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ON INNER PRODUCT OVER A LATTICE VECTOR SPACE

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ABSTRACT. This paper introduces the concept of lattice valued inner product space. A natural inner product structure for the space of n-tuples over Boolean algebras is generalised to n-tuples over a distributive lattice with 0 and 1. A dimension theorem for orthonormal basis of a Boolean vector space is generalised to a lattice vector space. Further, we prove that for any naturals m, n with $m < n$, any set of m orthogonal unit ortho vectors in a lattice vector space can be extended to an orthonormal basis (known in the Boolean algebra case).

1. Introduction

As lattice matrices become useful tools in various domains like the theory of switching circuits, graph theory, fuzzy systems, theoretical computer science, automata theory, optimizations etc, it is natural to study the lattice vector spaces in connection with lattice matrices. In this way the concept of a lattice vector space (or vector space over a distributive lattice) is introduced by G. Joy and K. V. Thomas in [**2**].

In comparison to usual linear algebra over fields, in this paper the notion of inner product space of a Boolean vector space [**4**] is extended to inner product space of a lattice vector space. As analogues to the Boolean-valued norm and orthogonality relations for Boolean vectors [**4**], the lattice valued norm and orthogonality relations for lattice vectors are introduced and studied.

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An important concept in our work is that unit ortho vector. These are lattice vectors (vectors whose components are lattice elements) of norm one whose components are mutually disjoint. We define an orthonormal basis of $V_n(L)$ in the usual way and it turns out that it must be made of unit ortho vectors. Our first main result is that all ortho normal bases for $V_n(L)$ have cardinality n and conversely, any orthonormal set of unit ortho vectors with cardinality n is a basis for $V_n(L)$. Secondly, any orthonormal set of unit ortho vectors in $V_n(L)$ can be extended to an orthonormal bases for $V_n(L)$.

2. Preliminaries

Throughout this paper $\mathbb{N}$ denotes the set of non zero natural numbers, we denote $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ etc as vectors and a, b, c, α, β etc as scalars and also zero vector as $\mathbf{0}$ and the vector $(1, 1, \ldots, 1)$ as $\mathbf{1}$.

We recall some basic definitions and results on lattice theory, lattice matrices and lattice vector spaces which will be used in our sequel. For details see [**1, 2, 3**].

A partially ordered set $(L, \leqslant)$ is a lattice if for all $a, b \in L$, the least upper bound of a, b and the greatest lower bound of a, b exist in L. For any $a, b \in L$, the least upper bound is denoted by $a \vee b$ and the greatest lower bound is denoted by $a \wedge b$ (or ab), respectively. An element $a \in L$ is called greatest element of L if $\alpha \leqslant a$, for all $\alpha \in L$. An element $b \in L$ is called least element of L if $b \leqslant \alpha$, forall $\alpha \in L$. We use 1 and 0 to denote the greatest element and the least element of L, respectively. A lattice L is a distributive lattice, if for any $a, b, c \in L$,

(1) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ hold.

Throughout this paper, unless otherwise stated, we assume that L is a distributive lattice with the greatest element 1 and the least element 0.

A lattice vector space V over L (or simply a lattice vector space) is a system $(V, L, +, \cdot)$, where V is a non-empty set, L is a distributive lattice with 1 and 0, '+' is a binary operation on V called addition and '$\cdot$' is a map

$$\cdot : L \times V \ni (a, \mathbf{x}) \mapsto a \cdot \mathbf{x} \in V,$$

called scalar multiplication such that the following properties hold:

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$,
(2) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$,
(3) there is an element $\mathbf{0}$ in V such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$,
(4) $\mathbf{x} + \mathbf{y} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{y} = \mathbf{0}$,
(5) $a \cdot (\mathbf{x} + \mathbf{y}) = a \cdot \mathbf{x} + a \cdot \mathbf{y}$,
(6) $(a \vee b) \cdot \mathbf{x} = a \cdot \mathbf{x} + b \cdot \mathbf{x}$,
(7) $(ab)\mathbf{x} = a \cdot (b \cdot \mathbf{x})$,
(8) $1 \cdot \mathbf{x} = \mathbf{x}$,
(9) $0 \cdot \mathbf{x} = \mathbf{0}$,

for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $a, b \in L$.

Throughout this paper V will denote an arbitrary vector space over a lattice L, $V_n(L)$ will denote the set of all n-tuples of elements of a lattice L.

Let V be a lattice vector space and W be a subset of V. Then W is a lattice vector subspace (or L subspace) of V, if W is itself a lattice vector space with the same operations in V.

The L sub space $W = \{\mathbf{0}\}$ is called the trivial L subspace of V. All L sub spaces of V other than V are call proper L subspaces of V.

Let V be a lattice vector space and S be a non-empty subset of V. Then a vector $\mathbf{y} \in V$ is a linear combination of the vectors in S, if there exists elements $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n \in \mathsf{S}$ and $a_1, a_2, \ldots, a_n \in L$ such that $\mathbf{y} = \sum_{i=1}^{n} a_i \mathbf{y}_i$.
We define any linear combination of the empty set of vectors to be $\mathbf{0}$. Let V be an lattice vector space and S be a subset of V. Then the span of S in V is the set of all linear combinations of the vectors in S. We use the notations span(S) to denote span of S in V. Here we remark that span of empty set is equal to $\{0\}$.

Let V be a lattice vector space and S be a non-empty subset of $V - \{\mathbf{0}\}$. Then S is said to be linearly independent, if every vector $\mathbf{u} \in$ span(S) can be expressed uniquely as a linear combination of the elements of S. Otherwise, the set S is said to be linearly dependent. Here uniqueness is in the following sence : for any $a_i, b_j \in L - \{0\}$ and $\mathbf{x}_i, \mathbf{y}_j \in \mathsf{S}$ with i =1,2,..., m and j = 1,2,..., k such that $\sum_{i=1}^{m}(a_i\mathbf{x}_i) = \sum_{j=1}^{k}(b_j\mathbf{y}_j)$, we have:
$m = k$, $\{a_1, a_2, \ldots, a_m\} = \{b_1, b_2, \ldots, b_k\}$ and $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$.

Let V be a lattice vector space and S be a subset of V. Then S is said to be a basis for V, if S is a spanning subset of V and S is linearly independent. Here onwards we shall call elements of a basis as basis vectors.

3. The Inner product space $V_n(L)$

In this section, we extend the concept of inner product over a Boolean algebra [4] to inner product over a lattice vector space and we show that if L is a finite distributive lattice with 0 and 1, then any orthonormal basis for $V_n(L)$, the lattice vector space of n - tuples over L, has cardinality n.

Definition 3.1. Let V be a lattice vector space. An inner product on V is a function $<,>: V \times V \to L$ which satisfy

(1) $< \mathbf{x}, \mathbf{x} > \geqslant 0$, for all $\mathbf{x} \in V$ and $< \mathbf{x}, \mathbf{x} > = 0$ iff $\mathbf{x} = 0_V$;
(2) $< \mathbf{x} + \mathbf{y}, \mathbf{z} > = < \mathbf{x}, \mathbf{z} > \vee < \mathbf{y}, \mathbf{z} >$, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$;
(3) $< \alpha\mathbf{x}, \mathbf{y} > = \alpha < \mathbf{x}, \mathbf{y} >$, for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in L$;
(4) $< \mathbf{y}, \mathbf{x} > = < \mathbf{x}, \mathbf{y} >$, for all $\mathbf{x}, \mathbf{y} \in V$.

