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SPECTRA AND FINE SPECTRA OF THE GENERALIZED UPPER DIFFERENCE OPERATOR WITH TRIPLE REPETITION Δ_3^{rs} OVER THE SEQUENCE SPACE bv

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Abstract. In this paper, we investigate the fine spectrum which is given by Goldberg, and some spectral decompositions which are sets that do not have to be disjointed for the generalized upper difference operator with triple repetition Δ_3^{rs} over the sequence space bv. Afterwards, we generalize these results for $n > 3$ ($n \in \mathbb{N}$).

Spectral theory is a standard and useful the mathematical tool of in various science. For example in aviation it may be determined whether the flow over a wing is uniform or turbulent by spectral values, in structural mechanics, spectral theory may determine whether an automobile is too noisy or whether a building will collapse in an earthquake, etc. It also has many applications in both mathematics and physics, including matrix theory, control theory, function theory, differential and integral equations, complex analysis, and quantum physics. For example, in quantum mechanics, it determines atomic energy levels and thus the frequency of a laser or the spectral signature of a star is obtained.

Band matrices occur in many fields and applications of mathematics. For example tridiagonal, or more general, banded matrices are used in telecomunication system analysis, finite difference methods for solving partial differential equations, linear recurrence systems with non-constant coeffcients, etc. Spectrum of band operators on the sequence space have been studied in recent years. Therefore, in this article, we examined the spectrum of the band operator. The band operator

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we examined is

$$\Delta_3^{rs} x = \Delta_3^{rs}(x_n) = (r_n x_n + s_n x_{n+1})_{n=0}^{\infty} \tag{0.1}$$

where $r_x = r_y$, $s_x = s_y$ for $x \equiv y(mod3)$. In this paper, we investigate the spectrum, the fine spectrum and subdivision of the spectrum of Δ_3^{rs} over the sequence space $bv = \{x = (x_n) : \sum_n |x_n - x_{n+1}| < \infty\}$.

1. Definitions and preliminaries

Let X and Y be the Banach spaces, and $L : X \to Y$ be a bounded linear operator. The range of L, the domain of L are denoted $R(L)$, $D(L)$ respectively. Set of all bounded linear operators on X into itself is denoted by $B(X)$.

Let $L : D(L) \subset X \to X$ be a linear operator. Herein X is a complex normed space. For $L \in B(X)$, $L_\lambda := \lambda I - L$ is denoted. Herein I is the identity operator and $\lambda \in \mathbb{C}$. Then a regular value of L is $\lambda \in \mathbb{C}$ such that $(R1)$ L_λ^{-1} exists; $(R2)$ L_λ^{-1} is bounded; $(R3)$ L_λ^{-1} is defined on a set which is dense in X. The set of all regular values is called as the resolvent set of L and is indicated with $\rho(L, X)$. In $\mathbb{C}$, the complement of $\rho(L, X)$ is called as the spectrum of L and is indicated with $\sigma(L, X)$.

The spectrum $\sigma(L, X)$ is the union of three disjoint sets as follows:

(1) The point spectrum $\sigma_p(L, X)$ is the set which L_λ^{-1} does not exist.

(2) The continuous spectrum $\sigma_c(L, X)$ is the set which the operator L_λ^{-1} is defined on a dense subspace of X and is unbounded.

(3) The residual spectrum $\sigma_r(L, X)$ is the set which the operator L_λ^{-1} exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$) is not dense in X than in this case L_λ^{-1} may be bounded or unbounded. From above definitions, we have

$$\sigma(L, X) = \sigma_p(L, X) \cup \sigma_c(L, X) \cup \sigma_r(L, X) \tag{1.1}$$

and

$$\sigma_p(L, X) \cap \sigma_c(L, X) = \emptyset,\ \sigma_p(L, X) \cap \sigma_r(L, X) = \emptyset,\ \sigma_r(L, X) \cap \sigma_c(L, X) = \emptyset.$$

The spectrum $\sigma(L, X)$ is also the union of three sets that are not necessarily disjoint as follows:

(1) The defect spectrum: $\sigma_\delta(L, X) := \{\lambda \in \sigma(L, X) : R(L_\lambda) \neq X\}$,

(2) The compression spectrum: $\sigma_{co}(L, X) := \left\{\lambda \in \mathbb{C} : \overline{R(L_\lambda)} \neq X\right\}$,

(3) The approximate point spectrum: $\sigma_{ap}(L, X) := \{\lambda \in \mathbb{C}$:there exists a sequence (x_k) in X such that $\|x_k\| = 1$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \|L_\lambda(x_k)\| = 0\}$.

The following Proposition is useful because the adjoint operator of the linear operator takes advantage of for calculating the partition of the spectrum of the linear operator in Banach spaces.

Proposition 1.1 ([**1**], Proposition 1.3). *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*
(a) $\sigma(L^*, X^*) = \sigma(L, X)$,

(b) $\sigma_c(L^*, X^*) \subseteq \sigma_{ap}(L, X)$,
(c) $\sigma_{ap}(L^*, X^*) = \sigma_\delta(L, X)$,
(d) $\sigma_\delta(L^*, X^*) = \sigma_{ap}(L, X)$,
(e) $\sigma_p(L^*, X^*) = \sigma_{co}(L, X)$,
(f) $\sigma_{co}(L^*, X^*) \supseteq \sigma_p(L, X)$,
(g) $\sigma(L, X) = \sigma_{ap}(L, X) \cup \sigma_p(L^*, X^*) = \sigma_p(L, X) \cup \sigma_{ap}(L^*, X^*)$.

Goldberg's classification of spectrum

If $L \in B(X)$, then there are three cases for $R(L)$:

(I) $R(L) = X$, (II) $R(L) \neq \overline{R(L)} = X$, (III) $\overline{R(L)} \neq X$

and three cases for L^{-1}:

(1) L^{-1} exists and continuous, (2) L^{-1} exists but discontinuous, (3) L^{-1} does not exist.

If these cases are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2 , II_3, III_1, III_2, III_3 (see [**2**]).

