JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
*J. Int. Math. Virtual Inst.*, **12**(1)(2022), 145-163
DOI: 10.7251/JIMVI2201145E

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

# EXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

**Şerife Müge Ege and Fatma Serap Topal**

ABSTRACT. Solution of fractional differential equations is an emerging area of present day research because such equations arise in various applied fields. In this paper, by using fixed point theorems in a cone, we discuss the existence and multiplicity of positive solutions to nonlinear fractional differential equations with an integral boundary condition. Finally, we also give an example to illustrate our main results.

## 1. Introduction

The study of fractional calculus has been used to model physical, engineering and economic processes, notions and phenomena. Fractional differential equations with boundary value problems, which emerge as a branch of differential equations are encouraged by the widespread applicability of fractional derivatives. Recently, fractional boundary value problems have attracted much more attention since there is extensive use in the fields of physics, chemistry, aerodynamics, polymer rheology, etc. Many papers and books on fractional calculus and fractional differential equations have appeared [**6, 10, 11**].

Many people pay attention to the existence and multiplicity of solutions or positive solutions for boundary value problems of nonlinear fractional differential equations by means of some fixed-point theorems [**13, 2, 5**] (such as the Schauder fixed-point theorem, the Guo-Krasnosel'skii fixed-point theorem,the Leggett-Williams fixed-point theorem). As far as we know, there are some papers devoted to the study of fractional differential equations with integral boundary conditions [**3, 4, 1**].

2010 *Mathematics Subject Classification.* Primary 34B15; Secondary 34B18; 34B40.
*Key words and phrases.* Fractional differential equation, cone theory, fixed point theorems, Riemann-Stieltjes integral.
Communicated by Daniel A. Romano.

Since the Riemann-Stieltjes integral is more general than the classical Riemann integral, it is a very useful tool in many research areas. In the study of differential equations, boundary value problems contain not only the classical Riemann integral boundary values, but also the Riemann-Stieltjes integral boundary values. Riemann-Stieltjes integral $\int_0^1 u(t)dA(t)$ become more significant when A is not differentiable or A is discontinuous. However, the best of our knowledge, the study of Riemann-Stieltjes integral boundary value problems of fractional differential equations is relatively scarce. Few researchers have studied on this class of problems.

Ahmad and Nieto [**7**] considered the fractional differential equation with integral boundary conditions

$$\begin{aligned}
&D_{0^+}^q x(t) = f(t, x(t), (\chi x)(t)), \quad t \in (0,1) \\
&\alpha x(0) + \beta x'(0) = \int_0^1 q_1(s)ds, \\
&\alpha x(1) + \beta x'(1) = \int_0^1 q_2(s)ds.
\end{aligned}$$

In [**5**], He, Jia, Liu and Chen studied the existence results for a class of high order fractional differential equation with integral boundary condition

$$\begin{aligned}
&D_{0^+}^\alpha u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1) \\
&u(0) = u'(0) = .... = u^{(n-2)}(0) = 0, \\
&D_{0^+}^\beta u(1) = \int_0^1 D_{0^+}^\beta u(t)dA(t),
\end{aligned}$$

where $n \geqslant 3$, $n-1 < \alpha \leqslant n$, $0 < \beta \leqslant 1$.

Inspired by above works, we will consider the following fractional boundary value problem

$$\begin{aligned}
&D_{0^+}^q u(t) + f(t, u(t)) = 0, \quad t \in (0,1) \\
&u''(0) = u'''(0) = .... = u^{(n-1)}(0) = 0, \\
&\alpha u(0) + \beta u'(0) = \int_0^1 h_1(s)u(s)dA(s), \\
&\gamma u(1) - \delta u'(1) = \int_0^1 h_2(s)u(s)dA(s),
\end{aligned} \tag{1.1}$$

where $n \geqslant 3$, $n-1 < q \leqslant n$, $\int_0^1 u(t)dA(t)$ denotes the Riemann- Stieltjes integrals with respect to $A$, in which $A(t)$ is monotone increasing function. $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ continuous function and $D_{0^+}^q$ is the Caputo fractional derivative.

Throughout this paper, we assume the following conditions hold:
(H1) $\dfrac{\delta}{\gamma} \geqslant \dfrac{\beta}{\alpha} \geqslant 1$ such that $\alpha, \beta, \gamma, \delta > 0$,
(H2)$f \in \mathbb{C}([0,1] \times [0,\infty], [0,\infty])$,
(H3) $h_1, h_2 \in \mathbb{C}([0,1], [0,\infty])$, $0 \leqslant \gamma v_1 + \alpha v_2 < \alpha$ $D := 1 - v_4 - v_1 + v_1 v_4 - v_2 v_3 > 0$,

where

$$v_1 = \frac{1}{\Delta}\int_0^1 [-\delta + \gamma(1-s)]h_1(s)dA(s),$$
$$v_2 = \frac{1}{\Delta}\int_0^1 [\alpha s - \beta]h_1(s)dA(s),$$
$$v_3 = \frac{1}{\Delta}\int_0^1 [-\delta + \gamma(1-s)]h_2(s)dA(s),$$
$$v_4 = \frac{1}{\Delta}\int_0^1 [\alpha s - \beta]h_2(s)dA(s) \text{ and } \Delta = \alpha\gamma - \alpha\delta - \gamma\beta.$$

The paper is organized as follows. In Section 2, present some background materials and preliminaries. In Section 3, we give some existence and multiplicity results for the boundary value problem (1.1) and also an example is given to illustrate main results.

## 2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper. Firstly, for convenience of the reader, we give some definitions and fundamental results of fractional calculus theory.

Definition 2.1. For a function $f$ given on the interval $[a, b]$, the Caputo derivative of fractional order $r$ is defined as

$$D^r f(t) = \frac{1}{\Gamma(n-r)}\int_0^t (t-s)^{n-r-1} f^{(n)}(s)ds, \quad n = [r] + 1,$$

where $[r]$ denotes the integer part of $r$.

Definition 2.2. The Riemann-Liouville fractional integral of order $r$ for a function $f$ is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s)ds, \quad r > 0,$$

where $[r]$ denotes the integer part of $r$.