Definition 3.2. A lattice vector space V together with an inner product $<,>$ is called a lattice inner product space and is denoted by $(V, <,>)$.

Example 3.1. Let $V = V_n(L) = \{\mathbf{x} \in V | \mathbf{x} = (x_1, x_2, \ldots x_n)\}$ be the set of all n-tuples. By defining "+" on V as

$$(x_1, x_2, \ldots x_n) + (y_1, y_2, \ldots y_n) = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$$

and scalar multiplication "$\cdot$" on V as

$$a(x_1, x_2, \ldots x_n) = (ax_1, ax_2, \ldots, ax_n),$$

$V_n(L)$ forms a vector space over L. For any $\mathbf{x}, \mathbf{y} \in V$, defining by $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^T\mathbf{y}$ or $< \mathbf{x}, \mathbf{y} >= \mathbf{y}^T\mathbf{x}$, the structure $(V_n(L), <, >)$ forms a lattice inner product space.

EXAMPLE 3.2. Let $V = M_n(L) = \{(a_{ij})_{n\times n} \mid a_{ij} \in L\}$ be the set of all $n \times n$ matrices. By defining "+" on V as

$$(a_{ij})_{n\times n} + (b_{ij})_{n\times n} = (a_{ij} \vee b_{ij})_{n\times n}$$

and scalar multiplication "$\cdot$" on V as

$$\alpha(a_{ij})_{n\times n} = (\alpha a_{ij})_{n\times n},$$

the structure $M_n(L)$ forms a lattice vector space over L. For any $\mathbf{A}, \mathbf{B} \in V$, let us define $< \mathbf{A}, \mathbf{B} > = Tr(\mathbf{A}\mathbf{B}^T)$ or $< \mathbf{A}, \mathbf{B} > = Tr(\mathbf{B}\mathbf{A}^T)$. Then the system $(M_n(L), <, >)$ forms an inner product space.

DEFINITION 3.3. A lattice vector space V over a distributive lattice L with 0 and 1 is said to be a normed vector space if there exist a mapping $\| \cdot \| : V \to L$ satisfying:

(1) $\|\mathbf{x}\| \geqslant 0$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$ and
(2) $\|a\mathbf{x}\| = a\|\mathbf{x}\|$ for all $\mathbf{x} \in V$ and $a \in L$.

PROPOSITION 3.1. *A lattice inner product space becomes a normed vector space by defining* $\|\mathbf{x}\| = < \mathbf{x}, \mathbf{x} >$

PROOF. Let V be a lattice inner product space. In order to prove V is normed, for any $\mathbf{x} \in V$ and $a \in L$, we need to show

(i) $\|\mathbf{x}\| \geqslant 0$, for all $\mathbf{x} \in V$ and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$ and
(ii). $\|a\mathbf{x}\| = a\|\mathbf{x}\|$, for all $a \in L$ and for all $\mathbf{x} \in V$.

(i) follows from the definition of inner product.

(ii) For $a \in L$, $\mathbf{x} \in V$, we have

$$\begin{aligned}\|a\mathbf{x}\| &=< a\mathbf{x}, a\mathbf{x} > \\ &= (a \wedge a) < \mathbf{x}, \mathbf{x} > \\ &= a < \mathbf{x}, \mathbf{x} > \\ &= a\|\mathbf{x}\|.\end{aligned}$$

□

REMARK 3.1. By Proposition 3.1, Example 3.1 and Example 3.2 are normed lattice vector spaces.

DEFINITION 3.4. Let V be an inner product space and $\mathbf{x}, \mathbf{y} \in V$. Then $\mathbf{x}$ is said to be orthogonal (or perpendicular) to $\mathbf{y}$ if $< \mathbf{x}, \mathbf{y} > = 0$.

DEFINITION 3.5. Let V be a normed lattice vector space. An element $\mathbf{x} \in V$ is said to be a unit vector if $\|\mathbf{x}\| = 1$.

DEFINITION 3.6. A subset M of V is called an orthogonal set if distinct elements in M are mutually orthogonal.

DEFINITION 3.7. A subset M of V is called an orthonormal set if M is orthogonal and each vector in M has norm 1.

DEFINITION 3.8. A vector $\mathbf{x} = (x_1, x_2, \ldots x_n)$ in $V_n(L)$ is said to be an orthovector if $x_i x_j = 0$, for all $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$.

We now give some properties of the norm in the normed space $V_n(L)$.

THEOREM 3.1. *Let* $\mathbf{x}, \mathbf{y} \in V_n(L)$. *Then*
(i) $\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x}\| \vee \|\mathbf{y}\|$.
(ii) $< \mathbf{x}, \mathbf{y} > \leqslant \|\mathbf{x}\|\|\mathbf{y}\|$.
(iii) *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are orthovectors and* $\|\mathbf{x}\| = \|\mathbf{y}\|$ *then* $< \mathbf{x}, \mathbf{y} > = \|\mathbf{x}\|\|\mathbf{y}\|$ *if and only if* $\mathbf{x} = \mathbf{y}$.

PROOF. (i). Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be any two vectors in $V_n(L)$. Then we have

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\| &= \vee_{i=1}^{n} (x_i \vee y_i) \\ &= (\vee_{i=1}^{n} x_i) \vee (\vee_{i=1}^{n} y_i) \\ &= \|\mathbf{x}\| \vee \|\mathbf{y}\| \end{aligned}$$

(ii). Consider

$$\begin{aligned} < \mathbf{x}, \mathbf{y} > &= \vee_{i=1}^{n} x_i y_i \\ &\leqslant \vee_{i,j=1}^{n} x_i y_j \\ &= (\vee_{i=1}^{n} x_i)(\vee_{i=1}^{n} y_i) \\ &= \|\mathbf{x}\|\|\mathbf{y}\|. \end{aligned}$$

(iii). Let $\mathbf{x}$ and $\mathbf{y}$ be orthovectors with $\|\mathbf{x}\|$=$\|\mathbf{y}\|$ and suppose that $< \mathbf{x}, \mathbf{y} >$ $= \|\mathbf{x}\|\|\|\mathbf{y}\|$. Then we have $< \mathbf{x}, \mathbf{y} > = \|\mathbf{x}\|$ and $< \mathbf{x}, \mathbf{y} > = \|\mathbf{y}\|$. Considering $< \mathbf{x}, \mathbf{y} > = \|\mathbf{x}\|$ we have $\vee_{i=1}^{n} x_i y_i = \vee_{i=1}^{n} x_i$. Now multiplying this equation with x_j on both sides we obtain $x_j y_j = x_j$, for all j. Similarly, by considering $< \mathbf{x}, \mathbf{y} >= \|\mathbf{y}\|$, we obtain $x_j y_j = y_j$ for all j. Hence $\mathbf{x} = \mathbf{y}$. □

REMARK 3.2. Note that the condition $\|\mathbf{x}\| = \|\mathbf{y}\|$ in the last statement of the Theorem 3.1 is necessary. If we let $\mathbf{x} = (x_1, 0, 0, \ldots, 0)$ and $\mathbf{y} = (y_1, 0, 0, \ldots, 0)$ with $x_1, y_1 \in L$ and $x_1 \neq y_1$ then $\mathbf{x}, \mathbf{y}$ are orthovectors of different norms and we can see that $< \mathbf{x}, \mathbf{y} > = \|\mathbf{x}\|\|\mathbf{y}\|$. Also the condition that $\mathbf{x}, \mathbf{y}$ are ortho vectors is necessary since $\mathbf{x} = (1, x_1)$ and $\mathbf{y} = (1, y_1)$, for $x_1, y_1 \in L$ with $x_1 \neq y_1$ then $\|\mathbf{x}\| = \|\mathbf{y}\|$=1 and $< \mathbf{x}, \mathbf{y} > = 1$.