$\sigma(L, X)$ can be divided into subdivisions $I_2\sigma(L, X) = \emptyset$, $I_3\sigma(L, X)$, $II_2\sigma(L, X)$, $II_3\sigma(L, X)$, $III_1\sigma(L, X)$, $III_2\sigma(L, X)$, $III_3\sigma(L, X)$. For example, if $T = \lambda I - L$ is in a given state, II_3 (say), then we write $\lambda \in II_3\sigma(L, X)$.

By the definitions given above and the introduction, the following table can be written

		1	2	3
		L_λ^{-1} exists and is bounded	L_λ^{-1} exists and is unbounded	L_λ^{-1} does not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L, X)$	–	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L, X)$	$\lambda \in \sigma_c(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_r(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$	$\lambda \in \sigma_p(L, X)$ $\lambda \in \sigma_{ap}(L, X)$ $\lambda \in \sigma_\delta(L, X)$ $\lambda \in \sigma_{co}(L, X)$

Table 1

By w, we denote the space of all sequences. Well-known examples of Banach sequence spaces are the spaces ℓ_∞, c, c_0 and bv of bounded, convergent, null and bounded variation sequences, respectively. Also by ℓ_p, bv_p we denote the spaces of all $p-$absolutely summable sequences and $p-$bounded variation sequences, respectively.

In this paper, we focus on sequence space

$$bv = \left\{ x = (x_n) : \sum_n |x_n - x_{n+1}| < \infty \right\}.$$

An equivalent norm on the sequence space bv is $\sum_n |x_n - x_{n-1}|$. The dual space bv^* of bv is norm isomorphic to the Banach space

$$bs = \left\{ x = (x_k) \in w : \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

The spectrum and fine spectrum of bounded linear operators on certain sequence spaces have been studied by many researchers. Here are some articles that have been studied on the spectrum of linear operators on bv and bv_p: In [**3**], Okutoyi examined on the spectrum of C_1 on bv. In [**4**], Yıldırım studied on the spectrum of the Rhaly operators on bv. In [**5**], on the spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv was examined by Furkan et. al. In [**6**], Akhmedov and Başar studied the fine spectra of the difference operator Δ over the sequence space bv_p $(1 \leqslant p < \infty)$. In [**7**], Bilgiç and Furkan calculated the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_p and bv_p $(1 \leqslant p < \infty)$. In [**8**], the spectrum of the operator $D(r,0,0,s)$ over the sequence spaces ℓ_p and bv_p was examined by Paul and Tripathy. In [**9**], Sawano and El-Shabrawy examined the fine spectra of the discrete generalized Cesàro operator on Banach sequence spaces.

2. Fine spectrum

The matrix representation corresponding to our operator is as follows:

$$\Delta_3^{rs} = \begin{bmatrix} r_0 & s_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & r_1 & s_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & r_2 & s_2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & r_0 & s_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & r_1 & s_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & r_2 & s_2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix} \quad (s_0, s_1, s_2 \neq 0). \tag{2.1}$$

As can be seen, this matrix is a triple repeating upper triangular band matrix. In this section, we going to calculate the fine spectra of the matrix Δ_3^{rs}.

Lemma 2.1 (Stieglitz and Tietz [**10**]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in (bv; bv)$ from bv to itself if and only if*

(i) $\sum\limits_{k=1}^{\infty} a_{nk} < \infty$*, for each n,*

(ii) $\sup\limits_{N>0} \sum\limits_{n=1}^{\infty} \left| \sum\limits_{k=0}^{N} (a_{nk} - a_{n-1,k}) \right| < \infty$*,*

Theorem 2.1. *$\Delta_3^{rs} : bv \to bv$ is a bounded linear operator.*

Proof. (i) It is clear.

(ii) Let $B_n^N = \sum\limits_{k=0}^{N} (a_{nk} - a_{n-1,k})$ and if we calculate B_n then we have

$$\begin{aligned}
B_1^N &= \sum_{k=0}^{N}(a_{1k}-a_{0k}) = (0-r_0)+(r_1-s_0)+(r_1-0)+(0-0)+\cdots+0+\cdots \\
B_2^N &= \sum_{k=0}^{N}(a_{2k}-a_{1k}) = (0-0)+(0-r_1)+(r_2-s_1)+(s_2-0)+(0-0)+\cdots+0+\cdots \\
&\vdots \\
B_n^N &= \sum_{k=0}^{N}(a_{nk}-a_{n-1,k}) = \underbrace{(0-0)+\cdots+(0-0)}_{(n-1)\text{ times}}+(0-r_{n-1})+(r_n-s_{n-1}) \\
&\quad +(s_n-0)+(0+0)+\cdots
\end{aligned}$$

Therefore we get

$$B_n^N = \begin{cases} 0 & , \quad N \leqslant n-1 \\ -r_{n-1} & , \quad N = n-1 \\ -r_{n-1}+r_n-s_{n-1} & , \quad N = n \\ r_n+s_n-(r_{n-1}+s_{n-1}) & , \quad N \geqslant n+1 \end{cases}.$$

Since

$$\sum_{n=1}^{\infty}\left|B_n^N\right| = \begin{cases} |r_0| & , \quad N=0 \\ |r_0+s_0| & . \quad N>0 \end{cases},$$

we have

$$\sup_{N>0}\sum_{n=1}^{\infty}\left|\sum_{k=0}^{N}(a_{nk}-a_{n-1,k})\right| = \sup_{N>0}\sum_{n=1}^{\infty}\left|B_n^N\right| = \max\{|r_0+s_0|,|r_0|\}.$$

So the conditions of the Lemma 2.1 are satisfied and it is a bounded linear operator. □

LEMMA 2.2 (Golberg [**2**, p.59]). *T has a dense range if and only if T^* is 1-1.*

LEMMA 2.3 (Golberg [**2**, p.60]). *T has a bounded inverse if and only if T^* is onto.*

Throughout this work, for convenience, we will denote the set

$$\{\lambda \in \mathbb{C} : |\lambda - r_0|\,|\lambda - r_1|\,|\lambda - r_2| \leqslant |s_0|\,|s_1|\,|s_2|\}$$

with M. Thus, we will denote the boundary and interior of the set M with ∂M and $\mathring{M}$ respectively.