Lemma 2.3. *Let $r > 0$. Then the differential equation $D^r x(t) = 0$ has solutions*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

*where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n$, $n = [r] + 1$.*

Lemma 2.4. *Let $r > 0$. Then*

$$I^r (D^r x)(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

*where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n$, $n = [r] + 1$.*

For finding a solution of the problem (1.1), we first consider the following fractional differential equation

$$
\begin{aligned}
&D^q_{0+}u(t) = -y(t), \quad t \in (0,1)\\
&u''(0) = u'''(0) = .... = u^{(n-1)}(0) = 0,\\
&\alpha u(0) + \beta u'(0) = \int_0^1 h_1(s)u(s)dA(s),\\
&\gamma u(1) - \delta u'(1) = \int_0^1 h_2(s)u(s)dA(s).
\end{aligned}
\tag{2.1}
$$

Lemma 2.5. *Let $n \geqslant 3$ and $n-1 < q \leqslant n$. Assume $y \in \mathbb{C}[0,1]$ and $(H1)$ holds, then the problem (2.1) has a unique solution $u(t)$ given by*

$$
u(t) = \int_0^1 G(t,s)y(s)ds + \int_0^1 H(t,s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s),
$$

*where*

$$
\begin{aligned}
H(t,s) = \frac{1}{\Delta D}\Big( &[\gamma(1-t)(1-v_4) - \delta + (\alpha t - \beta)v_3]h_1(s)\\
&+[\gamma(1-t)v_2 - \delta + (\alpha t - \beta)(1-v_1)]h_2(s)\Big)
\end{aligned}
$$

*and*

$$
G(t,s) = \frac{1}{\Delta\Gamma(q)}\begin{cases} (\alpha t - \beta)(\gamma(1-s)^{q-1} - \delta(q-1)(1-s)^{q-2}) - \Delta(t-s)^{q-1}, & s \leqslant t;\\ (\alpha t - \beta)(\gamma(1-s)^{q-1} - \delta(q-1)(1-s)^{q-2}), & t \leqslant s.\end{cases}
$$

Proof. According to Lemma 2.1, the general solution of fractional differential equation (2.1) can be given as

$$
u(t) = -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds + c_0 + c_1 t + ... + c_{n-1}t^{n-1},
$$

where $c_i \in \mathbb{R}$, $i = 1,2,...,n-1$.

Since $u''(0) = u'''(0) = ... = u^{(n-1)}(0) = 0$, we have $c_2 = c_3 = ... = c_{n-1} = 0$, so

$$
u(t) = -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds + c_0 + c_1 t,
$$

and

$$
u'(t) = -\frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}y(s)ds + c_1.
$$

From the boundary conditions, we get

$$\begin{aligned}
\alpha c_0+\beta c_1 &= \int_0^1 h_1(s)u(s)dA(s)\\
\gamma c_0+(\gamma-\delta)c_1 &= \frac{1}{\Gamma(q)}\int_0^1(1-s)^{q-1}y(s)ds-\frac{\delta}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}y(s)ds\\
&+ \int_0^1 h_2(s)u(s)dA(s).
\end{aligned}$$

Defining $\Delta=\alpha\gamma\left(1-\frac{\delta}{\gamma}-\frac{\beta}{\alpha}\right)$, we obtain

$$\begin{aligned}
c_0 &= \frac{1}{\Delta}\left(-\frac{\gamma\beta}{\Gamma(q)}\int_0^1(1-s)^{q-1}y(s)ds+\frac{\delta\beta}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}y(s)ds\right.\\
&+ \left.(\gamma-\delta)\int_0^1 h_1(s)u(s)dA(s)-\beta\int_0^1 h_2(s)u(s)dA(s)\right)
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= \frac{1}{\Delta}\left(\frac{\alpha\gamma}{\Gamma(q)}\int_0^1(1-s)^{q-1}y(s)ds-\frac{\alpha\delta}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}y(s)ds\right.\\
&- \left.\gamma\int_0^1 h_1(s)u(s)dA(s)+\alpha\int_0^1 h_2(s)u(s)dA(s)\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}y(s)ds+\frac{\gamma(\alpha t-\beta)}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}y(s)ds\\
&- \frac{\delta(\alpha t-\beta)}{\Delta\Gamma(q-1)}\int_0^1(1-s)^{q-2}y(s)ds+\frac{\gamma(1-t)-\delta}{\Delta}\int_0^1 h_1(s)u(s)dA(s)\\
&+ \frac{\alpha t-\beta}{\Delta}\int_0^1 h_2(s)u(s)dA(s).
\end{aligned}$$

Therefore, we have the form of $u(t)$ as

$$u(t)=\int_0^1 G(t,s)y(s)ds+\frac{\gamma(1-t)-\delta}{\Delta}\int_0^1 h_1(s)u(s)dA(s)+\frac{\alpha t-\beta}{\Delta}\int_0^1 h_2(s)u(s)dA(s),$$

where

$$G(t,s)=\frac{1}{\Delta\Gamma(q)}\begin{cases}(\alpha t-\beta)(\gamma(1-s)^{q-1}-\delta(q-1)(1-s)^{q-2})-\Delta(t-s)^{q-1}, & s\leqslant t;\\ (\alpha t-\beta)(\gamma(1-s)^{q-1}-\delta(q-1)(1-s)^{q-2}), & t\leqslant s.\end{cases}$$

Using the form of $u(t)$, we get

$$\begin{aligned}\int_0^1 h_1(s)u(s)dA(s) &= \int_0^1 h_1(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s)\\ &+ \int_0^1 h_1(s)\frac{\gamma(1-s)-\delta}{\Delta}\int_0^1 h_1(\eta)u(\eta)dA(\eta)dA(s)\\ &+ \int_0^1 h_1(s)\frac{\alpha s-\beta}{\Delta}\int_0^1 h_2(\eta)u(\eta)dA(\eta)dA(s)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 h_2(s)u(s)dA(s) &= \int_0^1 h_2(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s)\\ &+ \int_0^1 h_2(s)\frac{\gamma(1-s)-\delta}{\Delta}\int_0^1 h_1(\eta)u(\eta)dA(\eta)dA(s)\\ &+ \int_0^1 h_2(s)\frac{\alpha s-\beta}{\Delta}\int_0^1 h_2(\eta)u(\eta)dA(\eta)dA(s).\end{aligned}$$

Defining $v_1 := \frac{1}{\Delta}\int_0^1(\gamma(1-s)-\delta)h_1(s)dA(s)$, $v_2 := \frac{1}{\Delta}\int_0^1(\alpha s-\beta)h_1(s)dA(s)$, $v_3 := \frac{1}{\Delta}\int_0^1(\gamma(1-s)-\delta)h_2(s)dA(s)$ and $v_4 := \frac{1}{\Delta}\int_0^1(\alpha s-\beta)h_2(s)dA(s)$, we have

$$\begin{aligned}\int_0^1 h_1(s)u(s)dA(s) &= \int_0^1 h_1(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s) + v_1\int_0^1 h_1(s)u(s)dA(s)\\ &+ v_2\int_0^1 h_2(s)u(s)dA(s),\\ \int_0^1 h_2(s)u(s)dA(s) &= \int_0^1 h_2(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s) + v_3\int_0^1 h_1(s)u(s)dA(s)\\ &+ v_4\int_0^1 h_2(s)u(s)dA(s).\end{aligned}$$

So, we can get the exact form of $u(t)$ by solving following equation set:

$$\begin{aligned}(1-v_1)\int_0^1 h_1(s)u(s)dA(s) - v_2\int_0^1 h_2(s)u(s)dA(s) &= \int_0^1 h_1(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s)\\ -v_3\int_0^1 h_1(s)u(s)dA(s) + (1-v_4)\int_0^1 h_2(s)u(s)dA(s) &= \int_0^1 h_2(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s).\end{aligned}$$