The following corollary is an immediate consequence of Theorem 3.1.

COROLLARY 3.1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are unit orthovectors then* $< \mathbf{x}, \mathbf{y} >$ *=1 if and only if* $\mathbf{x} = \mathbf{y}$.

Towards the main result we now see that a basis must be made of unit vectors.

LEMMA 3.1. *The basis vectors of* $V_n(L)$ *are unit vectors.*

PROOF. Let S be a basis of $V_n(L)$. We first see that S does not contain $\mathbf{0}$, zero element of $V_n(L)$. Suppose $\mathbf{0} \in S$. Since $\mathbf{1}\in V_n(L)$ and S is a basis of $V_n(L)$. We have $\mathbf{1}=\sum_{i=1}^{n} a_i\mathbf{x}_i$, $a_i \in L, \mathbf{x}_i \in S$. If $a_1, a_2, \ldots, a_n$ are all distinct from 0, then $\mathbf{1}$ can also be written as linear combination of $\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. In any case $\mathbf{1}$ has two distinct linear combinations of elements of S. Which is a contradiction. Hence S does not contain $\mathbf{0}$.

Now let $\mathbf{u} \in \mathsf{S}$ then $\mathbf{u} = 1.\mathbf{u}$ which is a linear combination of elements of S. On the other hand $\|\mathbf{u}\|\mathbf{u} =< \mathbf{u}, \mathbf{u} > \mathbf{u} = (\vee_{i=1}^{n} u_i)\mathbf{u} = \mathbf{u}$. If $\|\mathbf{u}\| \neq 1$, then $1.\mathbf{u}$ and $\|\mathbf{u}\|\mathbf{u}$ are two distinct linear combinations for $\mathbf{u}$, a contradiction to the fact that S is a basis. Hence $\|\mathbf{u}\| = 1$. □

LEMMA 3.2. *An orthonormal set in $V_n(L)$ is a linearly independent set.*

PROOF. Let S be an orthonormal set in $V_n(L)$. To see that every element in $V_n(L)$ can be uniquely written as a linear combination of elements of S. For this, suppose $\mathbf{e} \in V_n(L)$ and $\mathbf{e} = \sum_{i=1}^{n} a_i\mathbf{x}_i = \sum_{i=1}^{k} b_j\mathbf{y}_j$, where $\mathbf{x}_i, \mathbf{y}_j \in \mathsf{S}, a_i, b_j \in L - \{0\}$, i=1 … n, j=1… k. If possible, for some j, $y_j \notin \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, $j \in \{1, 2, \ldots, k\}$. We have $b_j = < \mathbf{e}, \mathbf{y}_j >$, j= 1… k. For any j, consider $b_j =< \mathbf{e}, \mathbf{y}_j >=< \mathbf{y}_j, \mathbf{e} >=< \mathbf{y}_j, \sum_{i=1}^{m} a_i\mathbf{x}_i >= \vee_{i=1}^{m} a_i < \mathbf{y}_j, \mathbf{x}_i > = 0$. But for all j, b_j is non-zero. A contradiction arises. Hence $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ is contained in $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$. The reverse inclusion is obtained by symmetry. Then for each $i = 1 \ldots n$ there exist $j \in \{1, 2, \ldots, k\}$ such that $a_i =< \mathbf{e}, \mathbf{x}_i >=< \mathbf{e}, \mathbf{y}_j >= b_j$. which concludes the proof. □

We now define the following:

DEFINITION 3.9. A subset S of $V_n(L)$ is said to be an orthonormal basis of $V_n(L)$ if S is an orthonormal and spanning subset of $V_n(L)$.

REMARK 3.3. By Lemma 3.2, it follows that an orthonormal basis is a spanning set and linearly independent. Hence an orthonormal basis is a basis of $V_n(L)$. In general lattice vector spaces the converse is not true i.e., a basis of a lattice vector space need not be an orthonormal basis. For example, consider $L = \{0, a, b, c, d, 1\}$ where the Hasse diagram of L is shown below:

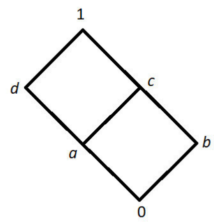

FIGURE 1

Now consider the lattice vector space

$$V = \{\alpha(a, b, d) + \beta(0, d, c) + \gamma(0, 0, b) : \alpha, \beta, \gamma \in L\}.$$

For this Lattice vector space, the set $B = \{(a,b,d),\ (0,d,c),\ (0,0,b)\}$ forms a basis but not orthogonal.

REMARK 3.4. For any natural n, $V_n(L)$ has at least one orthonormal basis. For example (see [**4**]), the canonical (or) standard basis of $V_n(L)$ is defined as the basis $\{\delta_1, \delta_2, \dots, \delta_n\}$ with $\delta_1 = (1, 0, \dots, 0)$, $\delta_2 = (0, 1, \dots, 0)$, …, $\delta_n = (0, 0, \dots, 1)$.

One more example is the following:

EXAMPLE 3.3. ([**4**]) Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be a unit orthovector. Let $\mathbf{e}_i = (x_i, x_{i+1}, \dots, x_n, x_1, x_2, \dots, x_{i-1})$ for all $i = 1, 2, \dots, n$. Then $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is an orthonormal subset of $V_n(L)$. Further, for each i the standerd basis vector δ_i can be written as follows: $\delta_i = x_i\mathbf{e}_1 + x_{i+1}\mathbf{e}_2 + \ldots + x_{i-1}\mathbf{e}_n$. Therefore,$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a spanning set and hence an orthonormal basis for $V_n(L)$.

REMARK 3.5. Linear independence in $V_n(L)$ is not an easy concept. Instead the concept of orthogonality can be used as a substitute. In fact, latter on, we shall see that the concept of a basis, i.e.,a linearly independent spanning subset, is identical to the concept of orthonormal basis.

The natural maps for our structure are:

DEFINITION 3.10. A mapping $T : V \to U$ of a lattice vector space V into a lattice vector space U is said to be linear if

$$T(a\mathbf{x} + \mathbf{y}) = aT(\mathbf{x}) + T(\mathbf{y}),$$

for all $a \in L$ and $\mathbf{x}, \mathbf{y} \in V$.

NOTE 3.2. If T is linear, then $T(\mathbf{0}) = \mathbf{0}$.

DEFINITION 3.11. A mapping $T : V \to U$ of a lattice vector space V into a lattice vector space U is said to be a linear isomorphism if T is linear and a bijection.

DEFINITION 3.12. A linear mapping $T : V \to V$ on a lattice vector space V is said to be a linear operator on V.

