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AN OVERVIEW ON SOME MODELS INVOLVED IN IMAGE RESTORATION

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Abstract. The main purpose of this work is to review different models used in image restoration over time. We start with the first classic model of Tikhonov to which progressive changes are brought in order to overcome certain defaults and get a restored image of the best quality. It seems that the space $W^{1,p(x)}$ is a good space for image restoration according to what we have gathered from recent work in this area.

1. Introduction

Often, some degradations affect an image during its transmission. These deteriorations can be deterministic related to the image acquisition modality. Image restoration can be seen as a significant step in image processing. The quality of the image is improved via this notion. Removing the noise or the blur or adding some information in the image are some goals of image restoration. Mathematically, the degradation is modeled in [**4**] by the following equation

$$u_0 = \varphi u + \eta \tag{1.1}$$

where u, u_0 are respectively the original image describing a real scene and the observed image of the same scene (which is the degraded version of u), both defined on a bounded region $\Omega \subset \mathbb{R}^2$. In equation (1.1) , η design a Gaussian white noise while φ is a linear blur operator which is not necessarily invertible, even if it is invertible its inverse is difficult to calculate numerically. To recover u from u_0,

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the main idea is to construct an approximation of u with the solution of the next problem

$$\inf_u \int_\Omega |u_0 - \varphi u|^2 dx. \tag{1.2}$$

In the case of neglection of the noise and if u is a minimum of (1.2), then u satisfies the following equation

$$\varphi^* u_0 = \varphi^* \varphi u. \tag{1.3}$$

By the fact that φ is not necessarily invertible, u can be obtained from (1.3).

An ill posed problem is the problem of image restoration. The direct method in the calculus of variations [**4, 20**] is the most used method in this area. Recently, authors have introduced different and interesting approaches in image processing [**6, 10, 19, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 51, 60, 65, 91**]. Besides, new approaches based on functionals with variable exponent have emerged and have greatly contributed to image restoration [**1, 8, 18, 46, 50, 57, 92**]. Several papers have dealt with variable exponent spaces and one can cite for examples [**39, 45, 63, 69, 70, 85, 86, 87**] and references therein. Let us recall the important notions concerning the $W^{1,p(x)}(\Omega)$ spaces. Let Ω be an open subset of $\mathbb{R}^N$ and set

$$C_+(\overline{\Omega}) := \{f : f \in C(\overline{\Omega}),\ f(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For all $p \in C_+(\overline{\Omega})$, let $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leqslant p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty$ and

$$L^{p(x)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\}.$$

On this space the so-called Luxemburg norm is given by

$$|u|_{p(x)} = \inf \Big\{ \gamma > 0 : \int_\Omega \Big| \frac{u(x)}{\gamma} \Big|^{p(x)} dx \leqslant 1 \Big\}.$$

The space $L^{p(x)}(\Omega)$ has common points with the space $L^p(\Omega)$ they are both Banach spaces, they are reflexive if and only if $1 < p^- \leqslant p^+ < \infty$. Moreover, if $p^+ < \infty$ then continuous functions are dense. On the other hand, if $0 < |\Omega| < \infty$ and $q_1(\cdot)$, $q_2(\cdot)$ are variable exponents such that $q_1(x) \leqslant q_2(x)$ a.e. $x \in \Omega$, Then, there exists a continuous embedding $L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega)$. That is the inclusion between Lebesgue spaces also generalizes naturally.

If $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, then the Hölder-type inequality says

$$\Big| \int_\Omega uv dx \Big| \leqslant \Big(\frac{1}{p^-} + \frac{1}{(p')^-} \Big) |u|_{p(x)} |v|_{p'(x)}, \quad u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega). \tag{1.4}$$

Moreover, if f_1, f_2 and $f_3 : \overline{\Omega} \to (1, \infty)$ are Lipschitz continuous functions such that

$$\frac{1}{f_1(x)} + \frac{1}{f_2(x)} + \frac{1}{f_3(x)} = 1$$

and for any functions $u_1 \in L^{f_1(x)}(\Omega), u_2 \in L^{f_2(x)}(\Omega), u_3 \in L^{f_3(x)}(\Omega)$, the following inequality is the generalized Hölder-type

$$\Big| \int_\Omega u_1 u_2 u_3 \, dx \Big| \leqslant \Big(\frac{1}{f_1^-} + \frac{1}{f_2{}^-} + \frac{1}{f_3{}^-} \Big) |u_1|_{f_1(x)} |u_2|_{f_2(x)} |u_3|_{f_3(x)}. \tag{1.5}$$

Inequalities (1.5) and (1.4) are both due to Orlicz in his paper [**63**].

The modular which is an important tool when we deal with the generalized Lebesgue spaces is defined by $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ such that

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx.$$

Proposition 1.1. *([**58**]) For all $u, v \in L^{p(x)}(\Omega)$, we have*

(1) $|u|_{p(x)} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho_{p(x)}(u) < 1$ (*resp.* $= 1, > 1$).
(2) $\min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \leqslant \rho_{p(x)}(u) \leqslant \max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+})$.
(3) $\rho_{p(x)}(u - v) \to 0 \Leftrightarrow |u - v|_{p(x)} \to 0$.

