JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
J. Int. Math. Virtual Inst., **12**(1)(2022), 103-121
DOI: 10.7251/JIMVI2201103I

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

COUNTEREXAMPLES FOR TOPOLOGICAL COMPLEXITY IN DIGITAL IMAGES

MELİH İS and İSMET KARACA

Abstract. Digital topology has its working conditions and sometimes differs from the usual topology. In the area of topological robotics, we provide important counterexamples in this study to emphasize this red line between a digital image and a topological space. We indicate that the results on topological complexities of certain path-connected topological spaces show alterations in digital images. We also give a result about the digital topological complexity number using the genus of a digital surface in discrete geometry.

1. Introduction

Digital topology makes it possible to analyze digital images by transferring topological properties into itself. This analyzing process provides advantages in technologies that include especially computer science and image analysis. After that digital topology is introduced by Rosenfeld with its most primitive concept, it has been involved in the studies of a wide variety of fields [**34**]. For instance, Peters has interesting ideas on visual pattern discovery using digital topology [**33**]. One of the most significant of these studies points to the subject of robotics, including algebraic topology over the past twenty years.

Farber assigns a positive integer called the topological complexity number TC(X) for each path-connected topological space X and makes inferences about the complexity of the motion area in which a robot is located with obstacles [**21**]. Afterwards, as the structure of the required topological space differs, the results of computing the topological complexity number also get vary. For example, if X is a

2010 *Mathematics Subject Classification.* Primary 68R01, 65D18; Secondary 55M30, 68U10.
Key words and phrases. Digital image, topological complexity number, digital topological complexity number, higher topological complexity, Betti number.
Communicated by Dusko Bogdanic.

connected Lie group, then the topological complexity number TC(X) of X equals its Lusternik-Schnirelmann category cat(X) of X [**22**]. Davis has different approaches to the topological complexity number of certain path-connected topological spaces such as a circle or a Klein bottle [**14, 15**]. Dranishnikov has remarkable results on TC and cat [**16, 17, 18**]. The higher topological complexity TC$_n$, defined by Rudyak, is the Schwarz genus of a special fibration and a positive integer such as cat and TC [**36**]. These are all homotopy invariants. Many essential concepts of the algebraic topology, for instance, cohomological cup-product, help to calculate these numbers [**21**]. Hence, it is the main task to have concrete ideas about the topological complexities of the digital images by making more use of the algebraic topology methods. For more information about the concepts, results and methods for topological robotics, refer to [**24**].

Digital topology has started to incorporate the algebraic topology methods in time and this process has not ended yet. One of the richest content of these methods is the cohomology rings of topological spaces. Ege and Karaca present the general framework related to the digital simplicial cohomology groups and introduce the cohomological cup-products in digital images [**19**]. In digital topology, researchers generally deal with the two-dimensional or three-dimensional digital images and they call them as digital curves and digital surfaces, respectively. Rosenfeld defines the digital curve and states that the digital image X is said to be a simple (closed) curve if each point of X has exactly two adjacent points in X [**35**]. On the other hand, the digital surface is introduced by Morgenthaler and Rosenfeld [**32**]. Digital surfaces are so important not only for digital topology but also for discrete and computational geometry. Chen uses one of the most important notions related to topology and geometry in digital images for the first time: digital manifolds. A digital manifold can be regarded as a discretization of a manifold. Other sayings, it is a combinatorial manifold which is defined in digital images [**12**]. Chen and Rong improve a powerful method for computing genus and the Betti numbers of a digital image [**13**].

In digital topology, not everything is the same as in the usual topology. For instance, Künneth Theorem does not work for digital images [**19**]. This leads to the fact that cohomological cup-product does not hold for digital images [**29**]. Therefore, the digital interpretations of cat, TC, and TC$_n$ have different results from the topological spaces [**2, 31, 29**]. In this study, we focus on such results and give counterexamples in digital topology. First, we remind the basic notions of the digital topology, give some concepts and results from discrete geometry, and present certain definitions of the main tools of robotics in Section 2. Thus, we recall a few deliberately selected results on mostly Farber's topological complexity numbers. The main goal of this study gives counterexamples for each properties that is valid in the usual topology but not in digital topology. This shows that digital topology cannot be thought as the same as the usual topology. They often have different impact areas in mathematics because the methods of algebraic topology can lose its influence such as in the example of Künneth Theorem and this is the strength of digital topology conditions.

In Example 3.1, we consider the digital image $X = [0,1]_{\mathbb{Z}}$. Since the closed interval $[0,1]$ is contractible in topological spaces, the higher topological complexity of $[0,1]$ for any n is 1. Moreover, Proposition 2.9 supports this result using the cohomological cup-product but in digital images, using the digital cup-product does not give the same result. Hence, the digital image $X = [0,1]_{\mathbb{Z}}$ is a counterexample of that Proposition 2.9 does not hold for digital images. Second, we show that the diagonal map of digital images does not coincide with the digital cup-product homomorphism. In Example 3.3, we choose a specific digital image whose Betti number is 2. In the usual topology, it is expected that the topological complexity number of a space is 3, when $b_1(X) = 2$ but we prove that it is possible that the digital topological complexity number of such an image is less than 3. We present a result about the digitally connected digital curves with considering the first Betti numbers. Example 3.5 is constructed on showing that the digital version of the topological complexity number of the wedge of $2-$sphere S^2 does not have to be 3. Finally, we study digital surfaces with genus g. In topology, the topological complexity number of a compact orientable surface of genus 1 is 3 whereas the topological complexity number of a compact orientable surface of genus 2 is 5. We prove that both of this results does not hold for digital surfaces. We show that the digital topological complexity number of digital simple closed surfaces of genus 0, 1 and 2 are 1, 1 and 2, respectively.

2. Preliminaries

This section consists of the fundamental notions in digital topology. In addition, remarkable points of the study of topological robotics are mentioned.

Let $\mathbb{Z}^m$ denote the set of all $m-$tuples of integers in the Euclidean space $\mathbb{R}^m$ and $X \subset \mathbb{Z}^m$. Then the pair (X, κ) is said to be a *digital image*, where κ is an adjacency relation for the points of X [**4**]. Let x and y be two different points in $\mathbb{Z}^m$ and k be a positive integer such that $k \leqslant m$. x and y are c_k-*adjacent* if there are at most k indices i such that $|x_i - y_i| = 1$ and for all other indices i such that $|x_i - y_i| \neq 1$, $x_i = y_i$ [**4**]. Therefore, in $\mathbb{Z}$, one has only $c_1 = 2$ adjacency. In $\mathbb{Z}^2$, there are two adjacencies such as $c_1 = 4$ and $c_2 = 8$. In $\mathbb{Z}^3$, there are three adjacencies such as $c_1 = 6$, $c_2 = 18$ and $c_3 = 26$.

Let X be a digital image in $\mathbb{Z}^m$. X is $\kappa-$*connected* if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_l\}$ of points of X such that $x = x_0$, $y = x_l$, and x_i and x_{i+1} are $\kappa-$adjacent, where $i = 0, 1, ..., l-1$ [**28**]. Let $f : X_1 \to X_2$ be a map such that X_1 has an adjacency relation κ_1 in $\mathbb{Z}^{m_1}$ and X_2 has an adjacency relation κ_2 in $\mathbb{Z}^{m_2}$. Then f is $(\kappa_1, \kappa_2)-$*continuous* if, for any κ_1-connected subset U_1 of X_1, $f(U_1)$ is κ_2-connected [**4**]. Moreover, if a bijective digital $(\kappa_1, \kappa_2)-$continuous map f has a $(\kappa_2, \kappa_1)-$continuous inverse f^{-1}, then f is said to be a $(\kappa_1, \kappa_2)-$*isomorphism* [**7**].