Defining $D := 1 - v_4 - v_1 + v_1v_4 - v_2v_3$, we get

$$\begin{aligned}\int_0^1 h_1(s)u(s)dA(s) &= \frac{1}{D}\Bigg((1-v_4)\int_0^1 h_1(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s)\\ &+ v_2\int_0^1 h_2(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s)\Bigg),\\ \int_0^1 h_2(s)u(s)dA(s) &= \frac{1}{D}\Bigg((1-v_1)\int_0^1 h_2(s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s)\\ &+ v_3\int_0^1 h_1(s)\int_0^1 G(s,\eta)y(\eta)y(\eta)d\eta dA(s)\Bigg).\end{aligned}$$

Finally, we obtain

$$u(t) = \int_0^1 G(t,s)y(s)ds + \int_0^1 H(t,s)\int_0^1 G(s,\eta)y(\eta)d\eta dA(s),$$

where

$$\begin{aligned}H(t,s) = \frac{1}{\Delta D}\Big(&[\gamma(1-t)(1-v_4) - \delta + (\alpha t - \beta)v_3]h_1(s)\\ &+[\gamma(1-t)v_2 - \delta + (\alpha t-\beta)(1-v_1)]h_2(s)\Big).\end{aligned}$$

□

Lemma 2.6. *Let $n \geqslant 3$ and $n-1 < q \leqslant n$. Assume $(H1)$ holds, then the Green's function for the problem* (2.1) *satisfies*

$$\left(1 - \frac{\alpha}{\beta}t\right)G(0,s) \leqslant G(t,s) \leqslant G(0,s). \tag{2.2}$$

Proof. When $0 \leqslant t \leqslant s \leqslant 1$, we have

$$\begin{aligned}\frac{G(t,s)}{G(0,s)} &= \frac{\gamma(\alpha t-\beta)(1-s)^{q-1} - \delta(q-1)(\alpha t-\beta)(1-s)^{q-2}}{\delta(q-1)\beta(1-s)^{q-2} - \gamma\beta(1-s)^{q-1}}\\ &= \frac{(\alpha t - \beta)\left[\gamma(1-s)^{q-1} - \delta(q-1)(1-s)^{q-2}\right]}{-\beta\left[\gamma(1-s)^{q-1} - \delta(q-1)(1-s)^{q-2}\right]}\\ &= 1 - \frac{\alpha}{\beta}t \leqslant 1,\end{aligned}$$

so $G(t,s) \leqslant G(0,s)$ and $G(t,s) = \left(1 - \frac{\alpha}{\beta}t\right)G(0,s)$.

When $0 \leqslant s \leqslant t \leqslant 1$, we have

$$\begin{aligned}\frac{G(t,s)}{G(0,s)} &= \frac{\gamma(\alpha t-\beta)(1-s)^{q-1} - \delta(q-1)(\alpha t-\beta)(1-s)^{q-2} - \Delta(t-s)^{q-1}}{\delta(q-1)\beta(1-s)^{q-2} - \gamma\beta(1-s)^{q-1}}\\ &= -\frac{\Delta(t-s)^{q-1}}{\delta(q-1)\beta(1-s)^{q-2} - \gamma\beta(1-s)^{q-1}} + 1 - \frac{\alpha}{\beta}t\\ &\geqslant 1 - \frac{\alpha}{\beta}t,\end{aligned}$$

so we can easily see that $G(t,s) \geqslant \left(1-\frac{\alpha}{\beta}t\right)G(0,s)$.

When differentiate twice $G(t,s)$ with respect to first variable, we get

$$\frac{\partial^2}{\partial t^2}\left(\frac{G(t,s)}{G(0,s)}\right) = \frac{1}{G(0,s)}\left(-\Delta(q-1)(q-2)(t-s)^{q-3}\right) \geqslant 0, \quad s,t \in [0,1].$$

So $\frac{G(t,s)}{G(0,s)}$ has maximum value at $t=s$ or $t=1$ as follows

$$\frac{G(s,s)}{G(0,s)} = 1-\frac{\alpha}{\beta}s \leqslant 1,$$

$$\begin{aligned}
\frac{G(1,s)}{G(0,s)} &= \frac{\gamma(\alpha-\beta)(1-s)^{q-1}-\delta(q-1)(\alpha-\beta)(1-s)^{q-2}-\Delta(1-s)^{q-1}}{\delta(q-1)\beta(1-s)^{q-2}-\gamma\beta(1-s)^{q-1}}\\
&= \frac{\gamma(\alpha-\beta)(1-s)-\delta(q-1)(\alpha-\beta)-\Delta(1-s)}{\delta(q-1)\beta-\gamma\beta(1-s)}\\
&= \frac{(\beta-\alpha)\left[\gamma(1-s)-\delta(q-1)\right]}{\beta\left[\gamma(1-s)-\delta(q-1)\right]}-\frac{\left[\alpha(\gamma-\delta)-\gamma\beta\right](1-s)}{\delta(q-1)\beta-\gamma\beta(1-s)}\\
&= 1-\frac{\alpha}{\beta}+\frac{\alpha(\gamma-\delta)-\gamma\beta}{\beta\left[\delta(q-1)-\gamma(1-s)\right]}(1-s)\\
&= \frac{\beta\delta(q-1)-\alpha\delta(q-1)+\alpha\delta(1-s)}{\beta\left[\delta(q-1)-\gamma(1-s)\right]}\\
&= \frac{(\beta-\alpha)\delta(q-1)+\alpha\delta(1-s)}{\beta\delta(q-1)-\beta\gamma(1-s)}.
\end{aligned}$$

Proving $\frac{(\beta-\alpha)\delta(q-1)+\alpha\delta(1-s)}{\beta\delta(q-1)-\beta\gamma(1-s)} \leqslant 1$, we assume that this is not true. In other words, let

$$\frac{(\beta-\alpha)\delta(q-1)+\alpha\delta(1-s)}{\beta\delta(q-1)-\beta\gamma(1-s)} > 1.$$

Since $0 < 1-s < 1$, $q-1 > 1$, $\delta \geqslant \gamma$ and $\delta(q-1) \geqslant \gamma(1-s)$, we get

$$\begin{aligned}
(\beta-\alpha)\delta(q-1)+\alpha\delta(1-s) &> \beta\delta(q-1-\beta\gamma(1-s)),\\
(q-1)\left[\delta(\beta-\alpha)-\beta\delta\right] &> -(\alpha\delta+\beta\gamma)(1-s)\\
-(q-1)\alpha\delta &> -(\alpha\delta+\beta\gamma)(1-s)\\
q-1 &< \left(1+\frac{\beta\gamma}{\alpha\delta}\right)(1-s)\\
1-s &> 1\\
s &< 0,
\end{aligned}$$

which is a contradiction.