Proposition 1.2. *([**24**]) Let p and q be measurable functions such that $p \in L^\infty(\Omega)$, and $1 \leqslant p(x)q(x) \leqslant \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\min(|u|_{p(x)q(x)}^{p^+}, |u|_{p(x)q(x)}^{p^-}) \leqslant ||u|^{p(x)}|_{q(x)} \leqslant \max(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+}).$$

The generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \Big\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \Big\}.$$

It is equipped with the norm

$$\|u\|_{1,p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Note that $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, equipped respectively with the above norms, are separable, reflexive and uniformly convex Banach spaces.

In this work, we will present the evolution of image restoration. It is clear that the evolution is done in the sense of correcting the defects resulting from the previous model and overcome them, or in the sense of giving new techniques that restore the image in better manner and with the minimum flaws.

Literature survey

Let us consider the image restoration problem. As we mentioned in the introduction we seek to find a minimizer u for the problem

$$\inf_u \int_\Omega |u_0 - \varphi u|^2 dx. \tag{1.6}$$

The classical way to solve such ill posed problems is to add a regularization term to the energy. In 1977, Tikhonov and Arsenin [**78**] proposed the first regularization term in literature for image restoration and studied the related problem

$$\inf_u \left(\frac{1}{2} \int_\Omega |u_0 - \varphi u|^2 dx \; + \; \tau \int_\Omega |\nabla u|^2 dx \right), \qquad \tau > 0 \tag{1.7}$$

where τ is positive number which is selected to equilibrate between the regularization and data fidelity terms. The associated Euler-Lagrange equation is linear and the L^2 norm of the gradient in energy (1.7) allows to remove the noise but unfortunately it penalizes too much the edges which are no longer preserved, so the obtained image does not contain noise but it is very blurry.

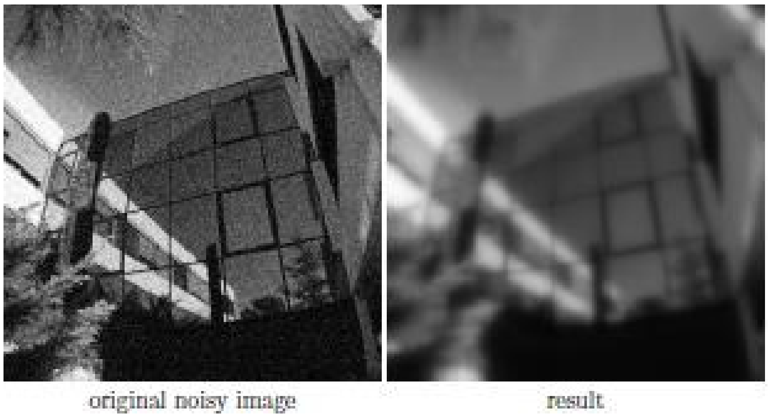

FIGURE 1. Restoration of the noisy "Borel building" image (additive Gaussian noise) edges are lost.

The Laplacian does not lead to a good restoration of the degraded image, given that it is a very strong smoothing operator which penalize too much the edges. For these reasons, the obtained image is very blurry. The anisotropic diffusion proposed in [**66, 83**] as an alternative to solve this problem, allowed to preserve details in the image but edges are still blurred.

To overcome this phenomenon, the idea cited in [**4**] was to use of the L^p norm of the gradient in the energy (1.7) and decrease p so that we preserve as much as possible the edges in the restored image. Among the first works that have been done in this direction [**73, 74**], who suggested to use the L^1 norm of the gradient of u, also named the total variation. We point out that an extensive studies have been done on the TV subjects. We cite for examples (e.g., [**2, 14**]) and many TV models were proposed. TV-based regularisation, TV-based diffusion, TV-based denoising are the most first topic studied in the literature. It has been proven that total variation approach is a very effective tool for image restoration especially in preserving edges. Rudin, Osher, and Fatemi [**73**] proposed the so-called TV-based regularization in the case $p = 1$ where they consider the problem

$$\inf_u \left(\frac{1}{2} \int_\Omega |u_0 - \varphi u|^2 dx \ + \ \tau \int_\Omega |\nabla u| dx \right), \qquad \tau > 0 \tag{1.8}$$

Assuming Neumann boundary conditions, the Euler- Lagrange equation is

$$\varphi^* \left(\varphi u - u_0 \right) - \tau \nabla . \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \tag{1.9}$$