A set $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} : a \leqslant z \leqslant b\}$ is called a digital interval from a to b [**6, 3**]. Let X and Y be two digital images such that $f, f' : X \to Y$ are $(\kappa_1, \kappa_2)-$continuous maps. If there exists a positive integer n and a map

$$G : X \times [0, n]_{\mathbb{Z}} \to Y$$

such that the following conditions hold, then f and $f^{'}$ are said to be *digitally* $(\kappa_1, \kappa_2)-$*homotopic* [**4**]. G is called a *digital* $(\kappa_1, \kappa_2)-$*homotopy* between f and $f^{'}$.

- For all $x \in X$, $G(x, 0) = f(x)$ and $G(x, n) = f^{'}(x)$;
- for all $x \in X$, the map $G_x : [0, n]_{\mathbb{Z}} \to Y$, defined by $G_x(t) = G(x, t)$, is $(2, \kappa_2)-$continuous for all $t \in [0, n]_{\mathbb{Z}}$;
- for all $t \in [0, n]_{\mathbb{Z}}$, the map $G_t : X \to Y$, defined by $G_t(x) = G(x, t)$, is $(\kappa_1, \kappa_2)-$continuous for all $x \in X$.

Let X be a digital image. If the identity map $id_X : X \to X$ is $(\kappa, \kappa)-$homotopic to a constant map

$$c : X \longrightarrow X$$
$$x \longmapsto c(x) = c_0,$$

for all $x \in X$, then (X, κ) is $\kappa-$*contractible* [**4**].

Let X_1 be a digital image with λ_1-adjacency and let X_2 be a digital image with λ_2-adjacency. Given two points (x_1, x_2) and $(x_1^{'}, x_2^{'})$ in the cartesian product digital image $X_1 \times X_2$, we say that (x_1, x_2) *and* $(x_1^{'}, x_2^{'})$ *are adjacent*, if one of the following conditions hold [**1**]:

- $x_1 = x_1^{'}$ and x_2 and $x_2^{'}$ are λ_2-adjacent; or
- x_1 and $x_1^{'}$ are λ_1-adjacent and $x_2 = x_2^{'}$; or
- x_1 and $x_1^{'}$ are λ_1-adjacent and x_2 and $x_2^{'}$ are λ_2-adjacent.

Let X be a digital image. If $f : [0, n]_{\mathbb{Z}} \to X$ is a $(2, \kappa)-$continuous map such that $f(0) = x$ and $f(n) = x^{'}$, then f is called a *digital path* from x to $x^{'}$ in X [**6**]. *A simple closed* $\kappa-$*curve* of $r \geqslant 4$ points in a digital image X is a sequence $\{g(0), g(1), ..., g(r-1)\}$ of images of the $\kappa-$path $g : [0, r-1]_{\mathbb{Z}} \to X$ such that $g(m)$ and $g(n)$ are $\kappa-$adjacent if and only if $n = (m \pm 1) mod\ r$ [**5**]. A *digital surface* is the set of surface points each of which has two adjacent components not in the surface in its neighborhood [**32**]. Let κ be an adjacency relation defined on $\mathbb{Z}^m$. A $\kappa-$*neighbor* of $x \in \mathbb{Z}^m$ is a point in $\mathbb{Z}^m$ that is $\kappa-$adjacent to x [**28**].

Theorem 2.1. [**13**] *If X is a closed digital surface, then the genus of the surface is*

$$g = 1 + \frac{(|M_5| + 2.|M_6| - |M_3|)}{8},$$

where M_i is a set of points with $i-$neighbors.

In Theorem 2.1, M_i is the set of surface-points each of which has i adjacent points on the surface. Let (X, κ) be a digital image in $\mathbb{Z}^2$. *The digital wedge union* $X \vee X$ is the disjoint union of two X with only one point x_0 in common and for any different elements x and y in X, x and y are not $\kappa-$adjacent to each other except the point x_0 [**26**].

Let PX denote the set of all digitally continuous paths $\alpha : [0, n]_{\mathbb{Z}} \to X$ in X. $\pi : PX \to X \times X$ is a digitally continuous map that takes any digitally continuous path α in X to the pair of its starting and ending points $(\alpha(0), \alpha(n))$.

Definition 2.2. [**31**] *Digital topological complexity number* $\mathrm{TC}(X, \kappa)$ is the minimal number l such that $U_1, U_2, ..., U_l$ is a cover of $X \times X$ and for all $1 \leqslant i \leqslant l$, there is a digitally continuous map $s_i : U_i \to PX$ such that $\pi \circ s_i = id_{U_i}$. If no such l exists we set $\mathrm{TC}(X, \kappa) = \infty$.

In the definition of the digital topological complexity number, the digital continuity of s_i is required. Due to this reason, the task is to define an adjacency relation between two digital paths. Let $\alpha_1 : [0, n_1]_{\mathbb{Z}} \to X$ and $\alpha_2 : [0, n_2]_{\mathbb{Z}} \to X$ be two digitally continuous paths in X. We say that α_1 and α_2 are $\lambda-$connected on PX, if for all t, $\alpha_1(t)$ and $\alpha_2(t)$ are $\lambda-$connected. Note that α_1 and α_2 can have different t. Without loss of generality, assume that $n_1 < n_2$. In this case, the steps of the shortest path are extended such that $\alpha_1(n_1 + l) = \alpha_1(n_1)$, where $0 \leqslant l \leqslant n_2 - n_1$. Hence, α_1 is synchronized with α_2.

If a map $p : (X, \kappa_1) \to (X^{'}, \kappa_2)$ has the digital homotopy lifting property for every digital image, then p is said to be a *digital fibration* [**20**]. For more detail information on fibrations in the digital setting, refer to [**20**].

Definition 2.3. [**29**] Let $p : X \to X^{'}$ be a digital fibration. *The digital Schwarz genus of* p is defined as the minimum number l such that $U_1, U_2, ..., U_l$ is a cover of $X^{'}$ and for all $1 \leqslant j \leqslant l$, there is a digitally continuous map $t_j : U_j \to X$, such that $p \circ t_j = id_{U_j}$.

Definition 2.4. [**29**] Let $[0, m]^i_{\mathbb{Z}}$ denote the $i-$th digital interval with the endpoint m. Given n digital intervals $[0, m_1]^1_{\mathbb{Z}}, \ldots, [0, m_n]^n_{\mathbb{Z}}$ and denote J_n with the wedge of the digital intervals for $n \geqslant 1$ and $n \in \mathbb{N}$, where $0_i \in [0, m_i]^i_{\mathbb{Z}}$, $i = 1, ..., n$, are identified. Let X be a digitally connected space. Then the *higher topological complexity of a digital image* X is

$$TC_n(X, \kappa) = genus_{\lambda_*, \kappa_*}(e_n),$$

where $e_n : (X^{J_n}, \lambda_*) \to (X^n, \kappa_*)$, $e_n(f) = (f_1(m_1), ..., f_n(m_n))$, is a fibration of digital images for a multipath $f = (f_1, \cdots, f_n)$.