Thus we get

$$\frac{G(t,s)}{G(0,s)} \leqslant 1,$$

so the result that $G(0,s) \geqslant G(t,s) \geqslant \left(1 - \frac{\alpha}{\beta}t\right) G(0,s)$ also holds while $s \leqslant t$. $\square$

LEMMA 2.7. *Let* $\xi \in \left[0, \frac{1}{2}\right]$, *we have*

$$\min_{t\in[\xi,1-\xi]} H(t,s) > \xi \max_{t\in[0,1]} H(t,s).$$

PROOF. From the definition of $H(t,s)$, we know that

$$\begin{aligned}
H(t,s) &= \frac{1}{\Delta D}([-\delta + \gamma(1-t)(1-v_4) + (\alpha t - \beta)v_3]h_1(s) \\
&+ [-\delta + \gamma(1-t)v_2 + (-\beta + \alpha t)(1-v_1)]h_2(s)) \\
&= \frac{1}{\Delta D}\left([\alpha v_3 - \gamma(1-v_4)]\, h_1(s) + [\alpha(1-v_1) - \gamma v_2]\, h_2(s)\right) t \\
&+ \frac{1}{\Delta D}\left([(-\delta + \gamma)(1-v_4) - \beta v_3]\, h_1(s) + [-\delta + \gamma v_2 - \beta(1-v_1)]\, h_2(s)\right).
\end{aligned}$$

If we denote $p(s) := \frac{1}{\Delta D}\left([\alpha v_3 - \gamma(1-v_4)]\, h_1(s) + [\alpha(1-v_1) - \gamma v_2]\, h_2(s)\right)$ and $q(s) := \frac{1}{\Delta D}\left([(\gamma - \delta)(1-v_4) - \beta v_3]\, h_1(s) + [-\delta + \gamma v_2 - \beta(1-v_1)]\, h_2(s)\right)$, we get

$$H(t,s) = p(s)t + q(s).$$

When $p(s) < 0$ with $0 < [\alpha v_3 - \gamma(1-v_4)]\, h_1(s) < [\gamma v_2 - \alpha(1-v_1)]\, h_2(s)$, due to monotonicity of $H(t,s)$, maximum value of $H(t,s)$ is $H(0,s) = q(s)$ and minimum value is $H(1,s) = p(s) + q(s)$.
Since

$$\begin{aligned}
[(-\delta+\gamma)(1-v_4) - \beta v_3] &= (-\delta+\gamma)\left(1 - \frac{1}{\Delta}\int_0^1 (\alpha s - \beta) h_2(s) dA(s)\right) \\
&- \frac{\beta}{\Delta}\int_0^1 [-\delta + \gamma(1-s)]\, h_2(s) dA(s) \\
&= -\delta + \gamma - \frac{1}{\Delta}\int_0^1 [-\delta\alpha s + \delta\beta + \alpha\gamma s - \gamma\beta - \delta\beta \\
&+ \gamma\beta - \gamma\beta s]\, h_2(s) dA(s) \\
&= -\delta + \gamma - \frac{1}{\Delta}\int_0^1 [\alpha(\gamma-\delta)s - \gamma\beta s]\, h_2(s) dA(s) < 0,
\end{aligned}$$

we get $q(s) > 0$ and

$$\begin{aligned}
p(s) + q(s) &= \frac{1}{\Delta D}\left([\alpha v_3 - \gamma(1-v_4) + (-\delta+\gamma)(1-v_4) - \beta v_3]\, h_1(s)\right. \\
&+ \left.[\alpha(1-v_1) - \gamma v_2 + \gamma v_2 \gamma v_2 - \delta - \beta(1-v_1)]\, h_2(s)\right) \\
&= \frac{1}{\Delta D}\left([(\alpha-\beta)v_3 - \delta(1-v_4)]\, h_1(s) + [(\alpha-\beta)(1-v_1) - \delta]\, h_2(s)\right) \\
&> 0.
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\min_{t\in[\xi,1-\xi]} H(t,s)}{\max_{t\in[0,1]} H(t,s)} &= \frac{H(1-\xi,s)}{H(0,s)} \\
&= \frac{(1-\xi)p(s)+q(s)}{q(s)} \\
&= \frac{(1-\xi)p(s)}{q(s)}+1 \\
&> \xi.
\end{aligned}$$

When $p(s) < 0$ with $[\alpha v_3 - \gamma(1-v_4)]h_1(s) < 0 < [\gamma v_2 - \alpha(1-v_1)]h_2(s)$, minimum value of $H(t,s)$ is $H(1,s)$ and

$$\begin{aligned}
H(1,s) &= \frac{1}{\Delta D}\left([\alpha v_3 - \gamma(1-v_4) + (\gamma-\delta)(1-v_4) - \beta v_3]\, h_1(s)\right. \\
&+ \left.[\alpha(1-v_1) - \gamma v_2 + (\alpha v_2 - \delta - \beta(1-v_1))]\right). \\
&= \frac{1}{\Delta D}\left([(\alpha-\beta)v_3 - \delta(1-v_4)]\, h_1(s)\right. \\
&+ \left.[(\alpha-\beta)(1-v_1) - \delta]\, h_2(s)\right) \\
&> 0.
\end{aligned}$$

Therefore, $\min\limits_{\xi<t<1-\xi} H(t,s) > \xi \max\limits_{0\leqslant t\leqslant 1} H(t,s)$.

On the other hand, when $p(s) \geqslant 0$, that is

$$[\alpha v_3 - \gamma(1-v_4)]\, h_1(s) \geqslant [\gamma v_2 - \alpha(1-v_1)]\, h_2(s) \geqslant 0,$$

then $\min_{0\leqslant t\leqslant 1} H(t,s) = H(0,s) > 0$. So, we have

$$\frac{\min_{\xi\leqslant t\leqslant 1-\xi} H(t,s)}{\max_{0\leqslant t\leqslant 1} H(t,s)} = \frac{H(\xi,s)}{H(1,s)} = \frac{\xi p(s)+q(s)}{p(s)+q(s)} \geqslant \xi.$$

□

The following fixed point theorems are fundamental and important to the proof of our main results.

THEOREM 2.8. [**12**] *Let $E = (E, \|\ .\ \|)$ be a Banach space, $A$ be a closed convex subset of $E$ and $T$ be a continuous map of $A$ into a compact subset of $A$. Then $T$ has a fixed point.*

THEOREM 2.9. [**8**] *Let $E = (E, \|\ .\ \|)$ be a Banach space, $P \subset E$ be a cone in $E$. Suppose that $\Omega_1$ and $\Omega_2$ are open subsets of $E$ with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*
*(1) $\|Tu\| \leqslant \|u\|$ for $u \in P \cap \partial\Omega_1, \|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_2$, or*
*(2) $\|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_1, \|Tu\| \leqslant \|u\|$ for $u \in P \cap \partial\Omega_2$ holds.*
*Then $T$ has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

Define the sets $P_c := \{u \in P :\| u \|< c\}$ and $P(\alpha, a, b) := \{u \in P : a \leqslant \alpha(u), \ \| u \|\leqslant b\}$ where $a, b, c > 0$ and $\alpha$ on $P$ is a nonnegative functional.