DEFINITION 3.13. A linear mapping $T : V \to U$ of a lattice vector space V into a lattice vector space U is said to be invertible if there is a linear mapping $S : U \to V$ such that $T \circ S = I_U$ and $S \circ T = I_V$, where $\circ$ is the composition of mappings and I_V, I_U are identity operators on V, U respectively.

NOTE 3.3. A linear mapping $T : V \to U$ on a lattice vector space V into a lattice vector space U has inverse if and only if T is a bijection. For any invertible linear mapping the inverse mapping is a unique operator which is denoted by T^{-1}.

We shall denote the direct product of n copies of L by L^n and it is a distributive lattice with 0 and 1. The elements of L^n are the same as the elements of $V_n(L)$, but the algebraic structures are different in fact L^n is a lattice and $V_n(L)$ is a lattice vector space.

LEMMA 3.3. *If $T : V_n(L) \to V_m(L)$ is a linear isomorphism between lattice vector spaces, then $T : L^n \to L^m$ is an isomorphism between lattices.*

Proof. Suppose $T : V_n(L) \to V_m(L)$ is a linear isomorphism then $T^{-1} : V_m(L) \to V_n(L)$ is also a linear mapping. As sets we have $V_k(L) = L^k$, for any natural k. We now show that $T : L^n \to L^m$ is an isomorphism between lattices. For this, we first see that T preserve the supremum in L^n. For, let $\mathbf{x}, \mathbf{y} \in L^n$. Since T is linear we have $T(\mathbf{x}+\mathbf{y}) = T(1 \cdot \mathbf{x}+\mathbf{y}) = 1 \cdot T(\mathbf{x})+T(\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y})$ but "+" in $V_n(L)$ is same as "$\vee$" in L^n. Therefore $T(\mathbf{x} \vee \mathbf{y}) = T(\mathbf{x}) \vee T(\mathbf{y})$. Since T^{-1} is also linear operator with the same argument we have $T^{-1}(\mathbf{x} \vee \mathbf{y}) = T^{-1}(\mathbf{x}) \vee T^{-1}(\mathbf{y})$. Secondly, we see that any operator S on $V_n(L)$ preserves the order on L^n. For, $\mathbf{u}, \mathbf{v} \in L^n$ and $\mathbf{u} \leqslant \mathbf{v}$ implies $\mathbf{u} \vee \mathbf{v} = \mathbf{v}$. Now $S(\mathbf{u}) \vee S(\mathbf{v}) = S(\mathbf{u} \vee \mathbf{v}) = S(\mathbf{v})$ consequently $S(\mathbf{u}) \leqslant S(\mathbf{v})$. Therefore any linear operator preserves the order on L^n. From these facts we have $\mathbf{x} \leqslant \mathbf{y}$ iff $T(\mathbf{x}) \leqslant T(\mathbf{y})$. Also, we can easily see that T preserves the infimum and $T(\mathbf{0}) = \mathbf{0}$, $T(\mathbf{1}) = \mathbf{1}$. Therefore T is a lattice homomorphism. Thus $T : L^n \to L^m$ is an isomorphism. □

Remark 3.6. The converse of the Lemma 3.3 does not hold, namely if $T : L^n \to L^m$ is a lattice automorphism, $T : V_n(L) \to V_m(L)$ need not be linear. For example consider the lattice $L = \{0, a, b, 1\}$ whose diagrammatical representation is as follows:

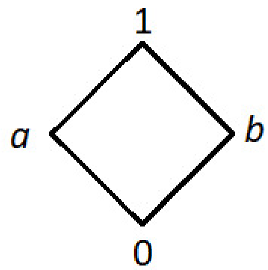

Figure 2

Consider the lattice L^2 (The direct product of L and L) Define $S : L \to L$ by $S(0) = 0$, $S(a) = b$, $S(b) = a$, $S(1) = 1$. Then $S : L \to L$ is an automorphism on L. By defining $T : L^2 \to L^2$ as $T(\alpha, \beta) = (S(\alpha), S(\beta))$, for all $\alpha, \beta \in L$, T is an automorphism on L^2 and T is not linear on $V_2(L)$. For, consider $T(a(1, 0)) = T(a, 0) = (S(a),\ S(0)) = (b, 0)$ and $aT(1, 0) = a(S(1),\ S(0)) = a(1, 0) = (a, 0)$. Therefore $T(a(1, 0)) \neq aT(1, 0)$. Hence T is not linear.

Theorem 3.4. *If L is a finite distributive lattice with 0 and 1 then any orthonormal basis for $V_n(L)$ has cardinality n.*

Proof. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m\}$ be an orthonormal basis for $V_n(L)$. Define $T : V_n(L) \to V_m(L)$ by $T(\mathbf{x}) = (< \mathbf{x}, \mathbf{e}_1 >, < \mathbf{x}, \mathbf{e}_2 >, \dots, < \mathbf{x}, \mathbf{e}_m >)$. Then T is linear and bijective from $V_n(L)$ onto $V_m(L)$. By Lemma 3.3 we have $T : L^n \to L^m$ is an isomorphism between lattices. Hence $m = n$ since L is finite. □

Remark 3.7. We shall show in the next section that this results holds for any distributive lattice L with 0 and 1

Definition 3.14. An orhonormal subset S of a lattice vector space V is said to be a unit ortho set, if all of its elements are unit orthovectors.

4. The Dimension Theorem

In this section, we show that any orthonormal basis of $V_n(L)$ has cardinality n. Conversely we show that any unit ortho orthogonal subset with cardinality n is a basis for $V_n(L)$.

We shall use the following notations: Given a subset $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ of m vectors in $V_n(L)$, we use the notation $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{nj})$ with $x_{ij} \in L$, $i = 1$ to n and $j = 1$ to m. Thus a set $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ can be regarded as a matrix $[x_{ij}]_{n\times m}$ whose columns are the elements of the set. We shall denote this matrix by $\mathbf{S}$.

We first prove that orthonormal bases possess a duality property.

Theorem 4.1. *Let $\mathsf{S}= \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be an orthonormal subset of $V_n(L)$. Then S is an orthonormal basis for $V_n(L)$ if and only if the set S^* of columns of $(x_{ji})_{m\times n}$ is an orthonormal subset of $V_m(L)$.*

Proof. Suppose $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be an orthonormal basis of $V_n(L)$, where $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{nj})$ with $x_{ij} \in L$, $i = 1 \ldots n$ and $j = 1 \ldots m$. Let $\mathsf{S}^* = \{\mathbf{x}^*_1, \mathbf{x}^*_2, \ldots, \mathbf{x}^*_n\}$, where $\mathbf{x}^*_i = (x_{i1}, x_{i2}, \ldots, x_{im})$, for $i = 1 \ldots n$. We will show that the set $\mathsf{S}^* = \{\mathbf{x}^*_1, \mathbf{x}^*_2, \ldots, \mathbf{x}^*_n\}$ is an orthonormal subset of $V_m(L)$. Since $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ is an orthonormal basis of $V_n(L)$ there exist elements a_1, a_2, …, $a_m \in L$ such that $\delta_1 = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_m\mathbf{x}_m$, where $\delta_1 = (1, 0, 0, \ldots, 0)$. which implies for $i \neq 1$ we have $\vee_{i=1}^{m} a_j x_{ij}$ =0, consequently for $i \neq 1$, $a_j x_{ij} = 0$, $j = 1 \ldots m$. On the other hand, $\vee_{i=1}^{m} a_j x_{1j} = 1$ and $x_{1k}x_{1j} = 0$ for $j \neq k$, so that we have $a_j x_{1j} = x_{1j}$, for all $j = 1 \ldots m$. It follows that $\vee_{i=1}^{m} a_j x_{1j} = \vee_{i=1}^{m} x_{1j} = 1$. Again from the established fact for $i \neq 1$, $a_j x_{ij} = 0$ we have $a_j x_{1j} x_{ij} = 0$,$i \neq 1$. Thus $x_{1j}x_{ij} = 0$,$i \neq 1$. Now by replacing δ_1 with δ_k for $k \in \{2, \ldots, n\}$ we can observe similarly that $\vee_{i=1}^{m} x_{kj} = 1$ and $x_{kj}x_{ij} = 0$, $i \neq k$, for all $j = 1 \ldots m$. Hence, the set of columns of $(x_{ji})_{m\times n}$ is an orthonormal subset of $V_m(L)$.