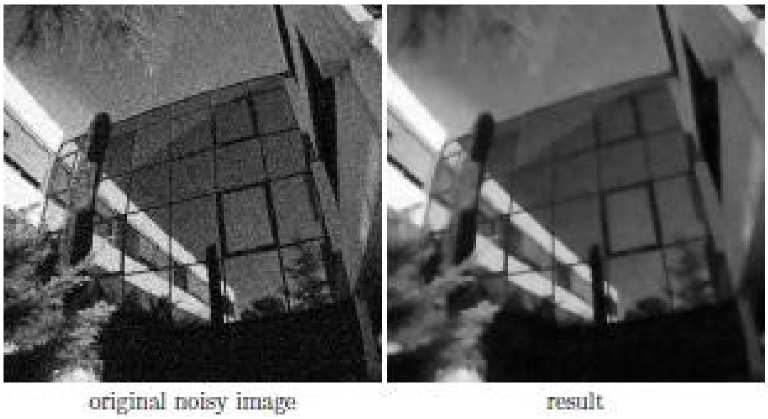

FIGURE 2. Result with half-quadratic minimization noise is removed, while discontinuities are retained.

Note that at point where $\nabla u = 0$, equation (1.9) is not defined and the standard way to avoid this discontinuity is to replace $|\nabla u|$ in (1.9) by $\sqrt{|\nabla u|^2 + \delta}$ which gives

$$\varphi^* (\varphi u - u_0) - \tau \nabla . \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right) = 0 \tag{1.10}$$

with δ a small positive number. The main difficulty in solving (1.10) is the linearization of the highly nonlinear term $-\tau \nabla . \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \delta}} \right)$. In practice, a large literature is devoted to the numerical study of (1.10)(e.g., [**71, 88**]), moreover authors proved that TV regularisation can very well remove noise and preserve sharp edges at the same time. However, this approach is criticized for the loss of contrast in the restored image.

Numerical methods have been used to solve this problem. In [**73**] authors used the gradient descent method to solve the parabolic equation $u_t = \varphi^* (\varphi u - u_0) - \tau \nabla . \left(\frac{\nabla u}{|\nabla u|} \right)$ with the constraint $u = u_0$ but this method is slowly convergent. An alternative approach called "lagged diffusivity fixed point iteration" denoted by FP was proposed by C. R. Vogel and M. E. Oman in [**82**] to solve (1.9), which consist to consider the following equation

$$\varphi^* \left(\varphi u^{n+1} - u_0 \right) - \tau \nabla . \left(\frac{\nabla u^{n+1}}{|\nabla u^n|} \right) = 0 \text{ with } u^0 = u_0. \tag{1.11}$$

unfortunately, even if this method is robust it is only linearly convergent. In [**79**] authors presented a better linearization technique which is globbly convergent and consists to introduce a new variable $\theta = \frac{\nabla u}{|\nabla u|}$ and then consider the equivalent system

$$\begin{cases} \varphi^* (\varphi u - u_0) - \tau \nabla . \theta = 0 \\ \theta |\nabla u| - \nabla u = 0 \end{cases} \tag{1.12}$$

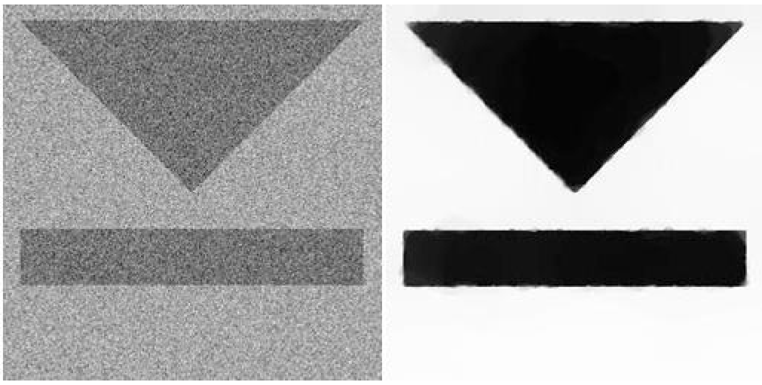

FIGURE 3. Primal-dual method denoising

An interesting adaptive total variation model was given by D. M. Strong and T. F. Chan in [**75**] where a control factor $\nu(x)$ was introduced to slow the diffusion at likely edges. That is to study the following minimization problem

$$J(u) = \frac{1}{2} \int_\Omega |u_0 - \varphi u|^2 dx \ + \ \tau \int_\Omega \nu(x) |\nabla u| dx, \qquad \tau > 0 \tag{1.13}$$