Theorem 2.10. [**9**] *Let $E = (E, \| \, . \, \|)$ be a Banach space, $P \subset E$ a cone of $E$ and $c > 0$ a constant. Suppose that there exists a nonnegative continuous concave functional $\alpha$ on $P$ with $\alpha(u) \leqslant\| u \|$ for $u \in \overline{P}_c$ and let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous map. Assume that there exist $a, b, c, d$ with $0 < a < b < d \leqslant c$ such that*
*(S1) $\{u \in P(\alpha, b, d) : \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ for all $u \in P(\alpha, b, d)$;*
*(S2) $\| Tu \|< a$ for all $u \in \overline{P}_a$;*
*(S3) $\alpha(Tu) > b$ for all $u \in P(\alpha, b, c)$ with $\| Tu \|> d$.*

*Then $T$ has at least three fixed points $u_1, u_2, u_3 \in P$ such that $\| u_1 \|< a, \alpha(u_2) > b, \| u_3 \|> a$ and $\alpha(u_3) < b$.*

## 3. Existence of positive solutions

Let us define $M := \max\{|H(t, s)| : t, s \in [0, 1]\}$. Let the Banach space $B = \mathbb{C}[0, 1]$ be equipped with the norm $\|u\| = \max_{0\leqslant t\leqslant 1} |u(t)|$ for $u \in B$. We now define a mapping $T : \mathbb{C}[0, 1] \to \mathbb{C}[0, 1]$ by

$$Tu(t) := \int_0^1 G(t, s)f(s, u(s))ds + \int_0^1 H(t, s) \int_0^1 G(s, \eta)f(\eta, u(\eta))d\eta dA(s).$$

Theorem 3.1. *Let (H1)-(H3) hold. If $R > 0$ satisfies*

$$\frac{Q\beta(\delta q - \gamma)(1 + M\,(A(1) - A(0)))}{\Gamma(q + 1)(\gamma\beta + \alpha\delta - \alpha\gamma)} \leqslant R$$

*where $Q > 0$ satisfies $Q \geqslant \max_{\|u\|\leqslant R} |f(t, u(t))|$, for $t \in [0, 1]$ then the problem (1.1) has a solution $u(t)$.*

Proof. Let $P_1 = \{u \in B : \|u\| \leqslant R\}$. We still apply Schauder's fixed point theorem. The solutions of problem (1.1) are the fixed points of the operator T. A standard argument guarantees that $T : P_1 \to B$ is continuous. Next we show $T(P_1) \subset P_1$.

For $u \in P_1$, we obtain

$$
\begin{aligned}
|Tu(t)| &= |\int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 H(t,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s)| \\
&\leqslant \int_0^1 |G(t,s)||f(s,u(s))|ds + \int_0^1 |H(t,s)|\int_0^1 |G(s,\eta)||f(\eta,u(\eta))|d\eta dA(s) \\
&\leqslant \int_0^1 G(0,s)Qds + \int_0^1 M\int_0^1 G(0,\eta)Qd\eta dA(s) \\
&= Q\int_0^1 \frac{1}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q)}\left[-\gamma\beta(1-s)^{q-1}+\delta(q-1)\beta(1-s)^{q-2}\right]ds \\
&+ \frac{MQ\,(A(1)-A(0))}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q)}\int_0^1 \left[-\gamma\beta(1-\eta)^{q-1}+\delta(q-1)\beta(1-\eta)^{q-2}\right]d\eta \\
&= \frac{Q}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q)}\left(\gamma\beta\frac{(1-s)^q}{q}|_0^1-\delta(q-1)\beta\frac{(1-s)^{q-1}}{q-1}|_0^1\right) \\
&+ \frac{MQ\,(A(1)-A(0))}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q)}\left(\gamma\beta\frac{(1-\eta)^q}{q}|_0^1-\delta(q-1)\beta\frac{(1-\eta)^{q-1}}{q-1}|_0^1\right) \\
&= \frac{Q}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q)}\frac{\beta(\delta q-\gamma)}{q}+\frac{MQ\,(A(1)-A(0))}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q)}\frac{\beta(\delta q-\gamma)}{q} \\
&= \frac{Q\beta(\delta q-\gamma)}{\Gamma(q+1)(\gamma\beta+\alpha\delta-\alpha\gamma)}(1+M\,(A(1)-A(0)))\leqslant R,
\end{aligned}
$$

for all $t \in [0,1]$.

This implies that $\|Tu\| \leqslant R$. A standard argument, by Arzela-Ascoli theorem, guarantees that $T : P_1 \to P_1$ is a compact operator. Hence $T$ has a fixed point $u \in P_1$ by Theorem 2.1. $\square$

We also assume throughout this section that $f_0 := \lim_{u\to 0+} \dfrac{f(t,u)}{u}$ and $f_\infty := \lim_{u\to\infty} \dfrac{f(t,u)}{u}$ exist uniformly in the extended reals. The case $f_0 = 0$, $f_\infty = \infty$ is called the superlinear and the case $f_0 = \infty$, $f_\infty = 0$ is called sublinear case. To prove our result, we will use the Theorem 2.2. $\square$

THEOREM 3.2. *Let (H1)-(H3) hold. If either the superlinear case or the sublinear case holds, the problem* (1.1) *has a positive solution.*

PROOF. Let we define a cone $P_2$ in $B$ by

$$P_2 = \{u \in B : u(t) \geqslant 0 \quad \textit{and} \quad \min_{t\in[\xi,1-\xi]} u(t) \geqslant \xi\|u\|\}.$$

It is easy to check that $P_2$ is a cone of nonnegative functions in $\mathbb{C}[0,1]$. We now show that $T : P_2 \to P_2$. First note that $u \in P_2$ implies $Tu(t) \geqslant 0$ on $[0,1]$ and

$$\begin{aligned}
\min_{\xi\leqslant t\leqslant 1-\xi} Tu(t) &= \int_0^1 \min_{\xi\leqslant t\leqslant 1-\xi} G(t,s)f(s,u(s))ds \\
&+ \int_0^1 \min_{\xi\leqslant t\leqslant 1-\xi} H(t,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s) \\
&\geqslant \min_{\xi\leqslant t\leqslant 1-\xi}\left(1-\frac{\alpha}{\beta}t\right)\int_0^1 G(0,s)f(s,u(s))ds \\
&+ \int_0^1 \xi \max_{0\leqslant t\leqslant 1} H(t,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s) \\
&\geqslant \xi\int_0^1 \max_{0\leqslant t\leqslant 1} G(t,s)f(s,u(s))ds \\
&+ \xi\int_0^1 \max_{0\leqslant t\leqslant 1} H(t,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s) \\
&\geqslant \xi\max_{0\leqslant t\leqslant 1}|Tu(t)| = \xi\|Tu\|.
\end{aligned}$$

Hence $Tu \in P_2$ and so $T : P_2 \to P_2$ which is what we want. Therefore $T$ is completely continuous.