Conversely, suppose S^* is an orthonormal subset of $V_m(L)$. We prove that S is an orthonormal basis for $V_n(L)$. For this it is enough to prove that S is spanning subset of $V_n(L)$. Since S^* is an orthonormal subset of $V_m(L)$ we have $\vee_{i=1}^{m} x_{ij} = 1$ and $x_{kj}x_{ij} = 0$,$i \neq k$, for all $j = 1 \ldots m$ which gives $\vee_{i=1}^{m} x_{ij}x_{kj} = \delta_{ik}$, for all $i, k =$ 1,2, … n. where δ_{ik}= 1 if $i = k$ and 0 if $i \neq k$ which further gives $\delta_k = \vee_{j=1}^{m} x_{kj}x_j$, for all $k = 1 \ldots n$. Since δ_k's are standard basis vectors of $V_n(L)$, x_j's are basis vectors for $V_n(L)$. Consequently S is a spanning subset of $V_n(L)$. Therefore S is orthonormal basis for $V_n(L)$. □

The following corollary is an immediate consequence of the Theorem 4.1.

Corollary 4.1. *An orthonormal basis of $V_n(L)$ is a unit ortho set.*

Corollary 4.2. *If an orthogonal unit ortho subset of $V_n(L)$ has exactly n elements then it is a basis.*

Proof. Suppose $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a unit ortho orthonormal subset of $V_n(L)$, where $\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{ni})$ with $x_{ij} \in L$, $i, j = 1$ to n. In order to prove S is basis of $V_n(L)$ it is enough to prove S is an orthonormal basis of $V_n(L)$. So, in view of Theorem 4.1 it suffices to prove that S^* is an orthonormal subset of $V_n(L)$. For this, since each $\mathbf{x}_i$ is a unit orthovector we have

$$\vee_{i=1}^{m} x_{ik} = 1 \text{ and } x_{ik}x_{jk} = 0,\ i \neq j, \text{ for all } k = 1 \ldots n \qquad (1)$$

and since S is orthonormal subset of $V_n(L)$ we have

$$\vee_{j=1}^{m} x_{ji} = 1 \text{ and } x_{ki}x_{kj} = 0,\ i \neq j, \text{ for all } k = 1 \ldots n. \qquad (2).$$

Now from (1) and (2), it follows $\vee_{j=1}^{m} x_{ij} = 1$ and $x_{ik}x_{jk} = 0$, $i \neq j$, for all $i, k = 1$ to n. So, S^* is orthonormal subset of $V_n(L)$. Hence S is a basis of $V_n(L)$. □

Remark 4.1. By symmetry, we can restate Theorem 4.1 as "S is orthonormal basis of $V_n(L)$ if and only if S^* is orthonormal basis of $V_m(L)$". We call S^* is the dual basis for S.

Example 4.1. Here we give an example of construction of dual basis. Let $L = \{0, a_1, a_2, a_3, a_4, a_5, 1\}$ be the lattice whose Hasse diagram is as follows:

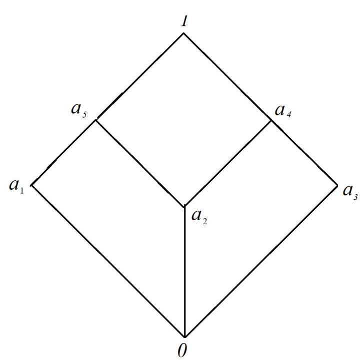

Figure 3

In L we have $a_ia_j = 0$, for i, j with $i \neq j$ in $\{1,2,3\}$ and $a_1 \vee a_2 \vee a_3 = 1$. Consider the following matrix:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

then the columns of the matrix A form an orthonormal set of $V_3(L)$, and the columns of the matrix A^* form an orthonormal set of $V_3(L)$. By theorem 4.1 the columns of Matrix A form an orthonormal basis for $V_3(L)$. Similarly, the columns of A^* form an orthonormal basis for $V_3(L)$. Therefore the columns of the matrix A form an orthonormal basis for $V_3(L)$ and the columns of A^* form the corresponding dual basis.

We now prove an important theorem concerning the construction of unit orthovectors.

THEOREM 4.2. *Let $n > 1$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two unit orthovectors in $V_n(L)$. Then $\mathbf{x}$ is orthogonal to $\mathbf{y}$ if and only if there exists a unit orthovector $\mathbf{z} = (z_1, z_2, \ldots, z_{n-1})$ in $V_{n-1}(L)$ such that $y_i = z_i(x_1 \vee x_2 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots x_n)$ for i=1 to $n-1$. If $\mathbf{x}$ is orthogonal $\mathbf{y}$ then we can always choose $\mathbf{z}$ with $z_i = y_n x_i \vee y_i$ for i=1 $\ldots n-1$.*

PROOF. Let $\mathbf{x}$ and $\mathbf{y}$ be any two orthogonal unit orthovectors.

Now we construct a unit orthovector $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ in $V_{n-1}(L)$ $z_i = y_n x_i \vee y_i$ for i=1 to $n-1$ as follows: $z_i = y_n x_i \vee y_i$ for i=1 $\ldots n-1$.

For $1 \leqslant i, j \leqslant n-1$, $i \neq j$, consider $z_i z_j = [y_n x_i \vee y_i][y_n x_j \vee y_j] = 0$.