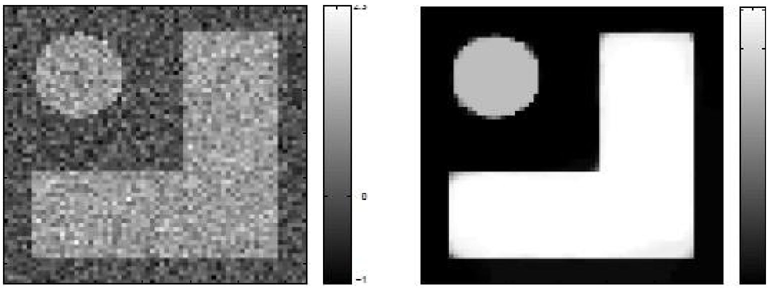

FIGURE 4. adaptive total variation denoising

The TV-based denoising is an approach defined in term of an optimization problem, and acts in two sense: reduce noise and preserve edges in an the restored image. According to [**66, 67, 68**], TVD approach has connections to the anisotropic diffusion with partial differential equations. Even if the total variation has achieved great success in image denoising but it has some disadvantages: flat region of the restored image may contain staircasing or blockly artifacts created artificially, textures tend to be too much smoothed and the contrast of the restored image may be lost [**8, 14, 22, 62, 72, 82, 84, 90**]. Due to properties of the space $BV(\Omega)$ of functions of bounded variation allowing discontinuities which are necessary for edges construction it is natural to study existence of slutions of some problems in this space. In this direction, authors in [**5, 80**] studied a general version of problem (1.8) where they considered the energy

$$J(u) = \frac{1}{2}\int_{\Omega} |u_0 - \varphi u|^2 dx \ + \ \tau \int_{\Omega} \psi(|\nabla u|)dx, \qquad \tau > 0 \tag{1.14}$$

Imposing some conditions on ψ, so that the smoothing is the same in all direction in places where the gradient is low and at the same time control the smoothing in the neighborhood of contours in order to preserve them. Authors in [**4**] calculated the relaxed functional of (1.14) in the BV_w^* and then use direct method of the calculus of variations to proof that the relaxed problem has a unique solution and then deduce existence of a unique solution for the original problem. Let recall that BV_w^* denote the space of functions of bounded variation endowed with weak topology. G. Aubert and P. Kornprobst claim [**4**], that in certain situations it is necessary to propose more adapted functional spaces than $BV(\Omega)$ due to the the staircase effect resulting from the minimization of the total variation, and also for the reason that naturel images are not totaly described by $BV(\Omega)$ functions [**42**]. Later on, the energy (1.14) have been used by L. Tang, Z. Fang in [**77**] to overcome the phenomenon of loss of contrast in the restored image. In the framework of TV, they proposed a forward-backward diffusion model called TV-FBD, based on variation methode and backward diffusion. There main result in [**77**] was the combination between the TV energy and the energy (1.14). In other words, they consider the energy

$$T(u) = \lambda \int_{\Omega} |\nabla u| dx + \frac{1}{2}\int_{\Omega} |u_0 - \varphi u|^2 dx + \tau \int_{\Omega} \psi(|\nabla u|)dx, \qquad \tau > 0, \lambda > 0 \tag{1.15}$$

Using the two-step spliting (TSS), they split the given problem (1.15) into two sub-problems

$$\min_u \left(\frac{1}{2}\int_{\Omega} |u_0 - \varphi u|^2 dx \ + \ \lambda \int_{\Omega} |\nabla u| dx \right), \qquad \lambda > 0 \tag{1.16}$$

and

$$\min_u \left(\tau \int_{\Omega} \psi(|\nabla u|)dx \right), \qquad \tau > 0 \tag{1.17}$$

then, by using the two problems alternatively, they succeeded in removing noise, simultaneously preserving contrast. But with this model textures are not well recovered, that is what lead authors to focus there research on how to incorporate texture representation in there model to improved it.

In fact, the idea of combined two energies comes a bit far back to Chambolle and Lions [**14**] who proposed to combin isotropic energy with TV-based diffusion

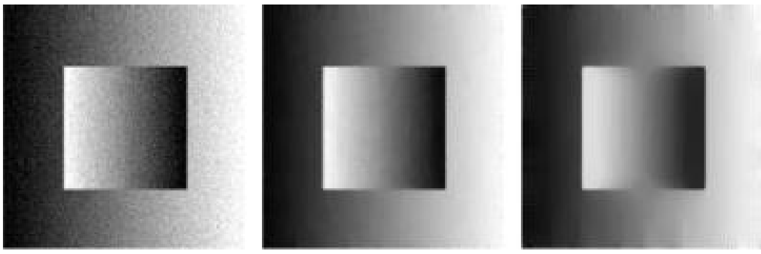

FIGURE 5. Left: noisy image, middle: denoised image with the proposed model, right: denoised image with TV model

energy and looked for minimizers of the energy

$$\min_{u\in BV(\Omega)}\left(\frac{1}{2\tau}\int_{|\nabla u|\leqslant\tau}|\nabla u|^2dx\ +\ \int_{|\nabla u|>\tau}|\nabla u|-\frac{\tau}{2}dx\right),\qquad \tau>0 \tag{1.18}$$

This model is effective for images where homogeneous regions are separted by distinct edges,they are well restored, but it is sensitive to the threshold τ for nonuniform and highly degraded. For this reason a good flexibility in the choice of the direction and the speed of flow is suggested.