THEOREM 2.2. *$\sigma_p(\Delta_3^{rs}, bv) = \mathring{M}$.*

Proof. Let λ be an eigenvalue of the operator Δ_3^{rs}. Then there exists $h \neq \theta = (0,0,0,...)$ in bv such that $\Delta_3^{rs} h = \lambda h$. Then for $x_n := \dfrac{\lambda - r_n}{s_n}$ we obtain

$$\begin{cases} h_{3n} &= t^n h_0, \\ h_{3n+1} &= x_0 t^n h_0, \\ h_{3n+2} &= x_0 x_1 t^n h_0, \end{cases} \quad , \ n \geqslant 0$$

where $t = x_0 x_1 x_2$. Thus we get

$$|h_{3k+i} - h_{3k+i+1}| = |D_i| \, |t|^k \, |h_0| \, , \ i = \overline{0,2}$$

where

$$D_r := \begin{cases} 1 - x_0 & , \quad i = 0 \\ x_0 - x_0 x_1 & , \quad i = 1 \\ x_0 x_1 - x_0 x_1 x_2 & , \quad i = 2 \end{cases}$$

Hereby we have

$$\sum_{n=1}^{\infty} |h_n - h_{n+1}| = \sum_{k=0}^{\infty} |h_{3k+r} - h_{3k+r+1}| = |D_r| \, |h_0| \sum_{k=0}^{\infty} |t|^k \, .$$

Since $D_r \, |h_0| \sum\limits_{k=0}^{\infty} |t|^k$ is convergent if and only if $|t| < 1$, $h = (h_n) \in bv$ is convergent if and only if $|\lambda - r_0| \, |\lambda - r_1| \, |\lambda - r_2| < |s_0| \, |s_1| \, |s_2|$ and so $\sigma_p(\Delta_3^{rs}, bv) = \mathring{M}$. □

If $T : bv \longmapsto bv$ is a bounded linear operator represented with a matrix A, then it is known that the adjoint operator $T^* : bv^* \longmapsto bv^*$ of T operator is represented by

$$A^* = \begin{pmatrix} \bar{\chi} & v_0 - \bar{\chi} & v_1 - \bar{\chi} & v_2 - \bar{\chi} & \cdots \\ u_0 & a_{00} - u_0 & a_{10} - u_0 & a_{20} - u_0 & \cdots \\ u_1 & a_{01} - u_1 & a_{11} - u_1 & a_{21} - u_1 & \cdots \\ u_2 & a_{02} - u_2 & a_{12} - u_2 & a_{22} - u_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$u_n = \lim_{m \to \infty} a_{m,n} \quad , \quad v_n = \sum_{m=0}^{\infty} a_{n,m}$$

and

$$\bar{\chi} = \lim_{n \to \infty} v_n .$$

The dual space bv^* of bv is norm isomorphic to the Banach space

$$bs := \left\{ x = (x_k) \in w : \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

In this section, we will take $r_n + s_n = r_{n+1} + s_{n+1} = c$, c is a constant, for $(\Delta_3^{rs})^*$ to exist, herein $r_x = r_y$, $s_x = s_y$, $x \equiv y \, (mod 3)$.

Theorem 2.3. $\sigma_p((\Delta_3^{rs})^*, bv^* \cong bs) = \emptyset$.

PROOF. Let λ be an eigenvalue of the operator $(\Delta_3^{rs})^*$ that is to say there exists $h \neq \theta = (0,0,0,...)$ in bs such that $(\Delta_3^{rs})^* h = \lambda h$.
Then, we obtain

$$ch_0 = \lambda h_0 \tag{2.2}$$

$$r_0 h_1 = \lambda h_1 \tag{2.3}$$

$$s_0 h_1 + r_1 h_2 = \lambda h_2 \tag{2.4}$$

$$s_1 h_2 + r_2 h_3 = \lambda h_3 \tag{2.5}$$

$$s_2 h_3 + r_0 h_4 = \lambda h_4 \tag{2.6}$$

$$\vdots$$

$$s_n h_{n+1} + r_{n+1} h_{n+1} = \lambda h_{n+1} \tag{2.7}$$

$$\vdots$$

Then if $h_0 \neq 0$, then from (2.2) $\lambda = c$ and from (2.3) $r_0 = c$. Also since $c = r_0 + s_0$ we get $s_0 = 0$ and this is a contradiction so $h_0 = 0$. Suppose that h_m be the first non-zero of the sequence (h_n) in this case if we take $n = m-1$ in (2.5) then $\lambda = r_m$ for

$$s_{m-1} h_{m-1} + r_m h_m = \lambda h_m$$

if we take $n = m$ in (2.7) since $\lambda = r_m$ we have

$$s_m h_m + r_{m+1} h_{m+1} = r_m h_{m+1}$$

hence

$$h_{m+1} = \left(\frac{s_{m+1}}{r_m - r_{m+1}} \right) h_m \tag{2.8}$$

if we take $n = m+1$ in (2.7) and since $\lambda = r_m$

$$h_{m+2} = \left(\frac{s_{m+1}}{r_m - r_{m+2}} \right) h_{m+1} \tag{2.9}$$

and if we take $n = m+2$ in (2.7) and since $\lambda = r_m$ we get

$$s_{m+2} h_{m+2} + r_{m+3} h_{m+3} = r_m h_{m+3}$$

and since $r_x = r_y$, $s_x = s_y$, $x \equiv y\,(mod 3)$, we have $s_{m+2} h_{m+2} = 0$ which implies $h_{m+2} = 0$ as $s_{m+2} \neq 0$. Therefore $h_{m+1} = 0$ from (2.9) then $h_m = 0$ from (2.8), a contradiction. Hereby, $\sigma_p((\Delta_3^{rs})^*, bv^* \cong bs) = \emptyset$. $\square$

THEOREM 2.4. $\sigma_r(\Delta_3^{rs}, bv) = \emptyset$.

PROOF. Owing to $\sigma_r(A, bv) = \sigma_p(A^*, bs) \backslash \sigma_p(A, bv)$, required result is given us by Theorems 2.2 and 2.3. $\square$

LEMMA 2.4.

$$\sum_{m=1}^{n}\left(\sum_{i=0}^{3m+t} f_i g_{mi}\right)=\sum_{i=0}^{2} f_i\left(\sum_{m=1}^{n} g_{mi}\right)+\sum_{i=1}^{n} f_{3i+t}\left(\sum_{m=i}^{n} g_{m,3i+t}\right)\ ,\ t=0,1,2$$

herein (f_k) *and* (g_{nk}) *are real numbers.*

PROOF. It is clear. □

THEOREM 2.5. $\sigma_c\left(\Delta_3^{rs}, bv\right)=\partial M$ *and* $\sigma\left(\Delta_3^{rs}, bv\right)=M$.