Consider

$$\begin{aligned}
\vee_{i=1}^{n-1} z_i &= \vee_{i=1}^{n-1} y_n x_i \vee y_i \\
&= [y_n x_1 \vee y_1] \vee \ldots [y_n x_{n-1} \vee y_{n-1}] \\
&= y_n(x_1 \vee x_2 \vee \ldots \vee x_{n-1}) \vee (y_1 \vee y_2 \vee \ldots \vee y_{n-1}) \vee 0 \\
&= y_n(x_1 \vee x_2 \vee \ldots \vee x_{n-1}) \vee (y_1 \vee y_2 \vee \ldots \vee y_{n-1}) \vee x_n y_n \\
&= y_n(x_1 \vee x_2 \vee \ldots \vee x_n) \vee (y_1 \vee y_2 \vee \ldots \vee y_{n-1}) \\
&= y_1 \vee y_2 \vee \ldots \vee y_n = 1.
\end{aligned}$$

Therefore $\mathbf{z}$ is a unit orthovector. Now we will show that $\mathbf{z}$ has the desired property. For, consider

$$\begin{aligned}
z_i(x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n) &= [y_n x_i \vee y_i][(x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots x_n)] \\
&= 0 \vee y_i(x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \ldots \vee x_n) \\
&= y_i(x_i \vee y_i(x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \ldots \vee x_n)) \\
&= y_i(x_1 \vee \ldots \vee x_n) \\
&= y_i, i \in \{1, 2, \ldots, n-1\}.
\end{aligned}$$

Conversely, suppose there is a unit ortho vector $\mathbf{z}$ in $V_{n-1}(L)$ such that $y_i = z_i(x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots x_n)$, i=1…$n-1$. We show that $\mathbf{x}$ is orthogonal to $\mathbf{y}$. Multiplying with x_i, i=1…$n-1$, on both sides of the equation $y_i = z_i(x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots x_n)$ we obtain $x_i y_i$=0, i =1 $\ldots n-1$. Consider $x_n y_n = 1.x_n y_n = (z_1 \vee z_2 \vee \ldots \vee z_{n-1}).x_n y_n = 0$. Therefore $x_i y_i = 0$, i=1 $\ldots n$. Hence $\mathbf{x}$ is orthogonal to $\mathbf{y}$. □

LEMMA 4.1. *The cardinality of an orthogonal unit ortho set of $V_n(L)$ is at most n.*

Proof. Suppose $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$, where $\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{ni})$, for all i= 1 ... m is an orthogonal unit ortho set in $V_n(L)$. We have to prove that $m \leqslant n$. We prove this theorem by induction on n. For $n = 1$, the only othonormal set is $\{1\}$. So, the result holds trivially. Now assume that the result is true for some $n = k$. Let $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$, where $\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{(k+1)i})$, for all i= 1 ... m is an orthogonal unit ortho set in $V_{k+1}(L)$. Then the matrix corresponding to S is $\mathbf{S} = [x_{ij}]_{(k+1)\times m}$. Now by Theorem 4.2 for each $j = 2, 3, \ldots, m$ there exists a unit ortho vector $\mathbf{z}_i = (z_{1i}, z_{2i}, \ldots, z_{ki})$ in $V_k(L)$ such that $x_{ij} = z_{ij}(x_{11} \vee x_{21} \vee \ldots \vee x_{(i-1)1} \vee x_{(i+1)1} \vee \ldots \vee x_{(k+1)1})$ for all $i = 1 \ldots k+1$ and $j = 2 \ldots m$. Let $j, l \in \{2, 3, \ldots, m\}$ with $j \neq l$ and $i \in \{1, 2, 3, \ldots, k+1\}$. Again by theorem 4.5 $z_{ij} = x_{i1}x_{(k+1)j} \vee x_{ij}$. Clearly $\vee_{i=1}^{k} z_{ij} = \vee_{i=1}^{k} x_{ij}$ =1.consider $z_{ij}z_{il} = [x_{i1}x_{nj} \vee x_{ij}][x_{i1}x_{nl} \vee x_{il}] = 0$. Similarly, we can have $z_{ik}z_{jk} = 0$, for all $i \neq j$. Therefore the set $\{\mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_m\}$ is an orhogonal unit orthovector set in $V_k(L)$. Then by induction hypothesis, we have $m - 1 \leqslant k$. Therefore $m \leqslant n$. □

We now prove the main result of this section:

Theorem 4.3. *The cardinality of an orthonormal basis of* $V_n(L)$ *is* n.

Proof. We prove this theorem by induction on n. The result is trivial for $n = 1$. Assume the result is true for some $n = k$. Suppose S is an orthonormal basis for $V_{k+1}(L)$. Then by Corollary 4.1, S is a unit ortho set. Consequently, by Lemma 4.1, the cardinality of S is less than or equal to $k+1$. Let us say the cardinality of S is m. Therefore $m \leqslant k+1$. Suppose if possible $m < k+1$. Then by Theorem 4.1 and Remark 4.1 we have S^* is an orthonormal basis for $V_m(L)$. Since $m < k+1$, we have $m \leqslant k$. Then by induction hyopothesis we have the cardinality $\mathsf{S}^* = m$. But by construction the cardinality of S^* is $k+1$. A contradiction. Hence the cardinality of S is $k+1$. Thus, if S is an orthonormal basis for $V_n(L)$ then the cardinality of S is n. □

Theorem 4.3 and Corollary 4.2 together gives the following result:

Corollary 4.3. *An orthogonal unit ortho set* S *is a basis for* $V_n(L)$ *if and only if the cardinality of* S *is* n.

We now see that any orthogonal unit ortho subset S of $V_n(L)$ can be extended to an orthonormal basis for $V_n(L)$.

We shall use the following concept:

Definition 4.1. A subset $\mathsf{M} \subseteq V_n(L)$ is called a subspace if it is generated by an othonormal set $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$. i.e., $\mathsf{M} = \{\sum_{i=1}^{m} a_i\mathbf{x}_i / a_1, a_2, \ldots a_m \in L\}$. Any orthonormal set S generating M is called an orthonormal basis for M.

Remark 4.2. We remark that we do not require orthonormal basis of sunbspaces to be unit ortho sets. In fact a subspace may not contain any orthogonal unit ortho basis. For example refer [**4**]. Thus we will some times use:

Definition 4.2. A subspace with a orthogonal unit ortho basis is called a unit ortho subspace.

We now concern on linear maps between two sub spaces.

DEFINITION 4.3. A linear map $T : \mathsf{M} \to \mathsf{N}$ between two subspaces M and N of $V_n(L)$ and $V_m(L)$ respectively, is called an isometry if $< T(\mathbf{x}), T(\mathbf{y}) >=< \mathbf{x}, \mathbf{y} >$, for all $\mathbf{x}, \mathbf{y} \in \mathsf{M}$.

LEMMA 4.2. *Let $\mathsf{M} \subseteq V_n(L)$ and $\mathsf{N} \subseteq V_m(L)$ be two subspaces. Let $T : \mathsf{M} \to \mathsf{N}$ be a linear map, then:*

(a) *The following conditions are equivalent:*

(i) *T is isometry,*

(ii) *There exists a orthonormal basis $\mathsf{S} = \{\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_k\}$ of M such that $\{T\mathbf{e}_i \ i = 1 \dots k\}$ is an orthonormal set of N.*

(iii) *For every orthonormal set $\mathsf{S} = \{\mathbf{e}_1, \dots \mathbf{e}_k\}$ of M, the set $\{T\mathbf{e}_1, \dots, T\mathbf{e}_k\}$ is orthonormal set of N.*

(b) *If $T : \mathsf{M} \to \mathsf{N}$ is an isometry, then T is injective.*

PROOF. (a) Sketch of the proof: $(i) \implies (iii) \implies (ii) \implies (i)$.

(i) implies (iii) and (iii) implies (ii) are obvious.