Othors proposed in [**1**] a similar model model which is a combination of fast growth with respect to low gradient and low growth when the gradient is large. They proved the existence of a minimizer in some Orlicz space for the following energie.

$$\min_u\left(\int_{|\nabla u|\leqslant 1}B\left(x,|\nabla u|\right)dx\ +\ \int_{|\nabla u|>1}A\left(|\nabla u|\right)\,dx+\frac{\tau}{2}\int_\Omega|u_0-\varphi u|^2dx\right),\ \tau>0 \tag{1.19}$$

where $A(s)=s\log^t(1+s)$, and $B(x,s)$ is an increasing C^2 function for almost everywhere $x\in\Omega$. It was proven numericaly that when t is close to 0 the restored image is well improved.

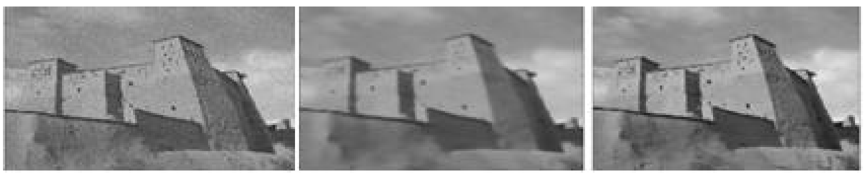

FIGURE 6. Left: noisy image, middle: denoised image with the proposed model (t = 1), right: denoised image with the proposed model (t = 0.00000001)

It seems that the model given in [**8**] can be considered as a model which combines two energies because it reaps both the advantages of isotropic and TV- diffusion, even if it is not written as a sum of these two energies. The proposed model is

$$\min_u \left(\frac{1}{2} \int_\Omega |u_0 - \varphi u|^2 dx \ + \ \int_\Omega \left| \nabla u \right|^{p(|\nabla u|)} dx \right) \tag{1.20}$$

where $\lim_{z \to 0} p(z) = 2,\ \lim_{z \to \infty} p(z) = 1,$ and p is monotonically decreasing. To capture edges, the $W^{1,1}(\Omega)$ space is used (the model acts as TV-bassed diffusion) and to reconstruct flat region, the space $W^{1,2}(\Omega)$ is used (isotropic diffusion), in the other regions they dealed with the space $W^{1,p}(\Omega),\ p \in (1,2)$. It is a very well model to reduce staircasing effect, but as long as they did not have sufficient mathematical tools, the mathematical study of this problem remains difficult. Y.Che, S. Levine and M. Rao were the first authors who introduced functional with variable exponent in image restoration. They established the existence of a unique pseudosolution for the partial differential equation with variable exponent that they have proposed in [**18**]. Moreover, they prsented experimental results to illustrate the effectiveness of the given model in image restoration. The model combines between a total variation (TV)-based regularization and Gaussian smoothing and the exponent was well choosen allowing the model to be an exelent tool for image restoration, denoising and enhancement.

The proposed energy was

$$\min_{u \in BV(\Omega) \cap L^2(\Omega)} \int_\Omega \psi\left(s, Du\right) + \frac{\tau}{2} \left(u_0 - u\right)^2 \, ds \tag{1.21}$$

$$\psi\left(s,t\right) = \begin{cases} \dfrac{1}{\alpha(s)} |t|^{\alpha(s)}, & |t| \leqslant \mu \\ |t| - \dfrac{\mu\alpha(s) - \mu^{\alpha(s)}}{\alpha(s)}, & |t| > \mu \end{cases} \tag{1.22}$$

with $\mu > 0,\ \ 1 < \alpha(s) \leqslant 2$, and

$$p(s) = \begin{cases} \alpha(s), & |\nabla u| < \mu \\ 1, & |\nabla u| \geqslant \mu \end{cases} \tag{1.23}$$

for

$$\alpha(s) = 1 + \frac{1}{1 + k|\nabla G_\sigma * u_0(s)|^2} \tag{1.24}$$

where $k > 0$, and G_σ the Gaussian filter. Under some additional assumptions and by using certain adequate mathematical tools they proved that a weak solution of