Note that for $n = 2$, the digital higher topological complexity number coincides with the digital topological complexity number [**29**]. The digital higher topological complexity is a homotopy invariant of digital images. TC_n is a natural lower bound for TC_{n+1}.

Definition 2.5. [**2**] *The digital Lusternik-Schnirelmann category*, $\mathrm{cat}_\kappa(X)$, of a digital image (X, κ) is defined to be the minimum number l such that there is a cover $U_1, U_2, ..., U_l$ of X such that for all $i = 1, ..., l$, each U_i is $\kappa-$contractible to a point in X.

Definition 2.6. [**30**] Let X be a digital image with c_k-adjacency and let $(X, \circ)$ be a group. Let $X \times X$ has also c_k-adjacency. $(X, c_k, *)$ is a c_k-*topological group*, if the maps

$$\begin{array}{ccc} \alpha : X \times X \longrightarrow X & \text{and} & \beta : X \to X \\ (x, x^{'}) \longmapsto x \circ x^{'} & & x \longmapsto x^{-1}, \end{array}$$

are digitally continuous, for all x, $x^{'} \in X$.

The digital version of a topological group is simply denoted as $(H, \kappa, *)$, and we read the triple $(H, \kappa, *)$ as a $\kappa-$topological group.

THEOREM 2.7. [**30**] *Let $(H, \kappa, \circ)$ be a $\kappa-$topological group such that (H, κ) is digitally connected and $n > 1$. Then*

$$TC_n(H, \kappa) = cat_{\kappa_*}(H^{n-1}),$$

where κ_* is an adjacency relation for H^{n-1}.

We use the cohomology groups of the digital image $MSS_6^{'}$ in the next chapter, so we completely need to know these values [**27**]. As a special example of the computation of cohomology groups of digital images, Proposition 2.8 answers it. Let us recall the proposition with its proof. (See [**10**] and [**11**] for more examples of digital curves and digital surfaces about computations in details of both homology and cohomology groups of the digital images.)

PROPOSITION 2.8. [**10**] *Let $MSS_6^{'}$ be a digital image which consists of 8 points p_0, p_1, p_2, p_3, p_4, p_5, p_6 and p_7 in $\mathbb{Z}^3$, where*

$$p_0 = (1,0,0), p_1 = (1,1,0), p_2 = (1,1,1), p_3 = (1,0,1),$$
$$p_4 = (0,0,1), p_5 = (0,1,1), p_6 = (0,1,0), p_7 = (0,0,0).$$

The digital cohomology groups of $MSS_6^{'}$ are

$$H^{q,6}(MSS_6^{'}) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^5, & q = 1 \\ 0 & q \neq 0, 1. \end{cases}$$

PROOF. Let $p_7 < p_4 < p_6 < p_5 < p_0 < p_3 < p_1 < p_2$. $C_0^6(MSS_6^{'})$ and $C_1^6(MSS_6^{'})$ are free abelian groups with bases $0-$simplexes $< p_0 >, ..., < p_7 >$ and $1-$simplexes

$$e_0 =< p_0p_1 >, e_1 =< p_0p_3 >, e_2 =< p_1p_2 >, e_3 =< p_6p_1 >$$
$$e_4 =< p_5p_2 >, e_5 =< p_3p_2 >, e_6 =< p_4p_3 >, e_7 =< p_7p_4 >$$
$$e_8 =< p_4p_5 >, e_9 =< p_6p_5 >, e_{10} =< p_7p_6 >, e_{11} =< p_7, p_0 >,$$

respectively. For $q > 1$, $C_q^6(MSS_6^{'}) = \{0\}$. Then the short exact sequence

$$0 \xrightarrow{\partial_2} C_1^6(MSS_6^{'}) \xrightarrow{\partial_1} C_0^6(MSS_6^{'}) \xrightarrow{\partial_0} 0$$

is obtained. Since

$$C^{0,6}(MSS_6^{'}) = Hom(C_0^6(MSS_6^{'}), \mathbb{Z}) \quad \text{and}$$
$$C^{1,6}(MSS_6^{'}) = Hom(C_1^6(MSS_6^{'}), \mathbb{Z}),$$

we have the sequence

$$0 \xrightarrow{\delta^{-1}} C^{0,6}(MSS_6^{'}) \xrightarrow{\delta^0} C^{1,6}(MSS_6^{'}) \xrightarrow{\delta^1} 0.$$

It is easy to see that

$$\partial_1(e_0) = p_1 - p_0,\ \ \partial_1(e_1) = p_3 - p_0,\ \ \partial_1(e_2) = p_2 - p_1,\ \ \partial_1(e_3) = p_1 - p_6,$$
$$\partial_1(e_4) = p_2 - p_5,\ \ \partial_1(e_5) = p_2 - p_3,\ \ \partial_1(e_6) = p_3 - p_4,\ \ \partial_1(e_7) = p_4 - p_7,$$
$$\partial_1(e_8) = p_5 - p_4,\ \ \partial_1(e_9) = p_5 - p_6,\ \ \partial_1(e_{10}) = p_6 - p_7,\ \ \partial_1(e_{11}) = p_0 - p_7.$$

So we find $0-$cochains

$$\delta^0 p_0^* = -e_0 - e_1 + e_{11},\ \ \delta^0 p_1^* = e_0 - e_2 + e_3,\ \ \delta^0 p_2^* = e_2 + e_4 + e_5,$$
$$\delta^0 p_3^* = e_1 - e_5 + e_6,\ \ \delta^0 p_4^* = -e_6 + e_7 - e_8,\ \ \delta^0 p_5^* = -e_4 + e_8 + e_9,$$
$$\delta^0 p_6^* = -e_3 - e_9 + e_{10},\ \ \delta^0 p_7^* = -e_7 - e_{10} - e_{11}.$$

Moreover, we get

$$\begin{aligned}\delta^0(\sum_{i=1}^{7} n_i p_i^*) &= e_0(-n_0+n_1) + e_1(-n_0+n_3) + e_2(-n_1+n_2) + e_3(n_1-n_6) \\ &+ e_4(n_2-n_5) + e_5(n_2-n_3) + e_6(n_3-n_4) + e_7(n_4-n_7) \\ &+ e_8(-n_4+n_5) + e_9(n_5-n_6) + e_{10}(n_6-n_7) + e_{11}(n_0-n_7).\end{aligned}$$

If $\delta^0(\sum_{i=1}^{7} n_i p_i^*) = 0$, then we find $n_0 = n_1 = ... = n_7 = n$. This shows that $Ker\delta^0$ is $\mathbb{Z}$. Therefore, we have that $Z^{0,6}(MSS_6^{'}) \cong \mathbb{Z}$. Since $Im\delta^{-1} = \{0\}$, we have that $B^{0,6}(MSS_6^{'}) \cong \{0\}$. Hence, we get $H^{0,6}(MSS_6^{'}) \cong \mathbb{Z}$. In addition, we have that

$$\begin{aligned}B^{1,6}(MSS_6^{'}) = Im\delta^0 &= \{t_0e_0 + t_1e_1 + t_2e_2 + t_3e_3 + t_4e_4 \\ &+ (t_2-t_1)e_5, +t_5e_6 + t_6e_7 + (-t_1+t_2-t_4+t_5)e_8 \\ &+ (t_2+t_3-t_4)e_9 + (t_0-t_1-t_3+t_5+t_6)e_{10} \\ &+ (-t_1+t_5+t_6)e_{11} \mid i = 0,1,...,6,\ \ \forall t_i \in \mathbb{Z}\} \cong \mathbb{Z}^7.\end{aligned}$$

By the definition, we see that $Z^{1,6}(MSS_6^{'}) = Ker\delta^1 \cong \mathbb{Z}^{12}$. This implies that

$$H^{1,6}(MSS_6^{'}) = Z^{1,6}(MSS_6^{'})/B^{1,6}(MSS_6^{'}) \cong \mathbb{Z}^5.$$

Finally, we get our result

$$H^{q,6}(MSS_6^{'}) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^5, & q = 1 \\ 0 & q \neq 0, 1. \end{cases}$$

□

Before ending the section, we list some results on computing the topological complexity numbers or the higher topological complexity numbers in topological spaces. Each of the results is invalid in digital images. The counterexamples of the digital images are exhibited in Section 3.