PROOF. Let $k=(k_n)\in bs$ be such that $(\Delta_3^{rs}-\lambda I)^* h=k$ for some $h=(h_n)$. Then we get following system of linear equations:

$$\begin{aligned}
(c-\lambda)h_0 &= k_0\\
(r_0-\lambda)h_1 &= k_1\\
s_0 h_1+(r_1-\lambda)h_2 &= k_2\\
&\vdots\\
s_2 h_{3n}+(r_0-\lambda)h_{3n+1} &= k_{3n+1}\\
s_0 h_{3n+1}+(r_1-\lambda)h_{3n+2} &= k_{3n+2}\\
s_1 h_{3n+2}+(r_2-\lambda)h_{3n+3} &= k_{3n+3}\\
&\vdots
\end{aligned}$$

where $n\geqslant 0$. Solving above equations, for $t=0,1,2; n=1,2,\ldots$, we have

$$h_{3n+t}=\frac{1}{r_{t+2}-\lambda}\left[\sum_{m=1}^{3n+t}(-1)^{3n+t-m}k_m\prod_{v=0}^{3n+t-m-1}\frac{s_{3n+t+1-\nu}}{r_{3n+t+1-\nu}-\lambda}\right].$$

Herein $r_x=r_y$, $s_x=s_y$ for $x\equiv y(mod3)$. Therefore we get

$$\begin{aligned}
\left|\sum_{m=0}^{3n+t} h_k\right| &= |h_0+h_1+h_2+h_3+\cdots+h_{3n+t}|\\
&= \left|h_0+h_1+h_2+\sum_{m=1}^{n} h_{3m+t}\right|\\
&\leqslant |h_0+h_1+h_2|\\
&\quad+\left|\sum_{m=1}^{n}\frac{1}{a_{t+2}-\lambda}\left[\sum_{i=0}^{3m+t}(-1)^{3m+t-i}k_i\prod_{\nu=0}^{3m+t-i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right]\right|\\
&\leqslant |h_0+h_1+h_2|\\
&\quad+\max_{l=0}^{2}\left|\frac{1}{a_l-\lambda}\right|\left|\sum_{m=1}^{n}\sum_{i=0}^{3m+t}(-1)^{3m+t-i}k_i\prod_{\nu=0}^{3m+t-i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right|
\end{aligned}$$

Thus the inequality is gotten;

$$\left|\sum_{k=0}^{3n+t} f_k\right| \leqslant |h_0+h_1+h_2|$$

$$+\max_{l=0}^{2}\left|\frac{1}{a_l-\lambda}\right|\left|\sum_{m=1}^{n}\sum_{i=0}^{3m+t}(-1)^{3m+t-i}k_i\prod_{\nu=0}^{3m+t-i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right| \tag{2.10}$$

In Lemma 2.4 if we take $f_i = k_i$ and $g_{mi} = (-1)^{3m+t-i}\prod_{\nu=0}^{3m+t-i-1}\frac{b_{3m+t+1-\nu}}{a_{3m+t+1-\nu}-\lambda}$ then we have

$$\begin{aligned}
&\left|\sum_{m=1}^{n}\sum_{i=0}^{3m+t}(-1)^{3m+t-i}k_i\prod_{\nu=0}^{3m+t-i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right|\\
= &\left|\sum_{i=0}^{2}k_i\left(\sum_{m=1}^{n}(-1)^{3m+t-i}\prod_{\nu=0}^{3m+t-i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right)\right.\\
&\left.+\sum_{i=1}^{n}k_{3i+t}\left(\sum_{m=i}^{n}(-1)^{3m-3i}\prod_{\nu=0}^{3m-3i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right)\right|\\
\leqslant &\left|\sum_{i=0}^{2}k_i\left(\sum_{m=1}^{n}(-1)^{3m+t-i}\prod_{\nu=0}^{3m+t-i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right)\right|\\
&+\left|\sum_{i=1}^{n}k_{3i+t}\left(\sum_{m=i}^{n}(-1)^{3m-3i}\prod_{\nu=0}^{3m-3i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right)\right|
\end{aligned}$$

Also since both $\prod_{\nu=0}^{3m+t-i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda} = w_e u^{3m-(i+t)}, t = 0,1,2$ where w_e, $e = 0,1,2$ constant, moreover

while $t=0,\begin{cases} w_0 = 1\\ w_1 = \frac{b_1}{(a_1-\lambda)}\\ w_{2}=\frac{b_1 b_0}{(a_1-\lambda)(a_0-\lambda)}\end{cases}$, while $t=1,\begin{cases} w_0 = \frac{b_2}{(a_2-\lambda)}\\ w_1 = \frac{b_2 b_1}{(a_2-\lambda)(a_1-\lambda)}\\ w_{2}=\frac{b_0 b_1 b_2}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}\end{cases}$ and while $t=2,\begin{cases} w_0 = \frac{b_0 b_2}{(a_0-\lambda)(a_1-\lambda)}\\ w_1 = \frac{b_0 b_1 b_2}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}\\ w_{2}=\frac{b_0^2 b_1 b_2}{(a_0-\lambda)^2(a_1-\lambda)(a_2-\lambda)}\end{cases}$ and setting

$u = \left(\frac{s_2 s_1 s_0}{(r_2-\lambda)(r_1-\lambda)(r_0-\lambda)}\right)^{1/3}$. Therefore the multiplication become $\prod_{\nu=0}^{3m-3i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda} = u^{3m-3i}$, the last equation turns into the sum

$$|w_e|\left|\sum_{i=0}^{2}k_i\left(\sum_{m=1}^{n}(-1)^{3m+t-i}u^{3m-(i+t)}\right)\right| + \left|\sum_{i=1}^{n}k_{3i+t}\left(\sum_{m=i}^{n}(-1)^{3m-3i}u^{3m-3i}\right)\right|.$$