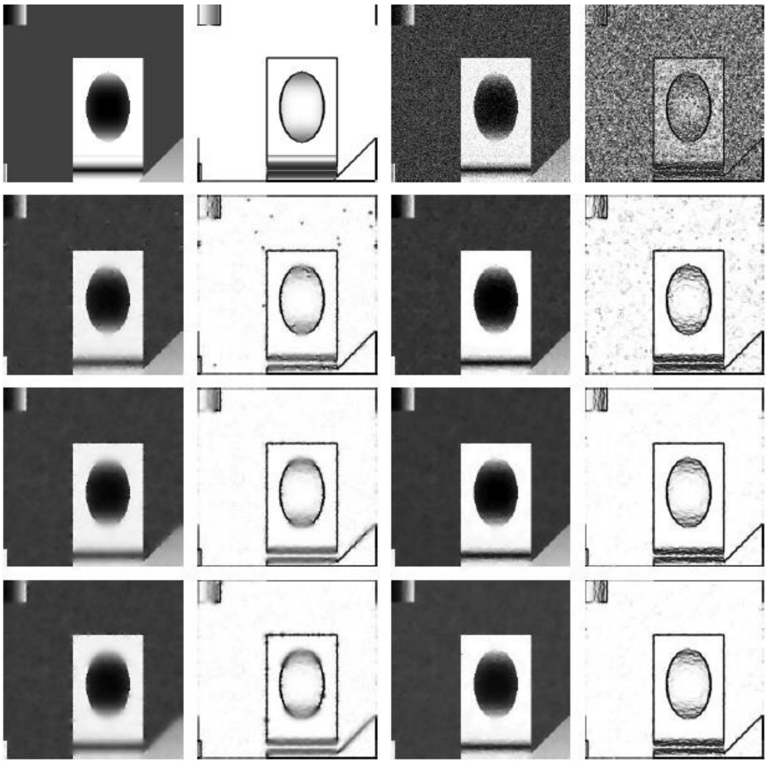

FIGURE 7. Top row: original piecewise smooth image and edge map, image + noise and edge map.Bottom three rows: First column from left: reconstructions with thresholds $\mu = 30, 50, 70$,respectively (1000 iterations). Second column: corresponding edge maps. Third column: reconstruction using TV-based diffusion only (2000 iterations) and the proposed model with thresholds $\mu = 30, 100$,respectively (1000 iterations). Fourth column: corresponding edge maps

the proposed problem satisfies the following equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - |\nabla u|^{p(x)-2}\Big[\,(p(x)-1)\,\Delta u + (2-p(x))\,|\nabla u| div\left(\dfrac{\nabla u}{|\nabla u|}\right) + \nabla p.\nabla u \log|\nabla u|\Big] \\ \qquad +\tau\,(u-u_0) = 0, \quad \text{in } \Omega\times[0,T] \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega\times[0,T] \\ u(0) = u_0. \end{cases}$$

Experimental results have confirmed that the strengths of this model is its way to act to accommodate the local information. Near edges the total variation is used and in uniform regions the model is isotropic, while in the other regions it acts as a Gaussian filter or total variation diffusion. Furthermore, the dependence of the anisotropic diffusion in ambiguous regions on the strength of the gradient enssures the no influence at the threshold.

Later on, F. Li, Z. Li and L. Pi [**54**] improved the model in [**18**] by weakining its formulation and changing the space of minimizers, they worked in $W^{1,p(x)}(\Omega)$ instead of $BV(\Omega)$. They have proposed the following minimization problem

$$\min_{u\in W^{1,p(x)}(\Omega)\cap L^2(\Omega)} \int_\Omega \Big(\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \frac{\tau}{2}\int_\Omega (u_0-u)^2\,dx\Big) \tag{1.25}$$

$$\begin{cases} p(x) = 1+\alpha(x), \\ \alpha(x) = 1 + \dfrac{1}{1+k|\nabla G_\sigma * u_0(x)|^2} \end{cases} \tag{1.26}$$

Where $k>0, \sigma>0, \tau>0$. Meanwhile, the new model is more flexible than (1.21) since it does not involve the threshold μ. According to (1.26), $p(x)$ can tends to 1 but not equal to 1 in region where gradient is large. When $p(x)\to 1$ edges will be preserved, if $p(x)\to 2$ the model approximates isotropic smoothing, and when $1<p(x)<2$, the diffusion is adjusted. Authors lefted existence of minimizers in

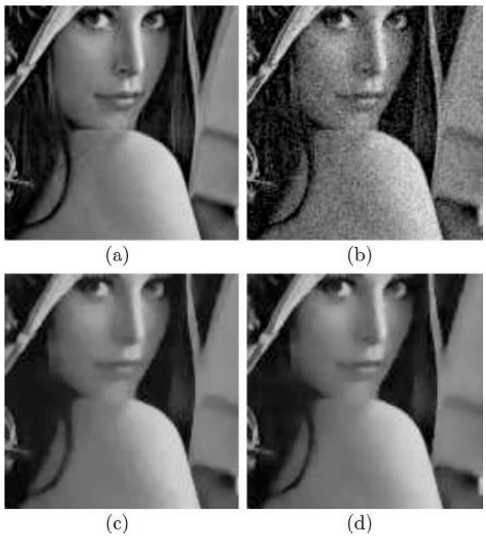

FIGURE 8. Comparison of the proposed model and the ROF model. (a) A part of Lena image; (b) the noisy image; (c) the restoration result by the ROF model; and (d) the restoration result by the proposed model.

