{L})$

LEMMA 3.1. *Let $\mathcal{L}$ be a lattice. Then, the upward directed graph $D(\mathcal{L})$ is uniquely determined for each $\mathcal{L}$.*

PROOF. Assume that $D_1(\mathcal{L})$ and $D_2(\mathcal{L})$ are different two upward directed graphs obtained from the lattice $\mathcal{L}$. Then, we have $A(D_1(\mathcal{L})) \neq A(D_2(\mathcal{L}))$. This means that we get $(u, v) \in A(D_1(\mathcal{L}))$ and $(u, v) \notin A(D_2(\mathcal{L}))$ or $(u, v) \in A(D_2(\mathcal{L}))$ and $(u, v) \notin A(D_1(\mathcal{L}))$ at least. Hence, we achieve the following results:

- Assume that $(u, v) \in A(D_1(\mathcal{L}))$ and $(u, v) \notin A(D_2(\mathcal{L}))$ are verified. If $(u, v) \notin$

$A(D_2(\mathcal{L}))$, then we obtain $u \vee v \neq v$. But this is a contradiction. We have $u \vee v = v$ because of $(u, v) \in A(D_1(\mathcal{L}))$.

• Assume that $(u, v) \in A(D_2(\mathcal{L}))$ and $(u, v) \notin A(D_1(\mathcal{L}))$ are verified. We also obtain a contradiction by using similar procedure in the previous step.

As a result, we conclude that the upward directed graph $D(\mathcal{L})$ is uniquely determined for each $\mathcal{L}$. □

Definition 3.2. Let $\mathcal{L}$ be a lattice. The downward directed graph of $\mathcal{L}$ is shown by $\overline{D}(\mathcal{L}) = (V, A)$ such that the set of vertices and the set of arcs corresponding the following sets, respectively:

$$V(\overline{D}(\mathcal{L})) = \{\ l \mid \ l \in L\}$$

and

$$A(\overline{D}(\mathcal{L})) = \{(l_1, l_2) | \ \ l_1 \wedge l_2 = l_2, \ \ l_1 \neq l_2 \ \ \textit{and} \ \ l_1, l_2 \in L\}.$$

In accordance with Definition 3.2, we can construct a downward directed graph of $\mathcal{L}$ by using Algorithm 2:

Algorithm 2: Constructing downward directed graph of lattice $\mathcal{L}$

Data: Lattice $\mathcal{L} = (L, \vee, \wedge)$
Result: Directed graph $\overline{D}(\mathcal{L})$

ConstructDownwardDigraph($\mathcal{L}$)
$V(\overline{D}(\mathcal{L})) = L$ `// initializing the vertex set to set` L
$A(\overline{D}(\mathcal{L})) = \emptyset$ `// initializing the arc set to empty set`
foreach x **in** L **do**
  foreach y **in** L **do** where $x \neq y$
    if $x \ \wedge \ y = y$ ***OR*** $y \ \wedge \ x = y$ **then**
      $A(\overline{D}(\mathcal{L})) = A(\overline{D}(\mathcal{L})) \cup (x, y)$
$\overline{D}(\mathcal{L}) = \big(V(\overline{D}(\mathcal{L})), A(\overline{D}(\mathcal{L}))\big)$
Return $\overline{D}(\mathcal{L})$

By the help of Definition 3.2 and Algorithm 2, we handle construction of downward directed graph on any lattice in Example 3.2.

Example 3.2. Let $\mathcal{L}$ be a lattice as given in Example 3.1. So, we obtain the downward directed graph in Figure 3:

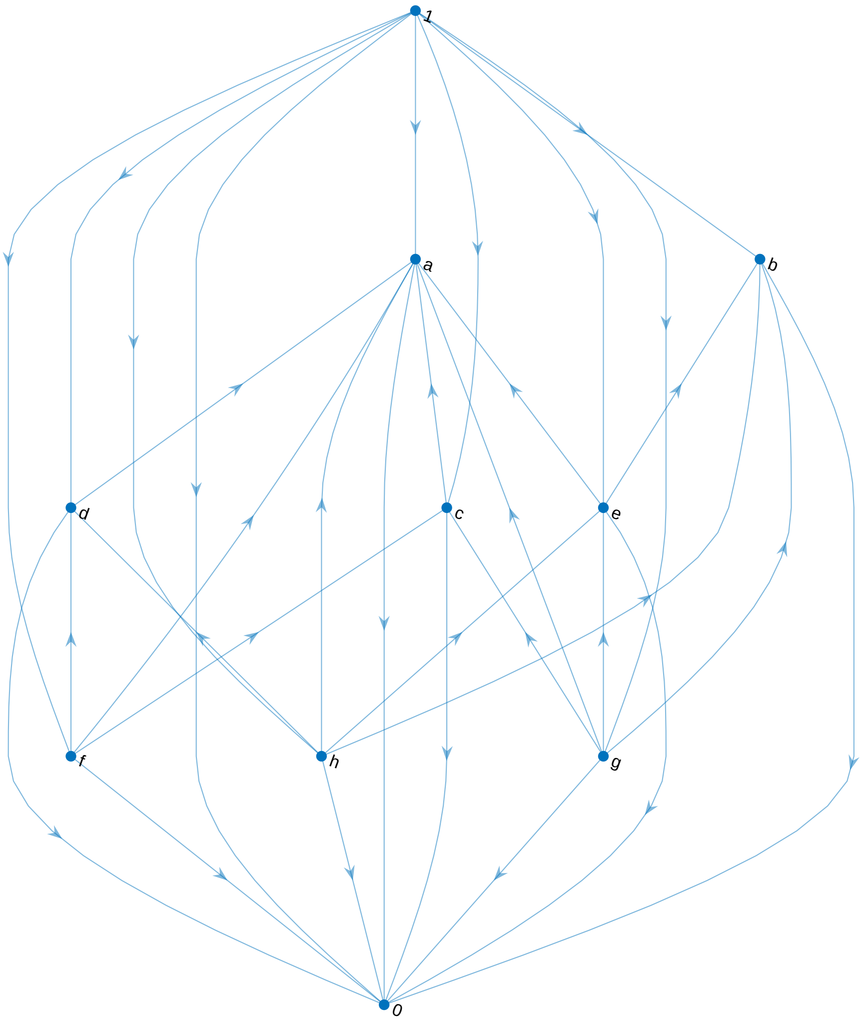

FIGURE 3. Downward directed graph $\overline{D}(\mathcal{L})$

LEMMA 3.2. *Let $\mathcal{L}$ be a lattice. Then, the downward directed graph $\overline{D}(\mathcal{L})$ is uniquely determined for each $\mathcal{L}$.*

PROOF. It can be proved by using similar technique in the proof of Lemma 3.1. □

DEFINITION 3.3. A directed graph D_1 is anti-isomorphic to a directed graph D_2 if there exists a bijective function $\psi : V(D_1) \to V(D_2)$ such that $(u, v) \in A(D_1)$ if and only if $(\psi(v), \psi(u)) \in A(D_2)$. The function ψ is called an anti-isomorphism from D_1 to D_2.

THEOREM 3.1. *Let $\mathcal{L}$ be a lattice. Then, the upward directed graph $D(\mathcal{L})$ is anti-isomorphic to downward directed graph $\overline{D}(\mathcal{L})$.*

PROOF. Let $\mathcal{L}$ be a lattice and let $D(\mathcal{L})$ and $\overline{D}(\mathcal{L})$ be upward directed graph and downward directed graph of $\mathcal{L}$, respectively.

The identity mapping f is defined between $V(D(\mathcal{L}))$ and $V(\overline{D}(\mathcal{L}))$ as follows:

$$\begin{aligned} f : V(D(\mathcal{L})) &\quad\rightarrow\quad V(\overline{D}(\mathcal{L})) \\ l_i &\quad\longmapsto\quad f(l_i) = l_i \end{aligned}$$

for each $l_i \in L$. It is clearly obtain that the mapping f is a bijection.