Now the consider only the sum $\left|\sum\limits_{i=1}^{n} k_{3i+t}\left(\sum\limits_{m=i}^{n}(-1)^{3m-3i}u^{3m-3i}\right)\right|$. Then

$$\left|\sum_{i=1}^{n} k_{3i+t}\left(\sum_{m=i}^{n}(-1)^{3m-3i}u^{3m-3i}\right)\right| = \left|\sum_{i=1}^{n} k_{3i+t}\left(\sum_{m=i}^{n}(-u)^{3k-3i}\right)\right|$$

from now on $(-u)^3 = z$ so

$$\begin{aligned}\left|\sum_{i=1}^{n} k_{3i+t}\left(\sum_{m=i}^{n}(-u)^{3m-3i}\right)\right| &= \left|\frac{1}{1-z}\right|\left|\sum_{i=1}^{n} k_{3i+t}\left(1-z\right)^{n-i+1}\right| \\ &\leqslant \left|\frac{1}{1-z}\right|\left|\sum_{i=1}^{n} k_{3i+t}\right| + \left|\frac{1}{1-z}\right|\left|\sum_{i=1}^{n} k_{3i+t}z^{n-i+1}\right|.\end{aligned}$$

Hence

$$\left|\sum_{i=1}^{n} k_{3i+t}\left(\sum_{m=i}^{n}(-1)^{3m-3i}u^{3m-3i}\right)\right| \leqslant \left|\frac{1}{1-z}\right|\|k\|_{bs} + \left|\frac{z^{n+1}}{1-z}\right|\left|\sum_{i=1}^{n} k_{3i+t}z^{-i}\right| \tag{2.11}$$

as long as $a_i = k_{3i+t}$, $b_i = z^{-i}$ and apply Abel's partial summation formula to sum $\sum\limits_{i=1}^{n} k_{3i+t}z^{-i} = \sum\limits_{i=1}^{n} \frac{k_{3i+t}}{z^i}$, now that $s_n = \sum\limits_{i=0}^{n} k_{3i+t}$, $\Delta b_i = \frac{r-1}{r^{i+1}}$ we obtain

$$\sum_{i=1}^{n} \frac{k_{3i+t}}{z^i} = \frac{1}{z^n}\sum_{i=1}^{n} k_{3i+t} + \sum_{i=1}^{n-1} \frac{z-1}{z^{i+1}}\sum_{m=0}^{i} k_{3m+t}.$$

Thus

$$\begin{aligned}\left|\frac{z^{n+1}}{1-z}\right|\left|\sum_{i=1}^{n} k_{3i+t}\left(z\right)^{-i}\right| &= \left|\frac{z^{n+1}}{1-z}\right|\left|\frac{1}{z^n}\sum_{i=1}^{n} k_{3i+t} + \sum_{i=1}^{n-1}\frac{z-1}{z^{i+1}}\sum_{m=0}^{i} k_{3m+t}\right| \\ &\leqslant \left|\frac{z}{1-z}\right|\left|\sum_{i=1}^{n} k_{3i+t}\right| + \left|\frac{(z-1)\,z^n}{1-z}\right|\left|\sum_{i=1}^{n-1}\frac{1}{z^i}\sum_{m=0}^{i} k_{3m+t}\right| \\ &\leqslant \left|\frac{z}{1-z}\right|\left|\sum_{i=1}^{n} k_{3i+t}\right| + \left|\frac{(z-1)\,z^n}{1-z}\right|\sum_{i=1}^{n-1}\frac{1}{|z|^i}\left|\sum_{m=0}^{i} k_{3m+t}\right| \\ &\leqslant \left|\frac{z}{1-z}\right|\|k\|_{bs} + \left|\frac{(z-1)\,z^n}{1-z}\right|\|k\|_{bs}\sum_{i=1}^{n-1}\frac{1}{|z|^i} \qquad (2.12)\end{aligned}$$

and we get

$$\left|\frac{z^{n+1}}{1-z}\right|\left|\sum_{i=1}^{n} k_{3i+t}z^{-i}\right| \leqslant \left[1 + (z-1)\,\frac{|z|^{n-1}-1}{|z|-1}\right]\left|\frac{z}{1-z}\right|\|k\|_{bs}. \tag{2.13}$$

Replacing (2.13) in (2.11),we have

$$\left|\sum_{i=1}^{n} k_{3i+t}\sum_{m=i}^{n}(-1)^{3m-3i}z^{3m-3i}\right| \leqslant \left|\frac{1}{1-z}\right|\|k\|_{bs} + \left[1+(z-1)\frac{|z|^{n-1}-1}{|z|-1}\right]\left|\frac{z}{1-z}\right|\|k\|_{bs}. \tag{2.14}$$

Replacing (2.14) in (2.12), we have

$$\left|\sum_{i=1}^{n} k_{3i+t}\left(\sum_{m=i}^{n}(-1)^{3m-3i}\prod_{\nu=0}^{3m-3i-1}\frac{s_{3m+t+1-\nu}}{r_{3m+t+1-\nu}-\lambda}\right)\right| \leqslant \left[1+|z|+(z-1)|z|\frac{|z|^{n-1}-1}{|z|-1}\right]\left|\frac{1}{1-z}\right|\|z\|_{bs}. \tag{2.15}$$

Finally replacing (2.15) in (2.10), we get

$$\begin{aligned}\left|\sum_{m=0}^{3n+t} h_m\right| \leqslant\ & |h_0+h_1+h_2| \\ & +\max_{l=0}^{2}\left|\frac{1}{a_l-\lambda}\right|\left\{\left|\sum_{i=0}^{2}k_i\left(\sum_{m=1}^{n}(-1)^{3m+t-i}\prod_{\nu=0}^{3m+t-i-1}\frac{s_{3m+t-\nu}}{r_{3m+t-\nu}-\lambda}\right)\right|\right. \\ & \left.+\left[1+|z|+(z-1)|z|\frac{|z|^{n-1}-1}{|z|-1}\right]\left|\frac{1}{1-z}\right|\|k\|_{bs}\right\}.\end{aligned}$$