the case $p=1$ as an open question. The answer of this question was given by P.Harjulehto, P.Hästö,V. Latvala and O. Toivanen, in there paper [**46**], where they have used the notion of $\Gamma-$ convergence for the proof of existence results in the space $BV^{(\cdot)}(\Omega) \cap L^s(\Omega)$ for the problem

$$\inf_{u \in BV^{(\cdot)}(\Omega)} \Big(\vartheta_{\in BV^{(\cdot)}(\Omega)} u + \tau \int_\Omega |u - u_0|^s \, dx \Big)$$

with

$$BV^{(\cdot)}(\Omega) := \Big\{ u \in BV(\Omega) \cap W^{1,p(\cdot)}(\Omega \backslash X) \Big\} , \quad \text{where} \;\; X := \{ x \in \Omega : \;\; p(x) = 1 \} .$$

$$\vartheta_{BV^{(\cdot)}(\Omega)} u = \|\nabla u\|_{X \cap A} + \int_{A \backslash X} |\nabla u|^{p(x)} \, dx$$

for every $A \subset \Omega, \tau > 0, u_0 \in L^s(\Omega)$. In a recent paper [**92**], authors gave some remarks on the model in [**18**]. According to there opinion, as there are no arguments which explain the choice of $p(x)$ in the previous model, then if the continuity of $p(x)$ is neglected, more general models with variable exponent can be constructed for image denoising. In particular, the restored image will have a better quality if $p(x)$ is constructed as an appropriate piecewise constant function the quality of the restored image is better. Unfortunately, the discontinuity of $p(x)$ lead to the indesired Lavrentiev phenomenon.

We point out that the presented models performs very well for denoising of images, while preserving edges. But, smaller details, such as textures are destroyed. We also recall that in all these models the authors focused on improving the regularity term. Following [**64, 81**], the restored image is formed from two component: cartoon information and texture information , the term $u_0 - u$ represents the texture or noise information. This very interesting description associated to Meyer results [**59**] allowed researchers to overcome the drawback of loss of texture in the restored image. In fact, Meyer proposed a new minimization method by replacing the L^2 norm of $(u_0 - u)$ by a weaker norm more appropriate to represent textured or oscillatory patterns.

$$\min_u \Big(\int_\Omega |\nabla u| \, dx \quad + \tau \|u_0 - u\|_* \Big) \tag{1.27}$$

But, due to the forme of the norm $\|\|_*$, the Euler-Lagrange equation with respect to u cannot be expressed directly and consequently the Problem (1.27) cannot be solved directly. In [**81**], L. Vese and S. Osher have poposed a practical approximation to the Problem (1.27) to overcome the above difficulty.

In [**64**], an other model used the H^{-1} norm to measure fidelity term. That is to minimize the following energy

$$\min_u \Big(\int_\Omega |\nabla u| \, dx \quad + \tau \|u_0 - u\|^2_{H^{-1}} \Big) \tag{1.28}$$

$\tau > 0, \|\|_{H^{-1}(\Omega)} = \int_\Omega |\nabla \Delta^{-1}(.)|^2 \, dx$. In [**44**], authors used Galerkin's method to establish existence and uniqueness of weak solutions in a suitable Sobolev spaces for the problem

$$\min_u \Big(\int_\Omega \frac{g(x)}{p(x)} |\nabla u|^{p(x)} \, dx \quad + \tau \|u_0 - u\|^2_{H^{-1}} \Big) \tag{1.29}$$

with $\tau > 0,\ u_0 \in L^2(\Omega), \quad g(x) = \dfrac{1}{1 + k_1 |\nabla G_{\sigma_1} * u_0|^2}, \ \ k_1 > 0, \ \ \sigma_1 > 0,$

$p(x) = 1 + \dfrac{1}{1 + k_2 |\nabla G_{\sigma_2} * u_0|^2}, \ \ k_2 > 0, \ \ \sigma_2 > 0$. The effectiveness of the model (1.29) in image restoration was shown by numerical experimental resultats.

However, in some cases the image may be not smooth enough it's why authors in [**56**] allowed the observed image to be an integrable function, and seeked to solve the Problem (1.29) in a more extensive space. They supposed that $u_0 \in L^1(\Omega)$, and succeeded to establish existence of renormalized solutions which are more suitable than weak solutions.

Recently, based on the work of Yaroslavsky [**89**], some new process of denoising models have been developed for image restoration. These are nonlocal methods which allawed to remove noise, preserve edges and take care of fine details , structures and textures. Nonlocal methods are based on the similarities between neighboring or overlapping patches in an image. Later, authors introduced nonlocal operators [**40, 41**], in a systematical study for nonlocal image processing. The minimization problem which can be considered in the general frame work of nonlocal variational problems

$$\min_u \left(\frac{1}{2p} \int_\Omega \int_\Omega \varphi(x,y) \big| u(y) - u(x) \big|^p d\mu(x) d\mu(y) + \tau \int_\Omega \int_\Omega \psi \left(\big| u(x) - u_0(x) \big| \right) d\mu(x) \right) \tag{1.30}$$

where φ is the weight function, $1 < p < \infty$, μ is a probablity mesure and τ is a fidelity parameter. A nonlocal $p-$ Laplacian deblurring and denoising images model [**52, 3**]. was proposed in the case $\psi(s) = \dfrac{s^2}{2}$. That is to look for minimizers for

$$\min_u \left(\frac{1}{2p} \int_\Omega \int_\Omega \varphi(x,y) \big| u(y) - u(x) \big|^p d\mu(x) d\mu(y) + \frac{\tau}{2} \int_\Omega \int_\Omega \big| u(x) - u_0(x) \big|^2 d\mu(x) \right) \tag{1.31}$$