Now, we give a mapping between the sets of arcs $A(D(\mathcal{L}))$ and $A(\overline{D}(\mathcal{L}))$ as follows:

$$\begin{aligned} \overline{f} : A(D(\mathcal{L})) &\quad\rightarrow\quad A(\overline{D}(\mathcal{L})) \\ (l_i, l_j) &\quad\longmapsto\quad \overline{f}(l_i, l_j) = (f(l_j), f(l_i)) \end{aligned}$$

for each $l_i, l_j \in L$. The mapping $\overline{f}$ is well-defined. We show that the mapping $\overline{f}$ is also a bijection. Assume that $\overline{f}(l_i, l_j) = \overline{f}(l_i^{'}, l_j^{'})$. Then, we have

$$\begin{aligned} \overline{f}(l_i, l_j) = \overline{f}(l_i^{'}, l_j^{'}) &\Rightarrow (f(l_j), f(l_i)) = (f(l_j^{'}), f(l_i^{'})) \\ &\Rightarrow f(l_i) = f(l_i^{'}) \ \ and \ \ f(l_j) = f(l_j^{'}) \\ &\Rightarrow l_i = l_i^{'} \ \ and \ \ l_j = l_j^{'} \\ &\Rightarrow (l_i, l_j) = (l_i^{'}, l_j^{'}). \end{aligned}$$

So, the mapping $\overline{f}$ is one-to-one.

For each $(l_i^{'}, l_j^{'}) \in A(\overline{D}(\mathcal{L}))$, we have $l_i^{'}, l_j^{'} \in L$ such that $f(l_i^{'}) = l_i^{'}$ and $f(l_j^{'}) = l_j^{'}$. Therefore, we obtain

$$(l_i^{'}, l_j^{'}) = (f(l_i^{'}), f(l_j^{'})) = \overline{f}(l_j^{'}, l_i^{'}).$$

Since $\mathcal{L}$ is a lattice and the mapping f is well-defined, we obtain that the mapping $\overline{f}$ is surjective. So, we conclude that the mapping $\overline{f}$ is a bijection.

Moreover, we have

$$(l_i, l_j) \in A(D(\mathcal{L})) \Leftrightarrow (f(l_j), f(l_i)) \in A(\overline{D}(\mathcal{L})).$$

As a result, we obtain an anti-isomorphism between $D(\mathcal{L})$ and $\overline{D}(\mathcal{L})$ from Definition 3.3. $\square$

EXAMPLE 3.3. We take the upward directed graph and downward directed graph in Example 3.1 and Example 3.2, respectively. We can easily see that there is an anti-isomorphism between $D(\mathcal{L})$ and $\overline{D}(\mathcal{L})$.

THEOREM 3.2. *The upward directed graphs produced from two isomorphic lattices are isomorphic.*

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two isomorphic lattices. Then, there exists a mapping $f : \mathcal{L}_1 \to \mathcal{L}_2$ such that $f(l_1 \vee l_2) = f(l_1) \vee f(l_2)$ and $f(l_1 \wedge l_2) = f(l_1) \wedge f(l_2)$ for each $l_1, l_2 \in L_1$.

Let $D(\mathcal{L}_1)$ and $D(\mathcal{L}_2)$ be two upward directed graphs of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The mapping g is defined between $V(D(\mathcal{L}_1))$ and $V(D(\mathcal{L}_2))$ as follows:

$$\begin{array}{rcl} g : V(D(\mathcal{L}_1)) & \to & V(D(\mathcal{L}_2)) \\ l_i & \longmapsto & g(l_i) = f(l_i) \end{array}$$

for each $l_i \in L_1$. Since the mapping f is an isomorphism, the mapping g is also an isomorphism.

The mapping $\overline{g}$ between the upward directed edges sets of $A(D(\mathcal{L}_1))$ and $A(D(\mathcal{L}_2))$ is defined as follows:

$$\begin{array}{rcl} \overline{g} : A(D(\mathcal{L}_1)) & \to & A(D(\mathcal{L}_2)) \\ (l_i, l_j) & \longmapsto & \overline{g}(l_i, l_j) = (f(l_i), f(l_j)) \end{array}$$

for each $l_i, l_j \in L_1$.

Let $(l_i, l_j) \in A(D(\mathcal{L}_1))$. Then, we have $l_i \vee l_j = l_i$ or $l_i \vee l_j = l_j$. Assume that $l_i \vee l_j = l_i$. Since the mapping f is an isomorphism, we obtain $f(l_i) = f(l_i \vee l_j) = f(l_i) \vee f(l_j)$. So, we attain that $\overline{g}(l_i, l_j) = (f(l_i), f(l_j)) \in A(D(\mathcal{L}_2))$. By using the isomorphism property of the mapping f, we conclude that

$$\begin{array}{rcl} (l_i, l_j) = (l_i^{'}, l_j^{'}) & \Leftrightarrow & l_i = l_i^{'} \quad and \quad l_j = l_j^{'} \\ & \Leftrightarrow & f(l_i) = f(l_i^{'}) \quad and \quad f(l_j) = f(l_j^{'}) \\ & \Leftrightarrow & (f(l_i), f(l_j)) = (f(l_i^{'}), f(l_j^{'})) \\ & \Leftrightarrow & \overline{g}(l_i, l_j) = \overline{g}(l_i^{'}, l_j^{'}). \end{array}$$

Therefore, we get the mapping $\overline{g}$ is well-defined and one-to-one.