Since $k=(k_n)\in bs$, $h=(h_n)\in bs$ if $|u|=\left|\dfrac{s_2s_1s_0}{(r_2-\lambda)(r_1-\lambda)(r_0-\lambda)}\right|^{1/3}<1$. Consequently, if for $\lambda\in\mathbb{C}$, $|r_2-\lambda|\,|r_1-\lambda|\,|r_0-\lambda|>|s_2|\,|s_1|\,|s_0|$, then $(h_n)\in bs$. Hereby, the operator $(\Delta_3^{rs}-\lambda I)^*$ is onto if $|\lambda-r_0|\,|\lambda-r_1|\,|\lambda-r_2|>|s_0|\,|s_1|\,|s_2|$. Then by Lemma 2.3, $\Delta_3^{rs}-\lambda I$ has a bounded inverse if $|\lambda-r_0|\,|\lambda-r_1|\,|\lambda-r_2|>|s_0|\,|s_1|\,|s_2|$. Therefore,

$$\sigma_c(\Delta_3^{rs},bv)\subseteq\{\lambda\in\mathbb{C}:|\lambda-r_0|\,|\lambda-r_1|\,|\lambda-r_2|\leqslant|s_0|\,|s_1|\,|s_2|\}.$$

Owing to $\sigma(L,X)$ is the disjoint union of $\sigma_p(L,X)$, $\sigma_r(L,X)$ and $\sigma_c(L,X)$, thence

$$\sigma(\Delta_3^{rs},bv)\subseteq\{\lambda\in\mathbb{C}:|\lambda-r_0|\,|\lambda-r_1|\,|\lambda-r_2|\leqslant|s_0|\,|s_1|\,|s_2|\}.$$

By Theorem 2.2, we have

$$\{\lambda\in\mathbb{C}:|\lambda-r_0|\,|\lambda-r_1|\,|\lambda-r_2|<|s_0|\,|s_1|\,|s_2|\}=\sigma_p\left(\Delta_3^{rs},bv\right)\subset\sigma(\Delta_3^{rs},bv).$$

Since, $\sigma(L,X)$ is closed and hence,

$$\overline{\{\lambda\in\mathbb{C}:|\lambda-r_0|\,|\lambda-r_1|\,|\lambda-r_2|<|s_0|\,|s_1|\,|s_2|\}}\subset\overline{\sigma(\Delta_3^{rs},bv)}=\sigma(\Delta_3^{rs},bv)$$

and

$$\{\lambda\in\mathbb{C}:|\lambda-r_0|\,|\lambda-r_1|\,|\lambda-r_2|\leqslant|s_0|\,|s_1|\,|s_2|\}\subset\sigma\left(U\left(r_0,r_1,r_2;s_0,s_1,s_2\right),bv\right).$$

Therefore, $\sigma(\Delta_3^{rs},bv)=M$ and so $\sigma_c(\Delta_3^{rs},bv)=\partial M$. $\square$

3. Subdivision of the spectrum

The articles in section 2, deal with spectrum decomposition described by Goldberg. In this section, the subdivision of the spectrum, which is also examined in [**11**]-[**21**] will be examined. For this, $r_0 = r_1 = r_0 = r$ and $s_0 = s_1 = s_0 = s$ must be taken in our operator.

LEMMA 3.1.

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1} f_n g_m\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1+m}^{\infty} f_n g_m\right)$$

where (f_n) *and* (b_m) *are positive real numbers.*

PROOF. It is clear. □

THEOREM 3.1. *If* $|\lambda - r| < |s|$, *then* $\lambda \in I_3\sigma(\Delta_3^{rs}, bv)$.

PROOF. Assume that $|\lambda - r| < |s|$ and hereby from Theorem 2.2, $\lambda \in \sigma_p(\Delta_3^{rs}, bv)$. So, λ satisfies Golberg's condition 3. We should show that $\Delta_3^{rs} - \lambda I$ is onto when $|\lambda - r| < |s|$.

Let $k = (k_n) \in bv$ be such that $(\Delta_3^{rs} - \lambda I)h = k$ for $h = (h_n)$. Then,

$$(r - \lambda)\, h_m + s h_{m+1} = k_m \ , \quad m \geqslant 0$$

Calculating h_m, we get

$$h_n = \frac{1}{s}\sum_{m=1}^{n} k_{m-1}\left(\frac{\lambda - r}{s}\right)^{n-m} + h_0\left(\frac{\lambda - r}{s}\right)^{n}, \quad n = 1, 2, 3, \ldots \tag{3.1}$$

We have to show that $h = (h_m) \in bv$. Setting $q := \left|\frac{\lambda - r}{s}\right|$.

$$\begin{aligned} |h_n - h_{n+1}| &= \left|\frac{1}{s}\sum_{m=1}^{n} k_{m-1} q^{n-m} + h_0 q^n - \frac{1}{s}\sum_{m=1}^{n+1} k_{m-1} q^{n+1-m} - h_0 q^{n+1}\right| \\ &= \left|\frac{q^n\,(1-q)}{s}\sum_{m=1}^{n} k_{m-1} q^{-m} + h_0\left(q^n - q^{n+1}\right) - \frac{1}{s}k_n\right| \end{aligned}$$

If we take $a_k = \frac{1}{q^m}$, $b_k = k_{m-1}$ and apply Abel's partial summation formula to sum $\sum_{m=1}^{n}\frac{k_{m-1}}{q^m}$, we obtain

$$\sum_{m=1}^{n}\frac{k_{m-1}}{q^m} = \frac{q^n - 1}{q^n\,(q-1)}k_{n-1} + \sum_{m=1}^{n-1}\frac{q^m - 1}{q^m\,(q-1)}\left(k_{m-1} - k_m\right)$$

since $s_n = \sum_{m=1}^{n} \left(\frac{1}{q}\right)^m$, $\Delta b_k = k_{m-1} - k_m$. Thus we have

$$|h_n - h_{n+1}| = \left| \tfrac{1}{s}\left(-q^n + 1\right) k_{n-1} + \tfrac{-q^n}{s} \sum_{m=1}^{n-1} \tfrac{q^m - 1}{q^m}\left(k_{m-1} - k_m\right) + h_0\left(q^n - q^{n+1}\right) - \tfrac{1}{s} k_n \right|,$$

$$|h_n - h_{n+1}| = \left| \tfrac{-q^n}{s} k_{n-1} + \tfrac{1}{s}\left(k_{n-1} - k_n\right) + \tfrac{-q^n}{s} \sum_{m=1}^{n-1} \tfrac{q^m - 1}{q^m}\left(k_{m-1} - k_m\right) + h_0\left(q^n - q^{n+1}\right) \right|.$$