Proposition 2.9. [**36**] *Let $\Delta_n : X \to X^n$ be the diagonal map on X. If there exist $v_i \in H^*(X^n; M_i)$ for which $(\Delta_n)_* v_i = 0$ and $v_1 \smile v_2 \smile ... \smile v_k \neq 0$, for*

$i = 1, ..., k$, in the cohomology $H^*(X^n; M_1 \otimes M_2 \otimes ... \otimes M_k)$, then we have that $TC_n(X) \geqslant k+1$.

Theorem 2.10. [**22**] *Let X be a connected graph. Then*

$$TC(X) = \begin{cases} 1, & \text{if } b_1(X) = 0, \\ 2, & \text{if } b_1(X) = 1, \\ 3, & \text{if } b_1(X) \geqslant 2, \end{cases}$$

where $b_1(X)$ denotes the first Betti number of X.

Lemma 2.11. [**22**] *Let X denote the wedge of $k-$spheres S^n. Then*

$$TC(X) = \begin{cases} 2, & \text{if } k = 1 \text{ and } n \text{ is odd}, \\ 3, & \text{if either } k > 1 \text{ or } n \text{ is even}. \end{cases}$$

Example 2.12. [**23**] Let $X = \sum_g$ be a compact orientable surface of genus g. Then

$$TC(X) = \begin{cases} 3, & \text{if } g \leqslant 1 \\ 5, & \text{if } g > 1. \end{cases}$$

3. Main results

The aim of this section is to emphasize the differences between the topological spaces and the digital images in the sense of the objects of topological robotics. The results and the examples in this section wish to indicate that the studies of the topological complexity numbers are so valuable and can be transfered into the digital images to get interesting results. This does not lose the value of the studies, contrarily it enriches the studies by carrying into the different platform. The results can be instructive for the future study of the topological complexity in digital image processing.

First, we show that Proposition 2.9 is not valid in digital images.

Example 3.1. Take $n = 3$ and $X = [0,1]_{\mathbb{Z}}$ in the Proposition 2.9. Let $\mathbb{M}$ be a field. The diagonal map $\Delta_3 : (X, 2) \to (MSS_6^{'}, 6)$ induces the digital homomorphism on digital cohomology with the coefficent $\mathbb{M}$ for the first dimension

$$(\Delta_3)_* : H^{1,6}(MSS_6^{'}; \mathbb{M}) \to H^{1,2}(X; \mathbb{M}).$$

Proposition 2.8 tells us that $(\Delta_3)_*$ is the map $\mathbb{M}^5 \to 0$ of digital images. This shows that $ker((\Delta_3)_*) = \mathbb{M}^5$. For any nonzero element $(v_1, v_2, v_3, v_4, v_5)$ of $\mathbb{M}^5$, we have that $(\Delta_3)_*(v_1, v_2, v_3, v_4, v_5) = 0$. Hence, we obtain $k \geqslant 1$. On the other hand, we shall show that $\mathrm{TC}_3(X, 2) = 1$. $([0,1]_{\mathbb{Z}}, 2, \circ)$ is a $2-$topological group, where $\circ$ is defined by

$$\circ(a, b) = \begin{cases} 0, & a = b \\ 1, & a \neq b, \end{cases}$$

for all a, $b \in [0,1]_{\mathbb{Z}}$. By Theorem 2.7, $\mathrm{TC}_3(X, 2) = \mathrm{cat}_4(X^2)$. Since X^2 is $4-$contractible, we obtain $\mathrm{TC}_3(X, 2) = 1$. As a result, $\mathrm{TC}_3(X, 2) < k + 1$.

Example 3.2. Let $\mathbb{M}$ be a field. Consider the digital cohomology induced by the diagonal map Δ_3, with the coefficient M for the first dimension, given in Example 3.1. Although $(\Delta_3)_*$ is the map $\mathbb{M}^5 \to 0$, the digital cup-product homomorphism

$$\smile_3 \colon H^{1,2}(X;\mathbb{M}) \otimes_{\mathbb{M}} H^{1,2}(X;\mathbb{M}) \otimes_{\mathbb{M}} H^{1,2}(X;\mathbb{M}) \to H^{3,2}(X;\mathbb{M})$$

is the map $0 \otimes_{\mathbb{M}} 0 \otimes_{\mathbb{M}} 0 \to 0$. This shows that $Ker(\smile_3) = 0$. Hence, $Ker(\smile_3)$ does not coincidence with $Ker((\Delta_3)_*)$. Thus, we conclude that

$$nil(Ker(\smile_3)) < TC_3(X,2) < nil(Ker(\Delta_3)_*).$$

As a result of this example, Proposition 3.2 of [**25**] becomes impractical in digital images because the diagonal map Δ_n cannot be identified with the digital cup-product homomorphism $\cup_n(X)$ in Definition 3.1 of [**25**].

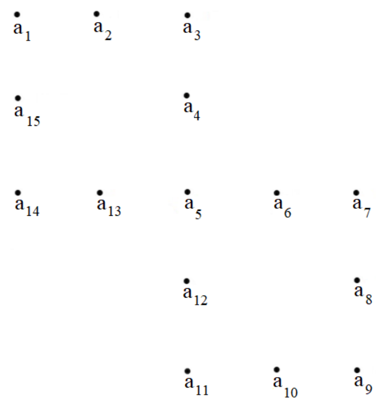

Figure 1. The digital image X.

In the next example, we show that Theorem 2.10 does not hold for digital images for the case $b_1(X) \geqslant 2$.

Example 3.3. Let X be a digital image as shown in the Figure 1. The first Betti number of X is 2 because X has two digital quadrilateral holes. On the other hand, we shall show that $\mathrm{TC}(X,4) = 2$. X is not $4-$contractible. Therefore, we have that $\mathrm{TC}(X,4) > 1$. Let $\alpha = \{a_1, a_{15}, a_{14}, a_{13}, a_5, a_6, a_7, a_8, a_9\}$ and $\beta = \{a_1, a_2, a_3, a_4, a_5, a_{12}, a_{11}, a_{10}, a_9\}$ be the subsets of X as shown in the Figure 2. $X \times X$ can be written as the union of the following sets

$$U_1 = \{(x,y) \in X \times X \ \mid \ (x,y) \in \beta \ \text{ or } \ (x,y) \in \beta \times \alpha \ \text{ or } \ (x,y) \in \alpha \times \beta\}$$

and

$$U_2 = \{(x,y) \in X \times X \ \mid \ (x,y) \in \alpha\}.$$

By the definition, the minimal number is 2 for the digital topological complexity number of X. Finally, we conclude that $\mathrm{TC}(X,4) = 2$.