Assume now that we are in the superlinear case $f_0 = 0$ and $f_\infty = \infty$.

Since $\lim\limits_{u\to 0^+}\dfrac{f(t,u)}{u} = 0$ uniformly on $[0,1]$, we may choose $r > 0$ such that

$$f(t,u) \leqslant \tau u, \quad 0 \leqslant u \leqslant r, \quad 0 \leqslant t \leqslant 1$$

where $\tau := \left[\dfrac{\beta(\delta q-\gamma)(1+M\,(A(1)-A(0)))}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q+1)}\right]^{-1}$.

Then if $\Omega_1$ is the ball in $B$ centered at the origin with radius $r$ and if $u \in P_2\bigcap\partial\Omega_1$ then we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 H(t,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s) \\
&\leqslant \int_0^1 G(0,s)f(s,u(s))ds + \int_0^1 M\int_0^1 G(0,\eta)f(\eta,u(\eta))d\eta dA(s) \\
&\leqslant \frac{\beta(\delta q-\gamma)}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q)q}\left(\int_0^1 \tau u(s)ds + M\,(A(1)-A(0))\int_0^1 \tau u(\eta)d\eta\right) \\
&\leqslant \frac{\beta(\delta q-\gamma)}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q+1)} r\tau(1+M\,(A(1)-A(0))) = r = \|u\|,
\end{aligned}$$

and so $\|Tu\| \leqslant \|u\|$ for all $u \in P_2\bigcap\partial\Omega_1$.

Next we use the assumption $\lim_{u\to\infty}\frac{f(t,u)}{u}=\infty$ uniformly on $[0,1]$. Let $t_0\in[\xi,1-\xi]$ and let

$$\mu:=\left[\xi\int_\xi^{1-\xi}G(t_0,s)ds\left(1+\int_\xi^{1-\xi}H(t_0,s)dA(s)\right)\right]^{-1}.$$

Then there is $\overline{r}$ such that $f(t,u)\geqslant\mu u$, $u\geqslant\overline{r}$. If we define $\widehat{r}:=\max\{2r,\frac{\overline{r}}{\xi}\}$ and $\Omega_2:=\{u\in B:\|u\|<\widehat{r}\}$ for $u\in P_2\bigcap\partial\Omega_2$, we have $\min_{\xi\leqslant t\leqslant 1-\xi}u(t)\geqslant\xi\|u\|=\xi\widehat{r}\geqslant\overline{r}$.

Therefore, for all $t\in[\xi,1-\xi]$ we have $f(t,u(t))\geqslant\mu u(t)\geqslant\mu\xi\widehat{r}=\mu\xi\|u\|$. Hence we get

$$\begin{aligned}
Tu(t_0) &= \int_0^1 G(t_0,s)f(s,u(s))ds+\int_0^1 H(t_0,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s)\\
&\geqslant \int_\xi^{1-\xi}G(t_0,s)f(s,u(s))ds\\
&+ \int_\xi^{1-\xi}H(t_0,s)\int_\xi^{1-\xi}G(s,\eta)f(\eta,u(\eta))d\eta dA(s)\\
&\geqslant \mu\xi\|u\|\left(\int_\xi^{1-\xi}G(t_0,s)ds+\int_\xi^{1-\xi}H(t_0,s)\int_\xi^{1-\xi}G(t_0,\eta)d\eta dA(s)\right)\\
&= \mu\xi\|u\|\int_\xi^{1-\xi}G(t_0,s)ds\left(1+\int_\xi^{1-\xi}H(t_0,s)dA(s)\right)\\
&= \|u\|=\widehat{r},
\end{aligned}$$

and so $\|Tu\|\geqslant\|u\|$ for all $u\in P_2\bigcap\partial\Omega_2$.

Consequently, by part (i) of Theorem 2.2, it follows that $T$ has a fixed point in $P_2\bigcap(\overline{\Omega_2}\backslash\Omega_1)$ and this implies that the problem (1.1) has a positive solution.

The sublinear case can be proven similarly. □

Theorem 3.3. *Let (H1)-(H3) hold. Also assume*

*(H5)* $\lim_{u\to 0^+}\frac{f(t,u)}{u}=+\infty$, $\lim_{u\to+\infty}\frac{f(t,u)}{u}=+\infty$ *for* $t\in[0,1]$,

*(H6) There exists constant* $\rho$ *such that* $f(t,u)\leqslant N\rho$ *for* $t\in[0,1]$, *where* $N\leqslant\tau$ *and* $\tau$ *is given as in the proof of Theorem* 3.2.

*Then the problem* (1.1) *has at least two positive solutions* $u_1$ *and* $u_2$ *such that* $0<\|u_1\|\leqslant\rho<\|u_2\|$.

Proof. Since $\lim_{u\to 0^+}\frac{f(t,u)}{u}=+\infty$, there exists $\rho_*\in(0,\rho_1)$ such that $f(t,u)\geqslant\mu_1 u$ for $0\leqslant u\leqslant\rho_*$ and $0<t<1$, where $\mu_1\geqslant\mu$, here $\mu$ is given in the proof Theorem 3.2

Set $\Omega_1 = \{u \in B : \|u\| < \rho_*\}$. For $u \in P \bigcap \partial\Omega_1$ and $t_0 \in [\xi, 1-\xi]$, we have

$$\begin{aligned}
Tu(t_0) &= \int_0^1 G(t_0, s) f(s, u(s)) ds + \int_0^1 H(t, s) \int_0^1 G(s, \eta) f(\eta, u(\eta)) d\eta dA(s) \\
&\geqslant \int_\xi^{1-\xi} G(t_0, s) f(s, u(s)) ds \\
&+ \int_\xi^{1-\xi} H(t_0, s) \int_\xi^{1-\xi} G(s, \eta) f(\eta, u(\eta)) d\eta dA(s) \\
&\geqslant \mu_1 \rho_* \left( \int_\xi^{1-\xi} G(t_0, s) ds + \int_\xi^{1-\xi} H(t_0, s) \int_\xi^{1-\xi} G(t_0, \eta) d\eta dA(s) \right) \\
&= \mu_1 \rho_* \int_\xi^{1-\xi} G(t_0, s) ds \left( 1 + \int_\xi^{1-\xi} H(t_0, s) dA(s) \right) \\
&\geqslant \rho_* = \|u\|,
\end{aligned}$$

and so $\|Tu\| \geqslant \|u\|$, for all $u \in P_2 \bigcap \partial\Omega_1$.