Then we get

$$\begin{aligned}
\sum_{n=1}^{\infty} |h_n - h_{n+1}| &= \sum_{n=1}^{\infty} \left| \tfrac{-q^n}{s} k_{n-1} + \tfrac{1}{s}\left(k_{n-1} - k_n\right) + \tfrac{-q^n}{s} \sum_{m=1}^{n-1} \tfrac{q^m - 1}{q^m}\left(k_{m-1} - k_m\right) + h_0\left(q^n - q^{n+1}\right) \right| \\
&\leqslant \sum_{n=1}^{\infty} \left| \tfrac{-q^n}{s} k_{n-1} \right| + \tfrac{1}{s} \sum_{n=1}^{\infty} n \left|\left(k_{n-1} - k_n\right)\right| + \left|-\tfrac{1}{s}\right| \sum_{n=1}^{\infty} \left| \sum_{m=1}^{n-1} q^n \tfrac{q^m - 1}{q^m}\left(k_{m-1} - k_m\right) \right| \\
&\qquad + \sum_{n=1}^{\infty} \left| h_0\left(q^n - q^{n+1}\right) \right| \\
&\leqslant \sum_{n=1}^{\infty} \left| \tfrac{-q^n}{s} k_{n-1} \right| + \tfrac{1}{s} \sum_{n=1}^{\infty} \left|\left(k_{n-1} - k_n\right)\right| + \left|-\tfrac{1}{s}\right| \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} |q|^n \tfrac{|q^m - 1|}{|q|^m} \left|k_{m-1} - k_m\right| \\
&\qquad + \sum_{n=1}^{\infty} \left| h_0\left(q^n - q^{n+1}\right) \right|.
\end{aligned}$$

We consider only the sum $\sum_{n=1}^{\infty} \sum_{m=1}^{n-1} |q|^n \frac{|q^m - 1|}{|q|^m} |k_{m-1} - k_m|$. If we take $a_n = |q|^n$, $b_m = \dfrac{|q^m - 1|}{|q|^m} |k_{m-1} - k_m|$ and from lemma 3.1

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{m=1}^{n-1} |q|^n \frac{|q^m - 1|}{|q|^m} |k_{m-1} - k_m| &= \sum_{m=1}^{\infty} \frac{|q^m - 1|}{|q|^m} |k_{m-1} - k_m| \sum_{n=m+1}^{\infty} |q|^n \\
&= |q| \frac{1 - |q|^n}{1 - |q|} \sum_{m=1}^{\infty} |q^m - 1| \, |k_{m-1} - k_m|
\end{aligned}$$

therefore,

$$\sum_{n=1}^{\infty} |q|^n \sum_{m=1}^{n-1} \frac{|q^m - 1|}{|q|^m} |k_{m-1} - k_m| \leqslant 2\,|q| \frac{1 - |q|^n}{1 - |q|} \, \|k_m\|_{bv}$$

since

$$\left|q^k - 1\right| \leqslant |q|^k + 1 < 2 \text{ for } |r| < 1$$

thus we have, (k_n) sequence is bounded and h_0 is constant. Since $|q| < 1$ the series, $\sum_{n=1}^{\infty} \left| \dfrac{-q^n}{s} k_{n-1} \right|$ and the series $\sum_{n=1}^{\infty} \left| h_0\left(q^n - q^{n+1}\right) \right|$ is convergent. Thus since $k = (k_n) \in bv$, $\lambda \in \sigma_p(\Delta_3^{rs}, bv)$ imply that the series $\sum_{n=1}^{\infty} |h_n - h_{n+1}|$ is convergent. Hence, $(h_n) \in bv$ if $|\lambda - r| < |s|$. Hereby, $\Delta_3^{rs} - \lambda I$ is onto. So, $\lambda \in I$. Hence the required result is gotten. □

Corollary 3.1. $III_1\sigma(\Delta_3^{rs}, bv) = III_2\sigma(\Delta_3^{rs}, bv) = \emptyset$.

Proof. $\sigma_r(L,X) = III_1\sigma(L,X) \cup III_2\sigma(L,X)$ in Table 1 and Theorem 2.4 gives the desired result where $r_0 = r_1 = r_0 = r$ and $s_0 = s_1 = s_0 = s$. □

Corollary 3.2. *$II_3\sigma(\Delta_3^{rs}), bv) = III_3\sigma(\Delta_3^{rs}, bv) = \emptyset$.*

Proof. $\sigma_p(L,X) = I_3\sigma(L,X) \cup II_3\sigma(L,X) \cup III_3\sigma(L,X)$ in Table 1 and Theorem 2.2 and Theorem 3.1 gives the desired result where $r_0 = r_1 = r_0 = r$ and $s_0 = s_1 = s_0 = s$. □

Theorem 3.2. *(a) $\sigma_{ap}(\Delta_3^{rs}, bv) = \{\lambda \in \mathbb{C} : |\lambda - r| \leqslant |s|\}$,*
(b) $\sigma_\delta(\Delta_3^{rs}, bv) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$,
(c) $\sigma_{co}(\Delta_3^{rs}, bv) = \emptyset$.

Proof. (a) In Table 1, we get

$$\sigma_{ap}(L,X) = \sigma(L,X) \backslash III_1\sigma\,(L,X)\,.$$

And so $\sigma_{ap}(\Delta_3^{rs}, bv) = \{\lambda \in \mathbb{C} : |\lambda - r| \leqslant |s|\}$ from Corollary 3.1.

(b) In Table 1, we get

$$\sigma_\delta(L,X) = \sigma\,(L,X) \backslash I_3\sigma(L,X)\,.$$

So using Theorem 2.5 and 3.1 with $r_0 = r_1 = r_0 = r$ and $s_0 = s_1 = s_0 = s$, the desired result is gotten.