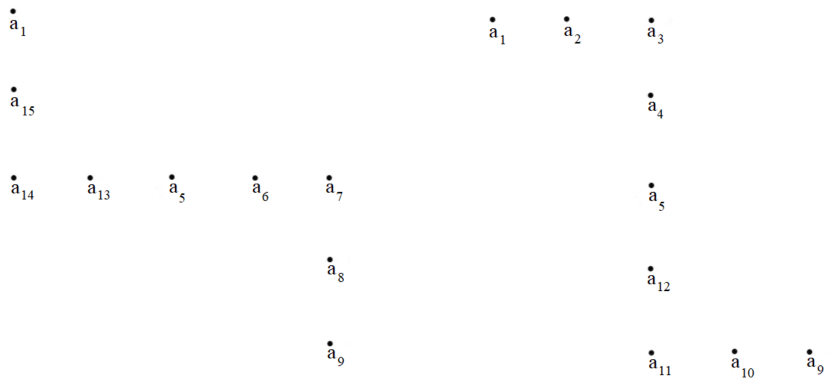

FIGURE 2. The digital images α on the left and β on the right in X.

THEOREM 3.4. *Let X be a digitally connected digital curve and $b_1(X)$ denote the first Betti number of X. Then*

$$TC(X) = \begin{cases} 1, & \text{if } b_1(X) = 0, \\ 2, & \text{if } b_1(X) \geqslant 1. \end{cases}$$

PROOF. Let X be a digitally connected digital curve. If $b_1(X) = 0$, then X has no quadrilateral holes in it. This means that X is $k-$contractible. Hence, we find that $\mathrm{TC}(X, \kappa) = 1$. If $b_1(X) = 1$, then there is only one hole in X and we cover $X \times X$ by two sets such as in Example 3.3. The digital homotopy invariance property of the digital topological complexity number concludes that $\mathrm{TC}(X, \kappa) = 2$. Consider the case $b_1(X) > 1$. Let $b_1(X) = n$, where $n > 1$. Then $X \times X$ is covered by two sets T_1 and T_2, where $T_1 = W_1 \times W_1$ and and $T_2 = W_1 \times W_2 \cup W_2 \times W_1 \cup W_2 \times W_2$ (see Figure 3 for the images W_1 and W_2). By the homotopy invariance property, we get $\mathrm{TC}(X, \kappa) = 2$, for $b_1(X) > 1$. □

Boxer defines the digital version of S^2 as the boundary of the digital $3-$cube I^3 [**6**], i.e.,

$$[-1, 1]^3_{\mathbb{Z}} \setminus \{(0, 0, 0)\} \subset \mathbb{Z}^3.$$

In general, $[-1, 1]^{n+1}_{\mathbb{Z}} \setminus \{0_{n+1}\}$, where 0_{n+1} is the origin of $\mathbb{Z}^{n+1}$, gives the digital setting of an $n-$sphere S^n.

The following example shows that Lemma 2.11 is not true in the digital setup of the topological complexity.

EXAMPLE 3.5. Take $k = 2$ and $n = 2$ in Lemma 2.11. Then we have the space $S^2 \vee S^2$. The digitally equivalent of this space is given by

$$X = [0, 2]^3_{\mathbb{Z}} \setminus \{(1, 1, 1)\} \bigvee [-2, 0]^3_{\mathbb{Z}} \setminus \{(-1, -1, -1)\}.$$

Here the wedge point is $(0, 0, 0)$. Lemma 2.11 says that $\mathrm{TC}(X, 6) = 3$. On the other hand, we find $\mathrm{TC}(X, 6) = 2$. Indeed, X is not $6-$contractible [**3**]. Therefore,

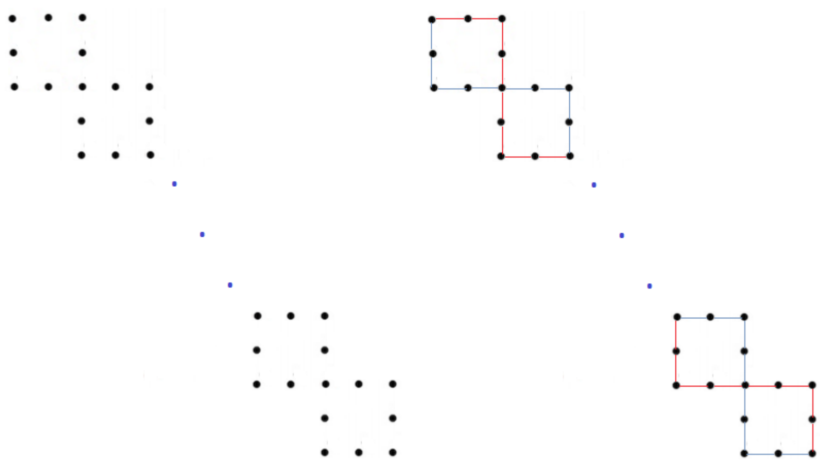

FIGURE 3. The left one is the digital image with the Betti number $n > 1$. The red and blue lines describe W_1 and W_2, respectively.

we get $\mathrm{TC}(X, 6) > 1$. Take two subsets A_1 and A_2 of X such that

$$A_1 = ([0,2]_{\mathbb{Z}} \times [0,2]_{\mathbb{Z}} \times \{0\}) \cup ([-2,0]_{\mathbb{Z}} \times [-2,0]_{\mathbb{Z}} \times \{0\})$$

and $A_2 = X \setminus A_1$. We set

$$B_1 = \{(x, y) \in X \times X : x, y \in A_1\}$$

and B_2 as the union of three subsets $\{(x, y) : x, y \in A_2\}$, $\{(x, y) : x \in A_1, y \in A_2\}$ and $\{(x, y) : x \in A_2, y \in A_1\}$ of $X \times X$. This gives us $\mathrm{TC}(X, 6) = 2$.

COROLLARY 3.6. *Let X be a digital image that consists of the wedge of r digital $n-$sphere in $\mathbb{Z}^{n+1}$ for $n \geqslant 1$. Then*

$$TC(X, c_l) = \begin{cases} 1, & \text{if } 1 < l \leqslant n+1 \\ 2, & \text{if } l = 1. \end{cases}$$

PROOF. Let $n = 1$. Homotopy invariance property of TC and Example 5.2 of [**30**] gives that $\mathrm{TC}(X, 4) = 2$. Since X is $8-$contractible [**26**], $\mathrm{TC}(X, 8) = 1$. If $n > 1$, then $\mathrm{TC}(X, c_1) = 2$ by using the method in Example 3.5. Moreover, the c_l-contractibility $(l > 1)$ of X from Proposition 3.5 of [**6**] demonstrates that $\mathrm{TC}(X, c_l) = 1$. □

EXAMPLE 3.7. Let X and Y be two digital images as shown in Figure 4. Chen shows that X is an example of the digital closed surface with genus 1 and Y is an example of the digital closed surface with genus 2 [**12**]. By Example 2.12, the digital topological complexity number of X is 3 and the topological complexity number of Y is 5. However, we shall show this is a contradiction. Take U_1 and U_2

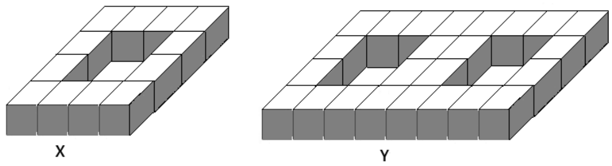

FIGURE 4. The digital images with genus 1 and genus 2, respectively.