Let $(l_i^{'}, l_j^{'}) \in A(D(\mathcal{L}_2))$. Since the mapping f is an isomorphism, we have $f(l_i) = f(l_i^{'})$ and $f(l_j) = f(l_j^{'})$ such that $l_i, l_j \in L_1$. Then, we obtain $(l_i^{'}, l_j^{'}) = (f(l_i), f(l_j)) = \overline{g}(l_i, l_j)$ for $(l_i, l_j) \in A(D(\mathcal{L}_1))$. This means that the mapping $\overline{g}$ is surjective. As a result, we conclude that $D(\mathcal{L}_1) \cong D(\mathcal{L}_2)$. □

Corollary 3.1. *The downward directed graphs produced from two isomorphic lattices are isomorphic.*

Proof. It is obtained by using similar technique in the proof of the Theorem 3.2. □

Besides Theorem 3.2 and Corollary 3.1, we give the Algorithm 3. This algorithm determines that two lattices are isomorphic or not. In accordance with this attempts, we can also decide two graphs induced from two lattices are isomorphic or not.

Algorithm 3: Determining of isomorphism between two lattices

Data: Two lattices $\mathcal{L}_1 = (L_1, \vee, \wedge)$ and $\mathcal{L}_2 = (L_2, \vee, \wedge)$, a mapping f
Result: The lattices are isomorphic or not.

Isomorphism($\mathcal{L}_1, \mathcal{L}_2, f$)
if $|L_1| \neq |L_2|$ **then**
  Print "Lattices are not isomorphic"
  Return
$L_2' = L_2$
foreach l_i **in** L_1 **do**
  $L_2' = L_2' \setminus f(l_i)$
  foreach l_j **in** L_1 **do** where $l_i \neq l_j$
    if $f(l_i) = f(l_j)$ **then**
      Print "Lattices are not isomorphic"
      Return
if $L_2' \neq \emptyset$ **then**
  Print "Lattices are not isomorphic"
  Return
foreach l_i **in** L_1 **do**
  foreach l_j **in** L_1 **do** where $l_i \neq l_j$
    if $f(l_i \ \vee_{L_1} \ l_j) \neq f(l_i) \ \vee_{L_2} \ f(l_j)$ *OR* $f(l_i \ \wedge_{L_1} \ l_j) \neq f(l_i) \ \wedge_{L_2} \ f(l_j)$ **then**
      Print "Lattices are not isomorphic"
      Return
Print "Lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic under the mapping f"

By the help of Theorem 3.2 and Corollary 3.1 and Algorithm 3, we obtain an isomorphism between given two lattices in Example 3.4.

Example 3.4. Let $\mathcal{L}_1$ be lattice $\mathcal{L}$ as given in Example 3.1 and we obtain the upward directed graph of $\mathcal{L}_1$ in Figure 2. Also, the lattice $\mathcal{L}_2 = (L_2, \wedge, \vee, 0, 1)$ with the following Hasse diagram (Figure 4) such that the set $L_2 = \{0, a', b', c', d', e', f', g', h', 1\}$.

The binary operations $\vee$ and $\wedge$ on $\mathcal{L}_2$ have Cayley tables (Table 3 and Table 4) as follows, respectively:

$\vee$	0	a'	b'	c'	d'	e'	f'	g'	h'	1
0	0	a'	b'	c'	d'	e'	f'	g'	h'	1
a'	a'	a'	d'	c'	d'	e'	f'	g'	h'	1
b'	b'	d'	b'	f'	d'	h'	f'	1	h'	1
c'	c'	c'	f'	c'	f'	1	f'	g'	1	1
d'	d'	d'	d'	f'	d'	h'	f'	1	h'	1
e'	e'	e'	h'	1	h'	e'	1	g'	h'	1
f'	f'	f'	f'	f'	f'	1	f'	1	1	1
g'	g'	g'	1	g'	1	g'	1	g'	1	1
h'	h'	h'	h'	1	h'	h'	1	1	h'	1
1	1	1	1	1	1	1	1	1	1	1