Since $\lim\limits_{u\to\infty} \dfrac{f(t,u)}{u} = +\infty$, there exists $\rho^* > \rho$ such that $f(t, u) \geqslant \mu_2 u$ for $u \geqslant \rho^*$ where $\mu_2 \geqslant \mu > \mu\xi$, here $\mu$ is given in the proof Theorem 3.2.

Set $\Omega_2 = \{u \in B : \|u\| < \rho^*\}$. For any $u \in P \bigcap \partial\Omega_2$, we get

$$\begin{aligned}
Tu(t_0) &= \int_0^1 G(t_0, s) f(s, u(s)) ds + \int_0^1 H(t, s) \int_0^1 G(s, \eta) f(\eta, u(\eta)) d\eta dA(s) \\
&\geqslant \int_\xi^{1-\xi} G(t_0, s) f(s, u(s)) ds \\
&+ \int_\xi^{1-\xi} H(t_0, s) \int_\xi^{1-\xi} G(s, \eta), f(\eta, u(\eta)) d\eta dA(s) \\
&\geqslant \mu_2 \rho^* \left( \int_\xi^{1-\xi} G(t_0, s) ds + \int_\xi^{1-\xi} H(t_0, s) \int_\xi^{1-\xi} G(t_0, \eta), d\eta dA(s) \right) \\
&= \mu_2 \rho^* \int_\xi^{1-\xi} G(t_0, s) ds \left( 1 + \int_\xi^{1-\xi} H(t_0, s) dA(s) \right) \\
&\geqslant \rho^* = \|u\|,
\end{aligned}$$

which yields $\|Tu\| \geqslant \|u\|$ for all $u \in P_2 \bigcap \partial\Omega_2$. Let $\Omega_3 = \{u \in B : \|u\| < \rho\}$. For $y \in P_2 \bigcap \partial\Omega_3$ from (H6), we obtain

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 H(t,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s) \\ &\leqslant \int_0^1 G(t,s)N\rho ds + \int H(t,s)\int_0^1 G(s,\eta)N\rho d\eta dA(s) \\ &\leqslant N\rho\left(\int_0^1 G(0,s)ds + \int_0^1 M\int_0^1 G(0,\eta)d\eta dA(s)\right) \\ &\leqslant \frac{N\rho\beta(\delta q-\gamma)}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q+1)}(1+M\left(A(1)-A(0)\right)) \\ &\leqslant \tau\rho\tau^{-1} \\ &= \rho = \|u\|, \end{aligned}$$

which yields $\|Tu\| \leqslant \|u\|$ for all $u \in P_2 \bigcap \partial\Omega_3$.

Hence, since $\rho_* \leqslant \rho < \rho^*$, it follows from Theorem 2.2 that $T$ has a fixed point $u_1$ in $P_2 \bigcap(\overline{\Omega_3} \setminus \Omega_1)$ and a fixed point $u_2$ in $P_2 \bigcap(\overline{\Omega_2} \setminus \Omega_3)$. Note both are positive solutions of the problem (1.1) satisfying $0 < \|u_1\| \leqslant \rho < \|u_2\|$. □

We also assume

$$\alpha(u) = min_{t\in[\xi,1-\xi]}|u(t)|.$$

Theorem 3.4. *Let (H1)-(H3) hold and also there exist constants* $A, B, C, D$ *with* $0 < A < B < C = D$ *such that the following conditions hold:*
*(H7)* $f(t,u) < KA$ *for all* $(t,u) \in [0,1]\times[0,A]$,
*(H8)* $f(t,u) \geqslant LB$ *for all* $(t,u)) \in [\xi, 1-\xi]\times[B,C]$,
*(H9)* $f(t,u) \leqslant KC$ *for all* $(t,u) \in [0,1]\times[0,C]$,

*where* $K \leqslant \tau$, $L \geqslant \mu$ *such that* $\tau$ *and* $\mu$ *are numbers given as in the proof of Theorem* 3.2.

*Then the problem* (1.1) *has at least three positive solutions* $u_1$, $u_2$ *and* $u_3$ *such that*
$\max\limits_{0\leqslant t\leqslant 1}|u(t)| < A$, $B < \min\limits_{t\in[\xi,1-\xi]}|u_2(t)| < \max\limits_{t\in[0,1]}|u_2(t| \leqslant C$, $A < \max\limits_{0\leqslant t\leqslant 1}|u_3(t)| \leqslant C$ *and* $\min\limits_{\xi\leqslant t\leqslant 1-\xi}|u_3(t)| < B$.

Proof. Let $y \in \overline{P_C} = \{u \in P : \|u\| \leqslant C\}$, then $\|y\| \leqslant C$. If $(H9)$ hold, then we get

$$\begin{aligned}
\|Tu\| &= \max_{0\leqslant t\leqslant 1} |Tu(t)| \\
&= \max_{0\leqslant t\leqslant 1} |\int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 H(t,s)\int_0^1 G(t,\eta)f(\eta,u(\eta))d\eta dA(s)| \\
&\leqslant \int_0^1 G(0,s)KCds + \int_0^1 M\int_0^1 G(0,\eta)KCd\eta dA(s)| \\
&\leqslant \tau C\frac{1}{(\gamma\beta+\alpha\delta-\alpha\gamma)\Gamma(q+1)}\beta(\delta q-\gamma) + \frac{M\tau C\left(A(1)-A(0)\right)}{\gamma\beta+\alpha\delta-\alpha\gamma}\beta(\delta q-\gamma) \\
&= \frac{\tau C\beta(\delta q-\gamma)}{\gamma\beta+\alpha\delta-\alpha\gamma}(1+M\left(A(1)-A(0)\right)) = C.
\end{aligned}$$

In the same way, we can show that if (H7) hold, then $T\overline{P_A} \subset P_A$. Hence condition (S2) of Theorem 2.3 is satisfied.

To show the condition (S1) of Theorem 2.3, we choose $u_0(t) = \dfrac{B+C}{2}$ for $t \in [0,1]$. It is easy to see that $u_0 \in P$, $\|u_0\| = \dfrac{B+C}{2} \leqslant C$ and $\alpha(u_0) = \dfrac{B+C}{2} < B$. That is $u_0 \in P(\alpha, B, D)$ and we have $B \leqslant u(t) \leqslant C$ for $t \in [\xi, 1-\xi]$. By (H8) and Lemma 2.4, we have