(c) From Proposition 1.1 (e), we get

$$\sigma_p(L^*, X^*) = \sigma_{co}(L,X).$$

Using Theorem 2.3 with $r_0 = r_1 = r_0 = r$ and $s_0 = s_1 = s_0 = s$, the desired result is gotten. □

Corollary 3.3. *(a) $\sigma_{ap}((\Delta_3^{rs})^*, bv^* \cong bs) = \{\lambda \in \mathbb{C} : |\lambda - r| = |s|\}$,*
(b) $\sigma_\delta((\Delta_3^{rs})^*, bv^* \cong bs) = \{\lambda \in \mathbb{C} : |\lambda - r| \leqslant |s|\}$.

Proof. By Proposition 1.1 (c) and (d), we obtain

$$\sigma_{ap}((\Delta_3^{rs})^*, bv^* \cong bs) = \sigma_\delta(\Delta_3^{rs}, bv)$$

and

$$\sigma_\delta((\Delta_3^{rs})^*, bv^* \cong bs) = \sigma_{ap}(\Delta_3^{rs}, bv).$$

from Theorem 3.2 (a) and (b) with $r_0 = r_1 = r_0 = r$ and $s_0 = s_1 = s_0 = s$ the required results are gotten. □

4. Results

We can generalize our operator

$$\Delta_n^{rs} = \begin{bmatrix} r_0 & s_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & r_1 & s_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & s_2 & \ddots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & r_{n-1} & s_{n-1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & r_0 & s_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & r_1 & s_1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

where $s_0, s_1, \ldots, s_{n-1} \neq 0$.

One can get parallel all our results obtained in the before section as follows.

THEOREM 4.1. *The following results for Δ_n^{rs} are ensured where*

$$R = \left\{ \lambda \in \mathbb{C} : \prod_{k=0}^{n-1} \left| \frac{\lambda - r_k}{s_k} \right| \leqslant 1 \right\},$$

$\mathring{R}$ be the interior of the set R and ∂R be the boundary of the set R

(1) $\sigma_p(\Delta_n^{rs}, bv) = \mathring{R}$,
(2) $\sigma_p((\Delta_n^{rs})^*, bv^* \cong bs) = \emptyset$,
(3) $\sigma_r(\Delta_n^{rs}, bv) = \emptyset$,
(4) $\sigma_c(\Delta_n^{rs}, bv) = \partial R$,
(5) $\sigma(\Delta_n^{rs}, bv) = R$.

References

[1] J. Appell, E. De Pascale, and A. Vignoli, *Nonlinear Spectral Theory*, Walter de Gruyter, Berlin, New York, 2004.

[2] S. Goldberg, *Unbounded Linear Operators*, McGraw Hill, New York, 1966.

[3] J. T. Okutoyi, *On the spectrum of C_1 as an operator on bv*, Commun. Fac. Sci. Univ. Ank. Series A_1, **41**, (1922), 197–207.

[4] M. Yildirim, *On the spectrum of the Rhaly operators on bv*, East Esian Math., textbf81(1), (2002), 21–41.

[5] H. Furkan, H. Bilgic, and K. Kayaduman, *On the spectrum of the generalized difference operator B(r,s) over the sequence spaces l_1 and bv*, Hokkaido Mathematical Journal, **35**, (2006), 893–904.

[6] A. M. Akhmedov and F. Başar, *The fine spectra of the difference operator Δ over the sequence space bv_p*, Acta Mathematica Sinica, English Series **23**(10), (2007), 1757–1768.

[7] H. Bilgiç and H. Furkan, *On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces l_p and bv_p,* $(1 < p < \infty)$, Nonlinear Analysis, **68**, (2008), 499–506.

[8] A. Paul and B. C. Tripathy, *The spectrum of the operator $D(r,0,0,s)$ over the sequence spaces ℓ_p and bv_p*, Hacet. J. Math. Stat., **43**(3), (2014), 425–434.

[9] Y. Sawano and S. R. El-Shabrawy, *Fine spectra of the discrete generalized Cesàro operator on Banach sequence spaces*, Monatshefte für Mathematik, 2020.

[10] M. Stieglitz and H. Tietz, *Matrixtranformationen von Folgeräumen Eine Ergebnisübersicht. Math. Z.*, **154**, (1977),1–16.

[11] N. Durna and M. Yildirim, *Subdivision of the spectra for factorable matrices on c_0*, GU J. Sci., **24**(1), (2011), 45–49.

[12] F. Başar, N. Durna, and M. Yildirim, *Subdivisions of the spectra for genarilized difference operator over certain sequence spaces*, Thai J. Math., **9** (1), (2011), 285–295.

[13] N. Durna, *Subdivision of the spectra for the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c*, ADYU Sci., **6**(1), (2016), 31–43.

[14] R. Das, *On the spectrum and fine spectrum of the upper triangular matrix $U(r_1, r_2; s_1, s_2)$ over the sequence space c_0*, Afr. Mat., **28**, (2017), 841–849.

[15] N. Durna, *Spectra of the upper triangular band matrix $U(r; 0; s)$ on the Hahn space*, Bull. Int. Math. Virtual Inst. **10**(1), (2020), 9–17.

[16] S. R. El-Shabrawy and S. H. Abu-Janah, *Spectra of the generalized difference operator on the sequence spaces and bv_0 and h*, Linear and Multilinear Algebra, **66**(1), (2018), 1691–1708.

[17] M. Yildirim and N. Durna, *The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on ℓ_p, $(1 < p < \infty)$*, J. Inequal. Appl., **2017**(193), (2017), 1–13.

[18] B. C. Tripathy and R. Das, *Fine spectrum of the upper triangular matrix $U(r,0,0,s)$ over the squence spaces c_0 and c*, Proyecciones J. Math., **37**(1), (2018), 85–101.

[19] N. Durna, M. Yildirim, and R. Kılıç, *Partition of the spectra for the generalized difference operator $B(r,s)$ on the sequence space cs*, Cumhuriyet Sci. J., **39**(1), (2018), 7–15.

[20] N. Durna, *Subdivision of spectra for some lower triangular doule-band matrices as operators on c_0*, Ukr. Mat. Zh., **70**(7), (2018), 913–922.

[21] M. Yildirim, M. Mursaleen, and Ç. Doğan, *The spectrum and fine spectrum of generalized Rhaly-Cesàro matrices on c_0 and c*, Operators and Matrices, **12**(4), (2018), 955–975.

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