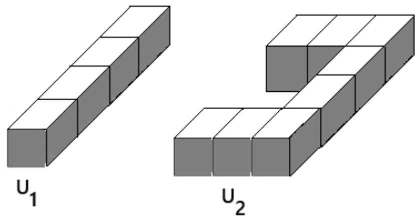

FIGURE 5. Divide the digital image X into 2 parts.

as shown in Figure 5. If we define $V_1 = \{(x, y) : x, y \in U_1\}$ and V_2 as the union of three digital sets

$$\{(x, y) : x \in U_1, y \in U_2\}, \quad \{(x, y) : x \in U_2, y \in U_1\}, \quad \{(x, y) : x, y \in U_2\},$$

then we get $\mathrm{TC}(X, 6) = 2$. Set T_1 and T_2 as the union of brown and yellow cubes, and the union of blue and yellow cubes, respectively as shown in Figure 6. If we define $W_1 = \{(x, y) : x, y \in T_1\}$ and W_2 as the union of three digital sets

$$\{(x, y) : x \in T_1, y \in T_2\}, \quad \{(x, y) : x \in T_2, y \in T_1\}, \quad \{(x, y) : x, y \in T_2\},$$

then we obtain that $\mathrm{TC}(Y, 6) = 2$.

The next result shows that Example 2.12 is not true for digital images.

COROLLARY 3.8. *Let X be a digital simple closed orientable surface of genus g with $6-$adjacency. Then $TC(X, 6) = 2$ for $g \leqslant 2$.*

PROOF. By Example 3.7, it is enough to show that $\mathrm{TC}(X, 6) = 1$ for the case $g = 0$. In the usual topological setting, a $2-$sphere S^2 is an example of a compact orientable surface. Moreover, the genus of S^2 is 0. We know that the boundary of a $3-$cube I^3 is the digital version of S^2 [**6**]. Denote this boundary as $X = [-1, 1]^3_{\mathbb{Z}} \setminus \{(0, 0, 0)\}$. Since X is not $6-$contractible, we get $\mathrm{TC}(X, 6) > 1$. Let

$$U_1 = ([-1, 1]_{\mathbb{Z}} \times [-1, 1]_{\mathbb{Z}} \times \{-1\})$$

and $U_2 = X \setminus U_1$. We set

$$V_1 = \{(x, y) \in X \times X : x, y \in U_1\}$$

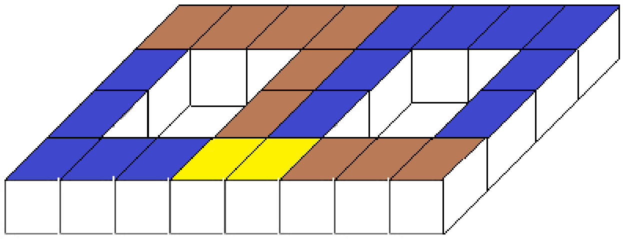

FIGURE 6. Divide the digital image Y into 2 parts: the first one consists of brown and yellow cubes, the second one consists of blue and yellow cubes.

and V_2 as the union of three subsets $\{(x,y) : x,y \in U_2\}$, $\{(x,y) : x \in U_1, y \in U_2\}$ and $\{(x,y) : x \in U_2, y \in U_1\}$ of $X \times X$. This means that $\mathrm{TC}(X,6) = 2$. Thus, using the homotopy invariance property for digital images gives the desired result. $\square$

Proposition 3.5 of [**6**] shows that $[-1,1]^3_{\mathbb{Z}} \setminus \{(0,0,0)\}$ is both 18 and 26−contractible. From this fact, we have the following result:

COROLLARY 3.9. *Let X be a digital simple closed orientable surface of genus 0. Then $TC(X,18) = 1 = TC(X,26)$.*

4. A counterexample as an application

Before stating the main theorem of this section on an application of digital topological complexity on the digital screen, we have some important results in digital images.

LEMMA 4.1. *If X is k−contractible, then $TC(X \times X, \kappa_*) = 1$, where κ_* is an adjacency relation on $X \times X$.*

PROOF. Let X be a κ−contractible digital image. Then for a positive integer r, we have a homotopy $F : X \times [0,r]_{\mathbb{Z}} \to X$ of maps for which $F(x,0) = id_X$ and $F(x,r) = c_X(x)$, where c_X is a constant map on X. Define

$$\begin{aligned} G : X \times X \times [0,r]_{\mathbb{Z}} &\longrightarrow X \times X \\ (x,y,t) &\longmapsto G(x,y,t) = (F(x,t), F(y,t)). \end{aligned}$$

Since F is digitally continuous, so is G. This is a digitally (contraction) homotopy between identity function $id_{X\times X}$ and the constant map $c_{X\times X}$ on $X \times X$. By Theorem 5.1 of [**31**], we get $\mathrm{TC}(X,\kappa) \leqslant \mathrm{cat}_{\kappa_*}(X \times X) \leqslant \mathrm{TC}(X \times X, \kappa_*)$. Corollary 3.3 and Corollary 3.7 of [**31**] say that $\mathrm{TC}(X \times X, \kappa_*) = 1$. $\square$

LEMMA 4.2. *Let (X,κ) be a connected image. If $r : X \to A$ is a digital retraction, then $TC(A,\kappa) \leqslant TC(X,\kappa)$.*

PROOF. Let $\mathrm{TC}(X,\kappa)=k$. Then we have the partition

$$X \times X = U_1 \cup U_2 \cup ...U_k$$

and for each U_i, $i=1,2,...,k$, $s_i : U_i \to X^{[0,m]_{\mathbb{Z}}}$ is digitally continuous such that $\pi \circ s_i = id_{U_i}$. Define

$$V_i = U_i \cap (A \times A)$$

for each i. Then $t_i : V_i \to A^{[0,m]_{\mathbb{Z}}}$ is clearly digitally continuous and satisfies the following:

$$\pi\big|_A \circ t_i(V_i) = (\pi \circ s_i)\big|_{V_i}(V_i) = id_{U_i}\big|_{V_i}(V_i) = id_{V_i}(V_i)$$

for each V_i. This proves that $\mathrm{TC}(A,\kappa) \leqslant k$. □

THEOREM 4.3. *Let (X,κ) and (Y,λ) be digital images such that $X \times Y$ has κ_*-adjacency. Let λ_* be an adjacency relation on $X \vee Y$. Then*

$$\max\{TC(X,\kappa), TC(Y,\lambda), cat_{\kappa_*}(X \times Y)\} \leqslant TC(X \vee Y).$$