$$\begin{aligned}
\alpha(Tu) &= \min_{\xi\leqslant t\leqslant 1-\xi} |Tu(t)| \\
&\geqslant \int_0^1 \min_{\xi\leqslant t\leqslant 1-\xi} G(t,s)f(s,y(s))ds \\
&+ \int_0^1 \min_{\xi\leqslant t\leqslant 1-\xi} H(t,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s) \\
&\geqslant \int_0^1 \min_{\xi\leqslant t\leqslant 1-\xi}\left(1-\frac{\alpha}{\beta}t\right)G(0,s)f(s,y(s))ds \\
&+ \int_0^1 \xi\max_{0\leqslant t\leqslant 1} H(t,s)\int_0^1 G(s,\eta)f(\eta,u(\eta))d\eta dA(s) \\
&\geqslant \xi\left(\int_0^1 \max_{0\leqslant t\leqslant 1} G(t,s)f(s,y(s))ds\right. \\
&+ \left.\int_0^1 \max_{0\leqslant t\leqslant 1} H(t,s)\int_0^1 \max_{0\leqslant t\leqslant 1} G(s,\eta)f(\eta,u(\eta))d\eta dA(s)\right) \\
&\geqslant \xi\left(\int_\xi^{1-\xi} \max_{0\leqslant t\leqslant 1} G(t,s)LBds\right. \\
&+ \left.\int_\xi^{1-\xi} \max_{0\leqslant t\leqslant 1} H(t,s)\int_0^1 \max_{0\leqslant t\leqslant 1} G(s,\eta)LBd\eta dA(s)\right)
\end{aligned}$$

$$\begin{aligned}&\geqslant \xi\mu B\int_{\xi}^{1-\xi}\max_{0\leqslant t\leqslant 1}G(t,s)ds\left(1+\int_{\xi}^{1-\xi}\max_{0\leqslant t\leqslant 1}H(t,s)dA(s)\right)\\&= B.\end{aligned}$$

Hence condition (S1) of Theorem 2.3 is satisfied. Since $D = C$, then condition (S1) implies condition (S3) of Theorem 2.3.

To sum up, all the hypothesis of Theorem 2.3 are satisfied. The proof is completed. □

To illustrate our main results, we give the following example.

EXAMPLE 3.5. We consider the fractional boundary value problem

$$\begin{aligned}&D_{0^+}^{\frac{9}{2}}u(t)+f(t,u(t))=0,\quad t\in(0,1)\\&u''(0)=u'''(0)=u^{iv}(0)=0,\\&u(0)+2u'(0)=\int_0^1 s^2u(s)dA(s),\\&u(1)-3u'(1)=\int_0^1 su(s)dA(s),\end{aligned}\tag{3.1}$$

where $A(s) = s^2 + 1$.

By direct calculation, we have $\Delta = -4, v_1 = \frac{7}{20}, v_2 = \frac{3}{20}, v_3 = \frac{11}{24}, v_4 = \frac{5}{24}, M \cong 2,32$ and $\tau \cong 2,43$. So, we can easily see that $0 < \gamma v_1 + \alpha v_2 = \frac{1}{2} < 1$, $D = 1 - v_4 - v_1 + v_1v_4 - v_2v_3 \cong 0,32 > 0$ and also $\frac{\delta}{\gamma} \geqslant \frac{\beta}{\alpha} > 1$. Thus the hypotheses (H1) and (H3) are hold.

**1.** Consider $f(t,u) = \frac{sinu}{t^2+1}$ for $t \in [0,1]$. The condition (H2) is satisfied for the function $f$. Since $f(t,u) \leqslant u$ for $t \in [0,1]$, we get $max_{\|u\|\leqslant R}|f(t,u(t))| \leqslant R$ and also we choose $Q = R$. Thus the inequality,

$$\frac{Q2\left(3\frac{9}{2}-1\right)(1+2,32(2-1))}{\Gamma\left(\frac{9}{2}+1\right)4}\leqslant R$$

is satisfied for $R = 1$. Then the problem (3.1) has a solution $u(t)$ by Theorem 3.1.

**2.** Consider $f(t,u) = u^{\frac{3}{2}} + \frac{u^2}{\sqrt{u^2+1}}$ for $t \in [0,1]$. The condition (H2) is satisfied for the function $f$. Since $f_0 = 0$ and $f_\infty = \infty$, we see that the superlinear case holds. Thus the problem (3.1) has a positive solution by Theorem 3.2.

**3.** Consider $f(t,u) = \sqrt{u} + u^4$ for $t \in [0,1]$. The condition (H2) is satisfied for the function $f$. Since $\lim_{u\to 0}\frac{f(t,u)}{u} = \infty$ and $\lim_{u\to\infty}\frac{f(t,u)}{u} = \infty$, then the condition (H5) holds. Also, choosing $\rho = 1$, we can easily see that the inequality

$$f(t,u)=\sqrt{u}+u^4\leqslant\sqrt{\rho}+\rho^4=2\leqslant 1N$$

is satisfied with $N = 2 \leqslant \tau = 2,43$, where $u \leqslant \rho$, then the problem (3.1) has at least two solutions such that $0 < \|u_1\| \leqslant 1 < \|u_2\|$, by Theorem 3.3.

## References


[1] B. Ahmad and J.J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions, Bound. Value Probl., (2009), 1-11.

[2] Z.B. Bai and H.S. Lü, Positive solutions of boundary value problems of nonlinear fractional differential equation, J. Math. Anal. Appl., **311** (2005), 495-505.

[3] A. Cabada, S. Dimitrijevic, T. Tomovic, and S. Aleksic, The existence of a positive solution for nonlinear fractional differential equations with integral boundary value conditions, Math. Methods Appl. Sci., **40**(6) (2016), 1880-1891.

[4] A. Cabada and Z. Hamdi, Nonlinear fractional differential equations with integral boundary value conditions, Appl. Math. Comput., **228** (2014), 251-257.

[5] J. He, M. Jia, X. Liu, and H. Chen, Existence of positive solutions for a high order fractional differential equation integral boundary value problem with sign nonlinearity, Advances in Difference Equations, **49** (2018), 1-17.

[6] R. Hilfer, Applications of Fractional Calculus in Physics. World Scientific Publishing Co., Singapore, (2000).

[7] T. Jankowski, Positive solutions to fractional differential equations involving Stieltjes integral conditions, Appl. Math. Comput., **241** (2014), 200-213.

[8] M.A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations Armstrong, Trans. Pergamon: Elmsford (1964).

[9] R.W. Leggett and L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana Univ. Math. J., **28** (1979), 673-688.

[10] I. Podlubny, Fractional Differential Equations. Mathematics in Science and Engineering, Academic Press, New York, (1999).

[11] J. Sabatier, O.P. Agrawal, and J.A. Tenreiro Machado, Advances in Fractional Calculus, Springer, (2007).

[12] D.R. Smart, Fixed Point Theorems, Cambridge University Press Cambridge (1974).

[13] W. Zhong and W. Lin, Nonlocal and multiple-point boundary value problem for fractional differential equations, Comput. Math. Appl., **59**(3) (2010), 1345-1351.


Received by editors 16.9.2021; Revised version 3.6.2022; Available online 15.6.2022.

Şerife Müge Ege, Department of Mathematics, Ege University, Izmir, Turkey
*Email address*: `serife.muge.ege@ege.edu.tr`

Fatma Serap Topal, Department of Mathematics, Ege University, Izmir, Turkey
*Email address*: `f.serap.topal@ege.edu.tr`