TABLE 3. Cayley table of $\vee$-operator of $\mathcal{L}_2$

$\wedge$	0	a'	b'	c'	d'	e'	f'	g'	h'	1
0	0	0	0	0	0	0	0	0	0	0
a'	0	a'	0	a'	a'	a'	a'	a'	a'	a'
b'	0	0	b'	0	b'	0	b'	0	b'	b'
c'	0	a'	0	c'	a'	a'	c'	c'	a'	c'
d'	0	a'	b'	a'	d'	a'	d'	a'	d'	d'
e'	0	a'	0	a'	a'	e'	a'	e'	e'	e'
f'	0	a'	b'	c'	d'	a'	f'	a'	d'	f'
g'	0	a'	0	c'	a'	e'	a'	g'	e'	g'
h'	0	a'	b'	a'	d'	e'	d'	e'	h'	h'
1	0	a'	b'	c'	d'	e'	f'	g'	h'	1

TABLE 4. Cayley table of $\wedge$-operator of $\mathcal{L}_2$

By the help of Cayley tables of $\mathcal{L}_2$, we obtain the lattice in Figure 4:

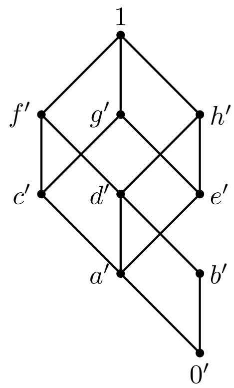

FIGURE 4. Hasse diagram of $\mathcal{L}_2$

As a result, we get the upward directed graph in Figure 5:

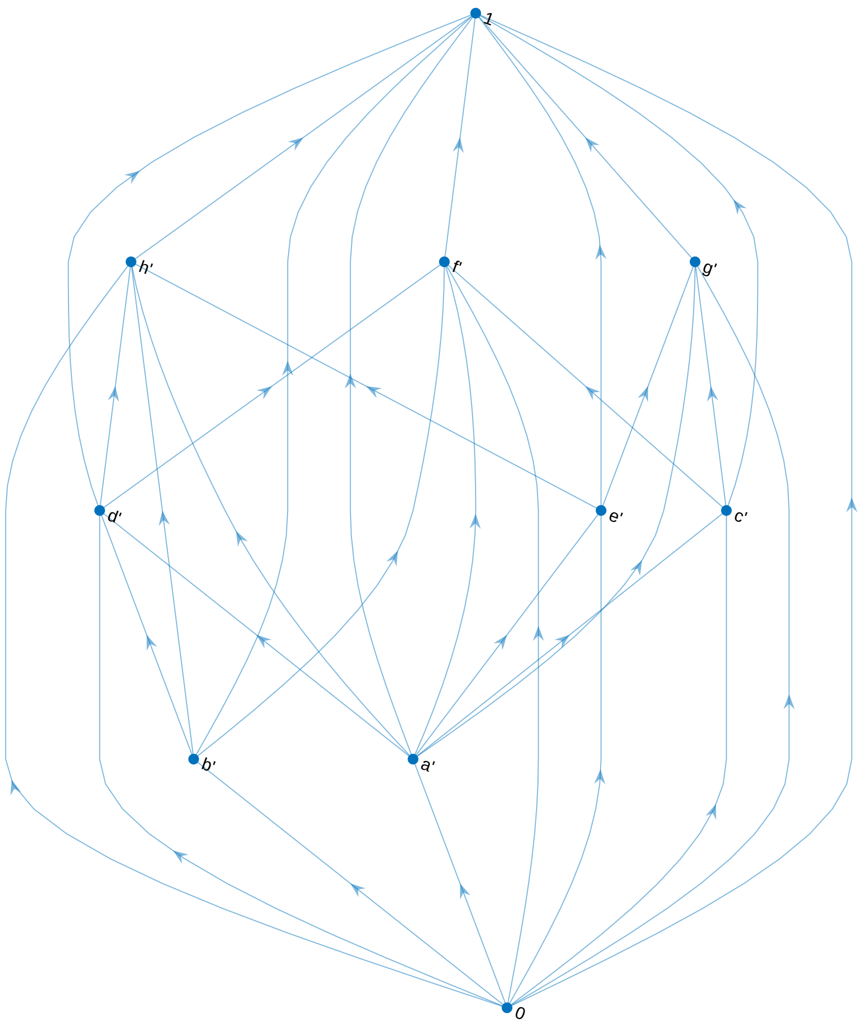

FIGURE 5. Upward directed graph $D(\mathcal{L}_2)$

Moreover, we define the following mapping from L_1 to L_2 as follows:

$$\begin{aligned} f : L_1 &\quad \to \quad L_2 \\ l_i &\quad \longmapsto \quad f(l_i) = \begin{cases} l_i', & l_i \notin \{e, d\}, \\ e', & l_i = d, \\ d', & l_i = e. \end{cases} \end{aligned}$$

In addition, we have $f(l_i \vee l_j) = f(l_i) \vee f(l_j)$ and $f(l_i \wedge l_j) = f(l_i) \wedge f(l_j)$ for all $l_i, l_j \in L_1$. Then, we obtain that

$$\mathcal{L}_1 \cong \mathcal{L}_2. \tag{3.1}$$

By the Isomorphism (3.1) and Theorem 3.2, we conclude that $D(\mathcal{L}_1) \cong D(\mathcal{L}_2)$.

Example 3.5. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices as given in Example 3.4. The downward directed graph $\overline{D}(\mathcal{L}_1)$ of $\mathcal{L}_1$ is given in Figure 3. Also, we get the downward directed graph $\overline{D}(\mathcal{L}_2)$ of $\mathcal{L}_2$ as Figure 6.

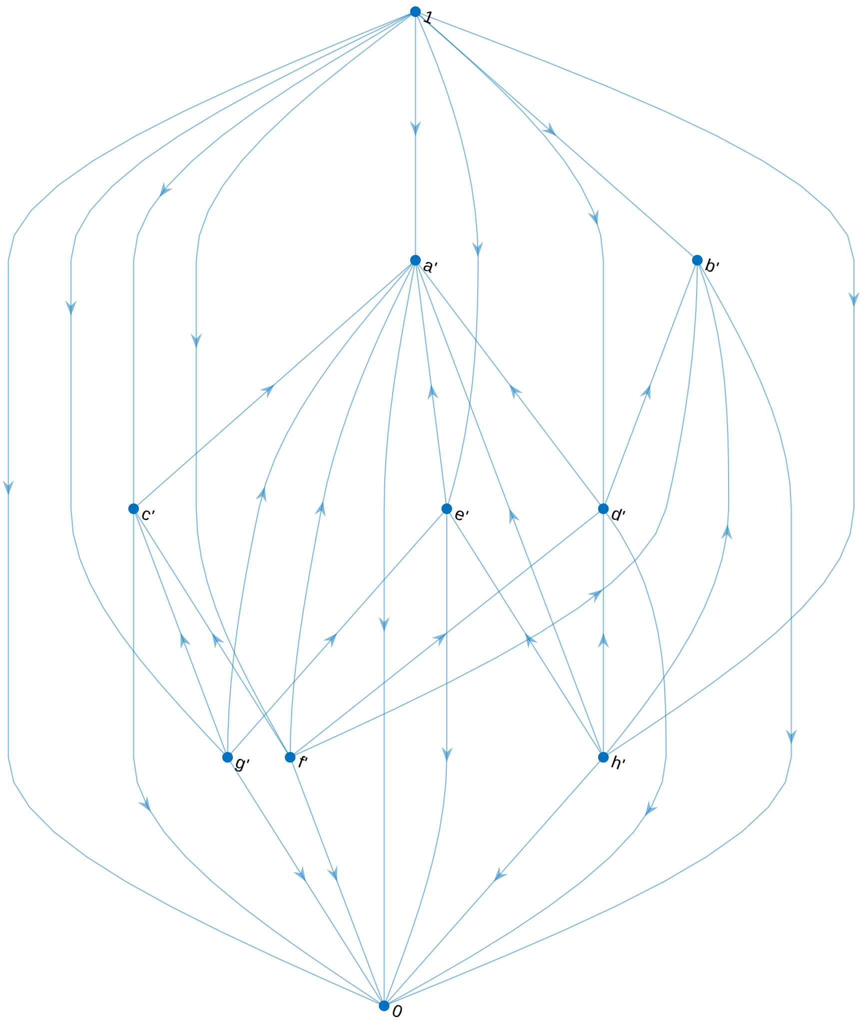

Figure 6. Downward directed graph $\overline{D}(\mathcal{L})$

Lemma 3.3. *If the lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic, then we have the following results:*
(i) The directed graph $D(\mathcal{L}_1)$ and the directed graph $\overline{D}(\mathcal{L}_2)$ are anti-isomorphic graphs.
(ii) The directed graph $D(\mathcal{L}_2)$ and the directed graph $\overline{D}(\mathcal{L}_1)$ are anti-isomorphic graphs.

Proof. It is straightforward from Theorem 3.1, Theorem 3.2 and Corollary 3.1. $\square$

Theorem 3.3. *Let $D(\mathcal{L})$ be an upward directed graph produced from $\mathcal{L}$ and u and v in $\mathcal{L}$. There is no path between u and v in $D(\mathcal{L})$ if and only if u and v are incomparable in $\mathcal{L}$.*

Proof. ($\Rightarrow$:) Assume that there is no path between u and v in $D(\mathcal{L})$. This means that $(u, v) \notin D(\mathcal{L})$ and $(v, u) \notin D(\mathcal{L})$. Therefore, we obtain the following results:
• if $(u, v) \notin D(\mathcal{L})$ is verified, then we get $u \vee v \neq v$,
and
• if $(v, u) \notin D(\mathcal{L})$ is verified, then we get $v \vee u \neq u$.
From these two results, we attain that $v \vee u \neq u \neq v$. As a result, we conclude that u and v are incomparable in $\mathcal{L}$.