PROOF. Let $\mathrm{TC}(X \vee Y)=k$. Since X is a digital retract of $X \vee Y$, we have a digital retraction $r_1 : X \vee Y \to X$ with $r_1(x)=x$ for all $x \in X$. Similarly, we have another retraction $r_2 : X \vee Y \to Y$ with $r_2(y)=y$ for all $y \in Y$ because of the fact that $Y \subset X \vee Y$ is a digital retract. Then by Lemma 4.2, we get $\mathrm{TC}(X,\kappa) \leqslant \mathrm{TC}(X \vee Y, \lambda_*)$ and $\mathrm{TC}(Y,\lambda) \leqslant \mathrm{TC}(X \vee Y, \lambda_*)$. Now it is enough to show that $\mathrm{cat}_{\kappa_*}(X \times Y) \leqslant k$. Assume that $\{U_1, U_2, \cdots, U_k\}$ is a covering for $X \times Y \subset (X \vee Y) \times (X \vee Y)$. For each postive integer m_i, $i=1,2,...,k$, consider the digital homotopy $F_i : U_i \times [0,m_i]_{\mathbb{Z}} \to X \vee Y$ with $F_i(x,y,0)=x$ and $F_i(x,y,m_i)=y$. Define a digital homotopy

$$H_i : U_i \times [0,m_i]_{\mathbb{Z}} \longrightarrow X \times Y$$
$$(x,y,t) \longmapsto H_i(x,y,t) = (r_1 \circ F_i(x,y,t), r_2 \circ F_i(x,y,m_i - t))$$

for each $i=1,2,...,k$. Let z_0 denote the wedge point of $X \vee Y$. Then we get

$$\begin{aligned} H_i(x,y,0) &= (r_1 \circ F_i(x,y,0), r_2 \circ F_i(x,y,m_i-0)) \\ &= (r_1(x), r_2(y)) \\ &= (x,y), \\ H_i(x,y,m_i) &= (r_1 \circ F_i(x,y,m_i), r_2 \circ F_i(x,y,m_i-m_i)) \\ &= (r_1(y), r_2(x)) = (z_0,z_0). \end{aligned}$$

This shows that for each i, U_i is digitally contractible. Consequently, we find that $\mathrm{cat}_{\kappa_*} \leqslant k$. □

Farber gives the first application to daily life problems for robotics on the robot arm example in Section 8 of [**21**]. In the digital setting of the application, we shall show that the result is depend on both the bar lengths of the robot arm and the adjacency relations. Note that the digital robot arm is made up entirely of points. We assume that the distance between any two adjacent points of the robot arm as 1. Then we consider the shortest distance between the endpoints of the arm. Thus, the bar length is denoted by such integers. For example, in Figure 7, the bar length is 1, but in Figure 8, it is 2.

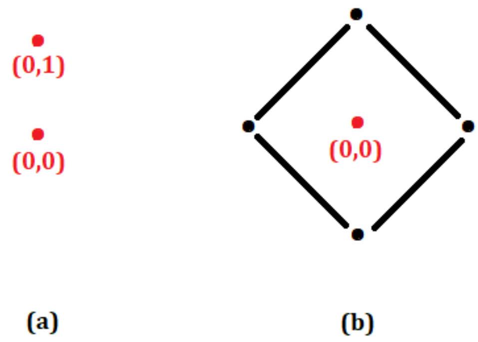

FIGURE 7. (a) The digitally connected robot arm with only one bar (consists of just two points) in $\mathbb{Z}^2$, and the length of the bar is equal to 1. (b) The digital configuration space X of the robot arm in (a).

Let L_i, $i = 1, ..., r$, denote each digital bar of the given robot arm in digital images. Here the bars consists of endpoints. Let the robot arm be digitally connected and the first bar L_1 be fixed. Assume that the length of L_i is 1. A configuration space in digital images is the digital image set of all possible states of a given system on the digital screen. Each bar can rotate 360° around its axis. Whereas we do not have a digitally connected configuration space with c_1-adjacency in $\mathbb{Z}^n$, we have a digitally connected configuration space with c_l-adjacency for $1 < l \leqslant n$. Assume that we have only one bar in the digital version of robot arm example and study in $\mathbb{Z}^2$ (see Figure 7 (a)). Then the configuration space of the digital robot arm X is given by in Figure 7 (b). This image is not $4-$connected, so TC number can not be computed. Unlike $4-$adjacency, the digital configuration space is $8-$connected. Moreover, $\text{TC}(X, 8) = 1$ by the fact that X is $8-$contractible. Note that there is no any obstacle on the screen during the motion of the robot arm. If there could be an obstacle, then the motion would be restricted. Thus, in $\mathbb{Z}^2$, a digital configuration space of a robot arm with r bars is digitally isomorphic to

$$X \times X \times \cdots \times X,$$

where X is digitally isomorphic to the image $\{(-1, 0), (0, 1), (1, 0), (0, -1)\}$ with $8-$adjacency. By Lemma 4.1, we first obtain the digital topological complexity of $X \times X$ is 1 due to the $8-$contractibility of X. By considering the general case of Lemma 4.1, we get

$$\text{TC}(X \times X \times \cdots \times X, \kappa_*) = 1,$$

where κ_* is an adjacency relation for $X \times X \times \cdots \times X$. The idea can be easily generalized for $n > 2$. Hence, the digitally continuous motion planning algorithm can always be constructed for the robot arm with c_l-adjacency for $l > 1$ on the digital screen.

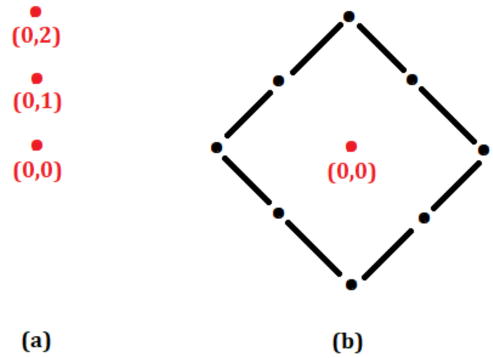

FIGURE 8. (a) The digitally connected robot arm with one bar but the length of the bar is 2 (it has three points) in $\mathbb{Z}^2$. (b) The digital configuration space of the robot arm Y given in (a).

Now assume that the length of each bar is greater than 1, i.e., each bar consists of at least three points such that the distance of endpoints in bars is at least 2. Consider the case in which the length is 2 (see Figure 8). The configuration space of the one bar robot arm $Y = \{(-2,0),(-1,1),(0,2),(1,1),(0,2),(1,-1),(0,-2),(-1,-1)\}$ is $8-$connected but not $8-$contractible. It is easy to see that $\mathrm{TC}(Y,8) = 2$ (similar construction to the Figure 5 can be considered). Similarly, $\mathrm{TC}(X,4)$ is not defined. If the length of each bar is greater than 2, the configuration space of the one bar robot arm is digitally homotopy equivalent to the case that the length of the bars is 2. Now assume that we have $r > 1$ digital bars in the robot arm X and the length of each one is greater than 1. Then the digital topological complexity of X is digitally isomorphic to the digital topological complexity of

$$Y \times Y \times \cdots \times Y.$$

By considering the similar construction to the digital image in Figure 3, we have that $\mathrm{TC}(Y \vee Y \vee \cdots \vee Y, 8) = 2$. Since $\mathrm{TC}(Y,8) = 2$, we find $\mathrm{cat}_{\kappa_*}(Y \times Y \times \cdots Y) = 2$ by using Theorem 4.3. Let the points of Y be named by rotating clockwise so that the first point is $a = (-2,0)$ and the last point is $h = (-1,-1)$. Then Y is a $8-$topological group under the group operation $\circ$ (see Table 1). The cartesian product of the digital topological groups is also a digital topological group. $\mathrm{TC}(Y \times Y \times \cdots \times Y, \kappa_*) = 2$ from Theorem 2.7. As a result, we get $\mathrm{TC}(X,8) = 2$ by using

$\circ$	a	b	c	d	e	f	g	h
a	h	a	b	c	d	e	f	g
b	a	b	c	d	e	f	g	h
c	b	c	d	e	f	g	h	a
d	c	d	e	f	g	h	a	b
e	d	e	f	g	h	a	b	c
f	e	f	g	h	a	b	c	d
g	f	g	h	a	b	c	d	e
h	g	h	a	b	c	d	e	f.