We show the implication (ii) $\implies$ (i). Suppose $\mathsf{S} = \{\mathbf{e}_1, \dots \mathbf{e}_k\}$ is an orthonormal basis of M such that $\{T\mathbf{e}_1, \dots, T\mathbf{e}_k\}$ is an orthonormal subset of N. Let $\mathbf{x}, \mathbf{y} \in \mathsf{M}$. Then we have $\mathbf{x} = \sum_{i=1}^{k} x_i\mathbf{e}_i$ and $\mathbf{y} = \sum_{i=1}^{k} y_i\mathbf{e}_i$ with $x_i, y_i \in L$ for $i = 1 \dots n$. Consider

$$
\begin{aligned}
< T(\mathbf{x}), T(\mathbf{y}) > &=< T(\sum_{i=1}^{k} x_i\mathbf{e}_i), T(\sum_{i=1}^{k} y_i\mathbf{e}_i) > \\
&=< \vee_{i=1}^{k} T(x_i\mathbf{e}_i), \vee_{j=1}^{k} T(y_j\mathbf{e}_j) > \text{(since T is linear)} \\
&= \vee_{i,j=1}^{k} < x_iT(\mathbf{e}_i), y_jT(\mathbf{e}_j) > \text{(by inner product property)} \\
&= \vee_{i,j=1}^{k} x_iy_j < T(\mathbf{e}_i), T(\mathbf{e}_j) > \\
&= \vee_{i=1}^{k} x_iy_i \\
&=< \mathbf{x}, \mathbf{y} >
\end{aligned}
$$

(b) Suppose T is an isometry and suppose $T(\mathbf{x}) = T(\mathbf{y})$, for $\mathbf{x}, \mathbf{y} \in \mathsf{M}$. Consider

$$
\begin{aligned}
x_i &=< \mathbf{x}, \mathbf{e}_i > \\
&=< T(\mathbf{x}), T(\mathbf{e}_i) > \\
&=< T(\mathbf{y}), T(\mathbf{e}_i) > \\
&=< \mathbf{y}, \mathbf{e}_i > \\
&= y_i
\end{aligned}
$$

for all i. Hence $\mathbf{x} = \mathbf{y}$. Therefore T is injective. □

DEFINITION 4.4. Let M and N be two subspaces of $V_m(L)$ and $V_n(L)$ respectively. A surjective isometry $T : \mathsf{M} \to \mathsf{N}$ is called an isomorphism, and in this case M and N are said to be isomorphic subspaces.

Remark 4.3. (a) The inverse of an isomorphism is an isomorphism, and the composition of two isomorphisms is again an isomorphism. Hence isomorphism is an equivalence relation.

(b) Isomorphisms map orthonormal bases to orthonormal bases: if $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is an orthonormal basis for a subspace M and $T : \mathsf{M} \to \mathsf{N}$ is an isomorphism then $\{T\mathbf{x}_1, T\mathbf{x}_2, \ldots, T\mathbf{x}_n\}$ is an orthonormal set since T is an isometry by Lemma 4.2. Moreover, if $\mathbf{v} \in \mathsf{N}$ then there exists $\mathbf{u} \in M$ such that $T(\mathbf{u}) = \mathbf{v}$. since $\mathbf{u} = \sum_{i=1}^{n} u_i\mathbf{x}_i$ for some $u_1, u_2, \ldots, u_n \in L$ we conclude that $\mathbf{v} = \sum_{i=1}^{n} u_i T(\mathbf{x}_i)$. Hence $\{T(\mathbf{x}_1), T(\mathbf{x}_2), \ldots T(\mathbf{x}_n)\}$ is an orthonormal spanning subset of N, hence a basis of N.

Theorem 4.4. *If M is a subspace of $V_n(L)$ then there exists $m \in \mathbb{N}$ and an isomorphism $T : \mathsf{M} \to V_m(L)$ such T map unit orthovectors to unit orthovectors, and if M is a unit ortho subspace then T and T^{-1} map unit orthovectors to unit orthovectors.*

Proof. Suppose M is a subspace of $V_n(L)$. Then there exists an orthonormal subset $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_m\}$ which generates M. We show that M is isomorphic to $V_m(L)$. Consider the standard basis $\{\delta_1, \delta_2, \ldots \delta_m\}$ for $V_m(L)$. Define $T : \mathsf{M} \to V_m(L)$ such that $T(\mathbf{x}) = (<\mathbf{x}, \mathbf{e}_1>, <\mathbf{x}, \mathbf{e}_2>, \ldots, <\mathbf{x}, \mathbf{e}_m>)$. Clearly T is well defined and linear. Consider $T(\mathbf{e}_i) = (<\mathbf{e}_i, \mathbf{e}_1>, <\mathbf{e}_i, \mathbf{e}_2>, \ldots, <\mathbf{e}_i, \mathbf{e}_m>) = (0, 0, \ldots, 1, 0, \ldots, 0) = \delta_i$. Therefore there exists an orthonormal set of $V_m(L)$. Then by Lemma 4.13 (???), T is isometry.

To show T is onto. Let $\mathbf{u} \in V_m(L)$ then $\mathbf{u} = (u_1, u_2, \ldots, u_m), u_i \in L$. Let $x = \sum_{i=1}^{m} u_i\mathbf{e}_i$ which belongs to M. Consider

$$\begin{aligned} T(\mathbf{x}) &= (<\mathbf{x}, \mathbf{e}_1>, <\mathbf{x}, \mathbf{e}_2>, \ldots, <\mathbf{x}, \mathbf{e}_m>) \\ &= (<\sum_{i=1}^{m} u_i\mathbf{e}_i, \mathbf{e}_1>, <\sum_{i=1}^{m} u_i\mathbf{e}_i, \mathbf{e}_2>, \ldots, <\sum_{i=1}^{m} u_i\mathbf{e}_i, \mathbf{e}_m>) \\ &= (u_1, u_2, \ldots, u_m) \\ &= \mathbf{u} \end{aligned}$$

Therefore T is surjective and isometry and hence T is an isomorphism.

Next, we see that T preserves unit orthovectors. Let $\mathbf{x} \in \mathsf{M}$ be a unit orthovector. Let $\{\delta_1, \delta_2, \ldots, \delta_n\}$ be the standard orthonormal basis for $V_n(L)$. Consider $<\mathbf{x}, \mathbf{e}_i> = <\mathbf{x}, \sum_{r=1}^{n} <\mathbf{e}_i, \delta_r> \delta_r> = \vee_{r=1}^{n} <\mathbf{e}_i, \delta_r><x, \delta_r>$, for all $i = 1 \ldots n$. Hence for $i \neq j$ and $i, j = 1, \ldots, m$ we have

$$\begin{aligned} <\mathbf{x}, \mathbf{e}_i><\mathbf{x}, \mathbf{e}_j> &= (\vee_{r=1}^{n} <\mathbf{e}_i, \delta_r><\mathbf{x}, \delta_r>)(\vee_{s=1}^{n} <\mathbf{e}_j, \delta_s <\mathbf{x}, \delta_s>) \\ &= \vee_{r,s=1}^{n} <\mathbf{e}_i, \delta_r><\mathbf{x}, \delta_r><\mathbf{e}_j, \delta_s><\mathbf{x}, \delta_s> \\ &= \vee_{r=1}^{n} <\mathbf{e}_i, \delta_r><\mathbf{x}, \delta_r><\mathbf{e}_j, \delta_r> \\ &\leqslant \vee_{r=1}^{n} <\mathbf{e}_i, \delta_r><\mathbf{e}_j, \delta_r> \\ &= <\mathbf{e}_i, \mathbf{e}_j> \\ &= 0 \end{aligned}$$

Consider

$$\begin{aligned}\vee_{i=1}^{n} < \mathbf{x}, \mathbf{e}_i > &= \vee_{i=1}^{n} < \sum_{i=1}^{m} x_i \mathbf{e}_i, \mathbf{e}_i > \\ &= \vee_{i=1}^{n} x_i \\ &= 1.\end{aligned}$$

Therefore $T(x)$ is unit orthovector.