($\Leftarrow$:) We assume that the elements u and v are incomparable in $\mathcal{L}$. Then, we attain $u \not\leq v$ and $u \not\leq v$. So, we get the following conclusions:
• if $u \not\leq v$ is verified, we get $u \vee v \neq v$,
and
• if $v \not\leq u$ is verified, then we get $v \vee u \neq u$.
From these two results, we attain that $(u, v) \notin D(\mathcal{L})$ and $(v, u) \notin D(\mathcal{L})$. As a result, we conclude that there is no path between u and $v \in D(\mathcal{L})$. $\square$

Corollary 3.2. *Let $\overline{D}(\mathcal{L})$ be a downward directed graph produced from $\mathcal{L}$ and u and v in $\mathcal{L}$. There is no path between u and v in $\overline{D}(\mathcal{L})$ if and only if u and v are incomparable in $\mathcal{L}$.*

Example 3.6. Let $\mathcal{L}$ be a lattice in Example 3.1 and let $D(\mathcal{L})$ be upward directed graph of $\mathcal{L}$ in Figure 2. Then, we conclude the following results:
• $(a, b) \notin A(D(\mathcal{L})) \Leftrightarrow a \not\leq b$ and $b \not\leq a$,
• $(b, c) \notin A(D(\mathcal{L})) \Leftrightarrow b \not\leq c$ and $c \not\leq b$,
• $(b, d) \notin A(D(\mathcal{L})) \Leftrightarrow b \not\leq d$ and $d \not\leq b$,
• $(b, f) \notin A(D(\mathcal{L})) \Leftrightarrow b \not\leq f$ and $f \not\leq b$,
• $(c, d) \notin A(D(\mathcal{L})) \Leftrightarrow c \not\leq d$ and $d \not\leq c$,
• $(c, e) \notin A(D(\mathcal{L})) \Leftrightarrow c \not\leq e$ and $e \not\leq c$,
• $(c, h) \notin A(D(\mathcal{L})) \Leftrightarrow c \not\leq h$ and $h \not\leq c$,
• $(d, e) \notin A(D(\mathcal{L})) \Leftrightarrow d \not\leq e$ and $e \not\leq d$,
• $(d, g) \notin A(D(\mathcal{L})) \Leftrightarrow d \not\leq g$ and $g \not\leq d$,
• $(e, f) \notin A(D(\mathcal{L})) \Leftrightarrow e \not\leq f$ and $f \not\leq e$,

• $(f, g) \notin A(D(\mathcal{L})) \Leftrightarrow f \nleq g$ and $g \nleq f$,
• $(f, h) \notin A(D(\mathcal{L})) \Leftrightarrow f \nleq h$ and $h \nleq f$,
• $(g, h) \notin A(D(\mathcal{L})) \Leftrightarrow g \nleq h$ and $h \nleq g$
where $L = \{0, a, b, c, d, e, f, g, h, 1\}$ is the universe of $\mathcal{L}$.