F. Karami, K. Sadik, and L. Ziad, in there recent work [**48**] proposed a new faster denoising process where they have combined between nonlocal imge denoising approch and the variable exponent $p(x)-$Laplacian. The resulting model inherts the power of the nonlocal approch in preseving texture and fine details. Moreover,

FIGURE 9. (a) Original image. (b) the noisy image . (c) Image restored using the new method. (d) Image restored using the TV model

the variable exponent enables fast convergence to the solution. The idea was to minimize

(1.32)

$$\min_u \left(\frac{1}{2p} \int_\Omega \int_\Omega \frac{\varphi(x,y)}{2p(x,y)} \left|u(y)-u(x)\right|^{p(x,y)} d\mu(x) d\mu(y) + \frac{\tau}{2} \int_\Omega \int_\Omega \left|u(x)-u_0(x)\right|^2 d\mu(x) \right)$$

Authors used semi group's theory to prove existence and uniqueness of solution for the Problem (1.33). The model was tested and its efficiency in reducing time of convergence was proven.

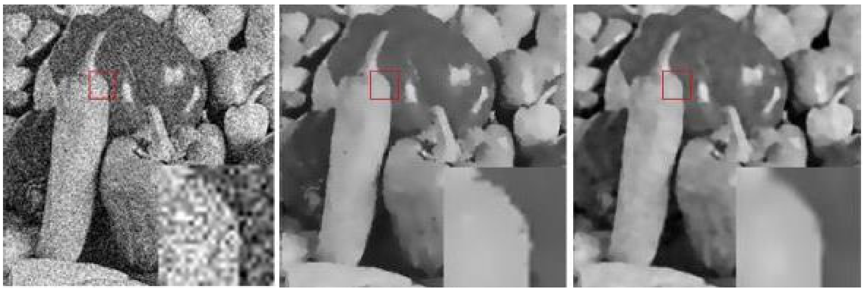

FIGURE 10. Left: the noisy image, middel Image restored using $p(x)-$Lplacian model, right: Image restored using the proposed model

FIGURE 11. LPL and NLPL image resoration

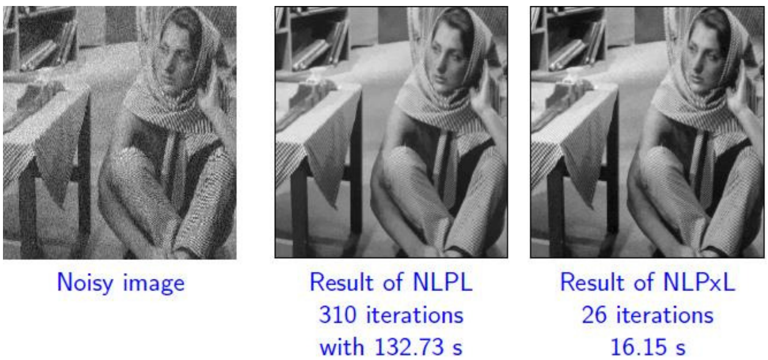

FIGURE 12. NLPL and NLPxL image resoration

Inspired by [**48**] and Meyer's comments [**59**] and related works regarding the fidelity term, authors in [**49**] considered the Problem (1.33) with an appropriate fidelity term. That is to seek minimizers for the following energy

(1.33)
$$\min_u \left(\frac{1}{2p} \int_\Omega \int_\Omega \frac{\varphi(x,y)}{2p(x,y)} \big|u(y)-u(x)\big|^{p(x,y)} d\mu(x) d\mu(y) + \frac{\tau}{q} \int_\Omega \int_\Omega \big|u(x)-u_0(x)\big|^q d\mu(x) \right)$$

where $1 < q \leqslant 2$. This model is more powerful than the others since it satisfies three properties at the same time, the variable exponent hepls in reducing the execution time, besides, the nonlocal approch and the weaker norm allow the preservation of textures and small details. The given model was tested and numerical simulations have proved the efficiency of the model in removing noise, preserving edges, moreover, the model handels better repetitive structures and textures in a reduce time.

FIGURE 13. First row: image corrupted by Gaussian noise with zero mean and variance $\sigma = 25$. Second row: restored images with the proposed model

Authors have compared the new proposed model to some others models.

FIGURE 14. Left: noisy Barbara image, middle: restored image by TV with L^1- fidelity term, right: restored by the proposed model

Conclusion

According to various presented works , the old idea to restor an image by adding some regularization term has changed. The fidelity term act also on the quality of the restored image. Moreover, nonlocal methods involving $p(x)-$Laplacian are more efficient. We wonder what would be the quality of the restored image when we combine the nonlocal methods with the $p(x)-$laplacian associated to the norm H^{-1} in fidelity and L^1 data.

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