Suppose M is a unit ortho subspace. Then M has a orthogonal unit ortho basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$. We show that $T^{-1} : V_m(L) \to \mathsf{M}$ preserves unit orthovectors. Let $(u_1, u_2, \ldots, u_m) \in V_m(L)$ be a unit orthovector. Since $T : \mathsf{M} \to V_m(L)$ is surjective, there exist $\mathbf{v} = (v_1, v_2, \ldots, v_m) \in \mathsf{M}$ such that $T(\mathbf{v}) = (u_1, u_2, \ldots, u_m)$. Which implies $(< \mathbf{v}, \mathbf{e}_1 >, < \mathbf{v}, \mathbf{e}_2 >, \ldots, < \mathbf{v}, \mathbf{e}_m >) = (u_1, u_2, \ldots, u_m)$. Which further implies $< \mathbf{v}, \mathbf{e}_j >= u_j$, for all $j= 1 \ldots m$. Now we show that $\mathbf{v}$ is a unit orthovector. Since $u_i u_j = 0$ we have $< \mathbf{v}, \mathbf{e}_i >< \mathbf{v}, \mathbf{e}_j >= 0$, for $i \neq j$. Consequently, $< \sum_{i=1}^{m} v_i \mathbf{e}_i, \mathbf{e}_i >< \sum_{i=1}^{m} v_i \mathbf{e}_i, \mathbf{e}_j >= 0$, for $i \neq j$. Hence $v_i v_j = 0$, for $i \neq j$. Consider

$$\begin{aligned}\vee_{i=1}^{n} v_i &= \vee_{i=1}^{n} < \sum_{i=1}^{m} v_i \mathbf{e}_i, \mathbf{e}_i > \\ &= \vee_{i=1}^{n} < \mathbf{v}, \mathbf{e}_i > \\ &= \vee_{i=1}^{n} u_i \\ &= 1.\end{aligned}$$

Therefore $\mathbf{v}$ is a unit orthovector. □

Corollary 4.4. *Any two orthonormal bases of a subspace* M *of* $V_n(L)$ *have same cardinality.*

Proof. Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ and $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ be two orthonormal bases of subspace M. By Theorem 4.4, there exist an isomorphism $T : \mathsf{M} \to V_m(L)$ and an isomorphism $S : \mathsf{M} \to V_k(L)$. By the same theorem we have $S^{-1} : V_k(L) \to \mathsf{M}$ is isomorphism. Therefore $T \circ S^{-1} : V_k(L) \to V_m(L)$ is an isomorphism and $T \circ S^{-1}$ maps orthonormal basis to orthonormal basis. Hence $m = n$. □

Remark 4.4. We call the common cardinality of orthonormal basis for a subspace M the dimension of M. It follows from Theorem 4.4 that if M has dimension m then M is isomorphic to $V_m(L)$.

Definition 4.5. For $\mathbf{x} \in V_n(L)$, we define $\mathbf{x}^{\perp} = \{\mathbf{y} \in V_n(L) / < \mathbf{x}, \mathbf{y} >= 0\}$.

Proposition 4.1. *If* $\mathbf{x} \in V_n(L)$ *is a unit orthovector then* $\mathbf{x}^{\perp}$ *is a unit ortho subspace of* $V_n(L)$ *of dimension* $(n-1)$*.*

Proof. Let $\mathbf{x} \in V_n(L)$. As in Example 3.3 we can extend the unit orthovector $\mathbf{x}$ to an orthonormal basis $\{\mathbf{x} = \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $V_n(L)$. If $< \mathbf{y}, \mathbf{x} >= 0$ then we have $< \mathbf{y}, \mathbf{e}_1 > = 0$ if and only if $y_1 = 0$, where $\mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{e}_i$. Hence it follows that $\mathbf{x}^{\perp} = \{\sum_{i=2}^{n} y_i \mathbf{e}_i / y_2, \ldots, y_n \in L, < \sum_{i=2}^{n} y_i \mathbf{e}_i, \mathbf{x} >= 0\}$ is the subspace of

$V_n(L)$ generated by the orthogonal unit ortho set $\{\mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $V_n(L)$. Thus the cardinality of $\mathbf{x}^{\perp}$ is $(n-1)$. $\square$

We now prove our main theorem

THEOREM 4.5. *If* $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ *is a orthogonal unit ortho set in* $V_n(L)$ *with* $m < n$ *then* S *can be extended to an orthonormal basis for* $V_n(L)$.

PROOF. We prove this theorem by using induction on n. The result is trivial for n=1. Assume the result is true for some n. Let $\mathsf{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$, where $m < n$, is an orthogonal unit ortho set in $V_{n+1}(L)$. We now see that S can be extended to an orthonormal basis of $V_{n+1}(L)$. By Proposition 4.1 and Theorem 4.4 there exist an isomorphism $T : \mathbf{x}_1^{\perp} \to V_n(L)$ such that T and T^{-1} preserve the unit orthovectors. Moreover $\{\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_m\} \subseteq \mathbf{x}_1^{\perp}$. Let $\mathbf{y}_i \in V_n(L)$ such that $T(\mathbf{x}_i) = \mathbf{y}_i$, for $i = 2, 3, \ldots m$. Then by Lemma 4.2, $\{\mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_m\}$ is an orthonormal set in $V_n(L)$ of cardinaliy $m - 1 < n$. Then by induction hypothesis, there exist unit orthovectors $\mathbf{y}_{m+1}, \mathbf{y}_{m+2}, \ldots, \mathbf{y}_{n+1}$ such that $\{\mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{n+1}\}$ is an orthonormal basis for $V_n(L)$. By Theorem 4.4, $\{T^{-1}(\mathbf{y}_2), T^{-1}(\mathbf{y}_3), \ldots, T^{-1}(\mathbf{y}_{n+1})\}$ is an orthogonal unit ortho set $V_{n+1}(L)$, which is a basis for $x_1^{\perp}$. Since $\mathbf{x}_i = T^{-1}(\mathbf{y}_i)$, for all $i =$ $2 \ldots m$, we conclude by Corollary 4.2, that $\{\mathbf{x}_1, T^{-1}(\mathbf{y}_2), T^{-1}(\mathbf{y}_3), \ldots, T^{-1}(\mathbf{y}_{n+1})\}$ is an othonormal basis $V_{n+1}(L)$ which extends S. $\square$

References

[1] G. Birkhoff. *Lattice Theory*, 3rd edition. American Mathematical Society, Providence, RI, 1967.

[2] G. Joy and K. V. Thomas. Lattice vector spaces and linear transformations. *Asian-Eur. J. Math.*, **12**(2)(2019), ID: 1950031

[3] Y. Give'on. Lattice matrices. *Inf. Control*, **7**(4)(1964), 477–484.

[4] S. Gudder and F. Latrémolière. Boolean inner product spaces and Boolean matrices. *Linear Algebra Appl.*, **431**(1-2)(2009), 272–296.

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