Corollary 3.3. *Assume that the elements k and m are comparable with u and v in $\mathcal{L}$, respectively. Then, there is no path between u and v in $D(\mathcal{L})$ if and only if the elements k and m are incomparable in $\mathcal{L}$.*

Proof. It is clearly obtained from Theorem 3.3. □

When we consider the Definition 2.6, we get the following results.

Theorem 3.4. *Let $\mathcal{L}$ be a lattice. Then, the underlying graph of $D(\mathcal{L})$ and the underlying graph of $\overline{D}(\mathcal{L})$ are the same.*

Proof. It is straightforward from Theorem 3.1 and Definition 2.6. □

Theorem 3.5. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two isomorphic lattices. Then, the underlying graphs of $D(\mathcal{L}_1)$, $D(\mathcal{L}_2)$, $\overline{D}(\mathcal{L}_1)$ and $\overline{D}(\mathcal{L}_2)$ are isomorphic.*

Proof. It can be proved by using Theorem 3.1 and Definition 2.6. □

4. Conclusion

In this paper, we presented the notions of upward directed graph and downward directed graph of lattices. Initially, we attained the upward directed graph and the downward directed graph are uniquely determined for each $\mathcal{L}$. We presented anti-isomorphic directed graph concept. Then, we proved that the upward directed graph and the downward directed graph for the same lattice are anti-isomorphic. We showed that the upward (downward) directed graphs produced from two isomorphic lattices are isomorphic. We also gave some fundamental results related these concepts. Moreover, we assisted all of these notions with algorithmic construction and examples. For future work, these concepts will be used for the lattices counterparts of different algebraic structures. Also, new concepts for these type directed graphs will be proposed and connected with this paper.

References

[1] B. Bollobás and I. Rival. The maximal size of the covering graph of a lattice. *Algebra Universalis*, **9(1)** (1979), 371—373.
[2] B. Brešar, M. Changat, S. Klavžar, M. Kovše, J. Mathews and A. Mathews. Cover-incomparability graphs of posets. *Order*, **25(4)** (2008), 335—347.
[3] D. Duffus and I. Rival. Path lengths in the covering graph of a lattice. *Discrete Mathematics*, **19(2)** (1977), 139—158.
[4] N. D. Filippov. Comparability graphs of partially ordered sets of different types. *Collq. Maths. Soc. János Bolyai*, **33** (1980), 373—380.
[5] E. Gedeonova. Lattices whose covering graphs are s-graphs, Colloq. Maths. Soc. Janos Bolyai. *Collq. Maths. Soc. János Bolyai*, **33** (1980), 407—435.
[6] S. K. Nimbhokar, M.P. Wasadikar and L. DeMeyer. Coloring of meet-semilattices. *Ars Combinatoria*, **84** (2007), 97–104.

[7] M. Wasadikar and P. Survase. Some properties of graphs derived from lattices. *Bull. Calcutta Math. Soc.*, **104** (2012), 125-–138.

[8] M. Wasadikar and P. Survase. Incomparability graphs of lattices. *Mathematical Modelling and Scientific Computation. ICMMSC 2012. Communications in Computer and Information Science*, **283** (2012), 78–85.

[9] M. Wasadikar and P. Survase. Incomparability graphs of lattices II. *Combinatorial Algorithms. IWOCA 2012. Lecture Notes in Computer Science, Springer-Verlag, Berlin, Heidelberg*, **7643** (2012), 148-–161.

[10] M. Wasadikar and P. Survase. Lattices, whose incomparability graphs have horns. *Journal of Discrete Algorithms*, **23** (2013), 63–75.

[11] H.P. Sankappanavar and S. Burris. A course in universal algebra. *Graduate Texts Math 78*, (1981).

[12] G. Chartrand, L. Lesniak and P. Zhang. Graphs and digraphs. *Vol. 22. London: Chapman and Hall*, (1996).

[13] F. Harary. Graph theory. *Perseus Books 15th print, 2001, Cambridge, Mass.* ISBN: 9780201410334.

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