Table 1. The group operation $\circ$ for $(Y, 8)$.

digital homotopy invariant property of TC, where X is the digital configuration space of the robot arm with r digital bars for which the length of each bar is greater than 1. This method can be easily generalized for $n > 2$. Finally, we state the following main theorem:

Theorem 4.4. *Let X denote the robot arm in $\mathbb{Z}^n$ with r digital bars. If the length of each bar is m, then*

$$TC(X, c_l) = \begin{cases} 1, & m = 1 \\ 2, & m > 1 \end{cases}$$

for $1 < l \leqslant n$.

Note that, independently of the length of the bars, $\mathrm{TC}(X, c_1)$ is not defined for a digital configuration space of the robot arm because X is not c_1-connected.

5. Conclusion

The subject of robotics is widely studied in every aspect of science. Algebraic topology has put these studies on a different and effective ground by means of configuration spaces. Our purpose in the future is to use digital topology instead of the usual topology and obtain remarkable results in the development of computer sciences. The evaluation of robotics' works in digital topology is still very new. First, the differences and the similarities should be determined. Therefore, determining the differences between the topological complexity number of a topological space and the digital topological complexity number of a digital image is the main goal of this study.

We begin by considering how the relation between cohomological cup-product homomorphism and the diagonal map works for digital images. Second, we take a connected graph as a topological space such that the first Betti number of the graph is greater than 1 and we observe the result in the digital setting. Our next example is the digital interpretation of the topological complexity number of the wedge of $k-$spheres. Finally, we deal with a compact orientable surface of genus g. After giving a counterexample in digital images, we reveal how the digital topological complexity numbers of the digital simple closed surface with genus g works in digital images.

Acknowledgments

The authors would like to thank the anonymous referees for their helpful comments. This work was partially supported by the Research Fund of the Ege University with the project number FDK-2020-21123. Moreover, the first author is granted as a fellowship by the Scientific and Technological Research Council of Turkey TUBITAK-2211-A.

References

1. C. Berge, *Graphs and Hypergraphs*, 2nd ed. North-Holland, Amsterdam, 1976.
2. A. Borat and T. Vergili, *Digital Lusternik-Schnirelmann category*, Turkish Journal of Mathematics **42** (2018), 1845–1852.
3. L. Boxer, *Digitally continuous functions*, Pattern Recognition Letters **15** (1994), 833–839.
4. L. Boxer, *A classical construction for the digital fundamental group*, Journal of Mathematical Imaging and Vision **10** (1999), 51–62.
5. L. Boxer, *Properties of digital homotopy*, Journal of Mathematical Imaging and Vision **22** (2005), 19–26.
6. L. Boxer, *Homotopy properties of sphere-like digital images*, Journal of Mathematical Imaging and Vision **24** (2006), 167–175.
7. L. Boxer, *Digital products, wedges, and covering spaces*, Journal of Mathematical Imaging and Vision **25** (2006), 169–171.
8. L. Boxer and I. Karaca, *Fundamental groups for digital products*, Advances and Applications in Mathematical Sciences **11(4)** (2012), 161–180.
9. L. Boxer and P.C. Staceker, *Fundamental groups and Euler characteristics of sphere-like digital images*, Applied General Topology **17(2)** (2016), 139–158.
10. G. Burak, *Dijital Kohomoloji Grupları*, PhD thesis, Adnan Menderes University, Aydın, Turkey, 2014.
11. G. Burak and I. Karaca, *Digital Borsuk-Ulam theorem*, Bulletin of the Iranian Mathematical Society **43** (2017), 477–499.
12. L. Chen, *Discrete surfaces and manifolds: A theory of digital-discrete geometry and topology*, Scientific & Practical Computing, Rockville, MD, 2004.
13. L. Chen and Y. Rong, *Digital topological method for computing genus and the Betti numbers*, Topology and its Applications **157(12)** (2010), 1931–1936.
14. D. Davis, *The symmetrized topological complexity of the circle*, New York Journal of Mathematics **23** (2017), 593–602.
15. D. Davis, *An approach to the topological complexity of the Klein Bottle*, Morfismos **21** (2017), 1–13.
16. A. Dranishnikov, *Topological complexity of wedges and covering maps*, Proceedings of the American Mathematical Society **142(12)** (2014), 4365–4376.
17. A. Dranishnikov, *On topological complexity of non-orientable surfaces*, Topology and its Applications **232** (2017), 61–69.
18. A. Dranishnikov and R. Sadykov, *On the LS-category and topological complexity of a connected sum*, Proceedings of the American Mathematical Society **147(5)** (2019), 2235–2244.
19. O. Ege and I. Karaca, *Cohomology theory for digital images*, Romanian Journal of Information Science and Technology **16(1)** (2013), 10–28.
20. O. Ege and I. Karaca, *Digital fibrations*, The Proceedings of the National Academy of Sciences, India, Section A: Pyhsical Sciences **87(1)** (2017), 109–114.
21. M. Farber, *Topological complexity of motion planning*, Discrete and Computational Geometry **29** (2003), 211–221.

22. M. Farber, *Instabilities of Robot Motion*, Topology and its Applications **140** (2004), 245–266.
23. M. Farber, *Topology of Robot Motion Planning*, in: Paul Biran, Octav Cornea, Francois Lalonde (ed.), *'Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology'*, Springer, 2006, 185–230.
24. M. Farber, *Invitation to Topological Robotics*, Zurich Lectures in Advanced Mathematics, EMS, 2008.
25. M. Grant, G. Lupton, and J. Oprea, *Spaces of topological complexity one*, Homology Homotopy and Applications **15(2)** (2013), 73–81.
26. S.E. Han, *Non-Product property of the digital fundamental group*, Information Sciences **171** (2005), 73–91.
27. S.E. Han, *Digital fundamental group and Euler characteristic of a connected sum of digital closed surfaces*, Information Sciences **177** (2007), 3314–3326.
28. G.T. Herman, *Oriented surfaces in digital spaces*, CVGIP: Graphical models and image processing **55** (1993), 381–396.
29. M. Is and I. Karaca, *The higher topological complexity in digital images*, Applied General Topology **21** (2020), 305–325.
30. M. Is and I. Karaca, *Certain topological methods for computing higher topological complexity in digital images*, Preprint arXiv:2103.00468 [math.AT].
31. I. Karaca and M. Is, *Digital topological complexity numbers*, Turkish Journal of Mathematics **42(6)** (2018), 3173–3181.
32. D.G. Morgenthaler and A. Rosenfeld, *Surfaces in three-dimensional images*, Informational Control **51** (1981), 227–247.
33. J.F. Peters, *Topology of Digital Images*, Visual Pattern Discovery In Proximity Spaces(Intelligent Systems Reference Library **63**), Springer, 2014.
34. A. Rosenfeld, *Digital topology*, American Mathematical Monthly **86** (1979), 76–87.
35. A. Rosenfeld, *Connectivity in digital pictures*, Journal of the ACM **17** (1970), 146–160.
36. Y. Rudyak, *On higher analogs of topological complexity*, Topology and Its Applications **157(5)** (2010), 916–920.

Received by editors 28.1.2022; Revised version 17.5.2022; Available online 30.5.2022.

Melih Is, Ege University, Faculty of Sciences, Department of Mathematics, Izmir, Turkey

Email address: melih.is@ege.edu.tr

Ismet Karaca, Ege University, Faculty of Science, Department of Mathematics, Izmir, Turkey

Email address: ismet.karaca@ege.edu.tr