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FERMATEAN FUZZY SUBGROUPS

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ABSTRACT. Fermatean fuzzy environment is the modern tool for handling uncertainty in many decisions making problems. In this paper, we introduce the notion of Fermatean fuzzy subgroup (FFSG) as a generalization of intuitionistic fuzzy subgroup and Pythagorean fuzzy subgroup. We investigate various properties of our proposed fuzzy subgroup. Also, we introduce Fermatean fuzzy coset and Fermatean fuzzy normal subgroup (FFNSG) with their properties. Further, we define the notion of Fermatean fuzzy level subgroup and establish related properties of it. Finally, we discuss the effect of group homomorphism on Fermatean fuzzy subgroup.

1. Introduction

Group theory has various applications in many fields of Mathematics such as Algebraic geometry, Cryptography, Harmonic analysis, Algebraic number theory, etc. Uncertainty is a part of our daily life. There is an uncertainty in almost every problem we face day to day. Fuzzy set theory was first invented by Zadeh [**39**] to handle uncertainty in real life problems. Using the concept of fuzzy set, Rosenfeld [**28**] first defined the notion of fuzzy subgroup. Using t-norm the notion of fuzzy subgroup was redefined by Anthony and Sherwood [**5**, **6**]. Das [**15**] introduced the concept of fuzzy level subgroup. Choudhury et al. [**12**] proved various properties of fuzzy subgroups and fuzzy homomorphism. Dixit et al. [**16**] discussed fuzzy level subgroups and union of fuzzy subgroups. The notion of anti-fuzzy subgroups was first proposed by Biswas [**10**]. Ajmal and Prajapati [**2**] gave the idea of fuzzy normal subgroup, fuzzy coset and fuzzy homomorphism. Many more results on fuzzy

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subgroup ware introduced by Mukherjee [22, 23] and Bhattacharya [7]. In recent years many researchers studied various properties of fuzzy subgroups. Tarnauceanu [36] classified fuzzy normal subgroup of finite groups. Onasanya [25] reviewed some anti fuzzy properties of fuzzy subgroups. Shuaib [30] and Shaheryar [31] studied the properties of omicron fuzzy subgroup and omicron anti fuzzy subgroup. Addis [1] developed fuzzy homomorphism theorems on groups. Atanassov [4] invented Intuitionistic fuzzy set. Intuitionistic fuzzy subgroup was first studied by Biswas [11]. Zhan and Tan [21] introduced intuitionistic fuzzy M-group. Furthermore, researchers developed intuitionistic fuzzy subgroup in many ways [9, 21, 35]. Picture fuzzy set can be treated as an immediate generalization of intuitionistic fuzzy set by togethering three components namely positive, neutral and negative. With the advancement of time, different kinds of research works under picture fuzzy environment were performed by several researchers [14, 17, 19, 18, 33].

Yager [37] introduced Pythagorean fuzzy set, where the sum of square of the membership degree and non membership degree lies between 0 and 1. Pythagorean fuzzy set is more fruitful in many decision making problems. This concept is perfectly designed to represent vagueness and uncertainty in mathematical way and to produce a formalized tool to handle imprecision to real problems. Naz et al. [24] proposed a novel approach to decision making problem using Pythagorean fuzzy set. Akram and Naz [3] applied complex Pythagorean fuzzy set in decision making problems. we have identified and proved several of these properties, particularly those involving the operation $A \to B$ defined as pythagorean fuzzy implication with other operations [32]. Ejegwa [20] gave an application of Pythagorean fuzzy set in career placements based on academic performance using max-min-max composition. Some results related to it were given by Peng [26] and Yang [27]. Bhunia et al. [8] represented the notion of Pythagorean fuzzy subgroup (PFSG) as a generalization of intuitionistic fuzzy subgroup and investigated various properties of Pythagorean fuzzy subgroup. Also, they introduced Pythagorean fuzzy coset and Pythagorean fuzzy normal subgroup (PFNSG) with their properties. Further, they define the notion of Pythagorean fuzzy level subgroup and establish related properties of it. Senapti and Yager [29] coined the Fermatean fuzzy set (FFS) with its comparison measures. FFS can characterize more complex uncertain information by redefining the constraint condition $0 \leq \mu^3 + \nu^3 \leq 1$. In other words, IFS and PFS are two special forms of FFS, which means that the FFSs are able to handle higher levels of uncertainties. Senapti and Yager [29] gave an example: For understanding the FFS better, we give an instance to illuminate the understandability of the FFS: We can definitely get 0.9 + 0.6 > 1, and, therefore, it does not follow the condition of intuitionistic fuzzy sets. Also, we can get $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 > 1$, which does not obey the constraint condition of Pythagorean fuzzy set. However, we can get $(0.9)^3 + (0.6)^3 = 0.729 + 0.216 = 0.945 \le 1$. which is good enough to apply the Fermatean fuzzy set to control it. we have developed some new operators for fermatean fuzzy sets [34].

Fermatean fuzzy set gives a modern way to model vagueness and uncertainty with high precision and accuracy compared to intuitionistic fuzzy set. Group symmetry plays a vital role to analyse molecule structures. Isotope molecules decays with a certain rate, so the fuzzy sense comes into it. If decay rate follows the criteria of Fermatean fuzzy environment then we can not opt for intuitionistic fuzzy subgroup and Pythagorean fuzzy subgroup to analyse the structure of that isotope at certain time. For this type of situations where we can not opted for intuitionistic fuzzy subgroup and Pythagorean fuzzy subgroup, it is very important to introduce Fermatean fuzzy subgroup as a bigger class of intuitionistic fuzzy subgroup and Pythagorean fuzzy subgroup. But till now no algebraic structure is defined on Fermatean fuzzy environment. In this article, we proved that intuitionistic fuzzy subgroup and Pythagorean fuzzy subgroup is a subclass of Fermatean fuzzy subgroup.

This paper is organized as follows: Fermatean fuzzy subgroup is described in Section 3. Fermatean fuzzy coset and Fermatean fuzzy normal subgroup are discussed in Section 4. Fermatean fuzzy level subgroup and its properties are given in Section 5. Finally, effect of group homomorphism on Fermatean fuzzy subgroup is discussed in Section 6 and a conclusion is given in Section 7.

2. Preliminaries

In this section, we recap some definitions and concepts which are very much important to develop later sections.

DEFINITION 2.1. ([39]) Let C be a crisp set. Then $\mu : C \to [0,1]$ is called a fuzzy subset of C. Here $\mu(m)$ is called degree of membership of $m \in C$.

DEFINITION 2.2. ([28]) Let $\mu : C \to [0,1]$ be a fuzzy subset of a group (C,*)Then μ is said to be a fuzzy subgroup of (C,*) if the following conditions hold:

(i) $\mu(m * n) \ge \mu(m) \land \mu(n)$ for all $m, n \in C$,

(ii) $\mu(m^{-1}) \ge \mu(m)$ for all $m \in C$.

DEFINITION 2.3. ([4]) Let C be a crisp set. An intuitionistic fuzzy set (IFS) I on C is defined by $I = \{(m, \mu(m), \nu(m)) | m \in C\}$ where $\mu(m) \in [0, 1]$ and $\nu(m) \in [0, 1]$ are the degree of membership and non membership of $m \in C$ respectively, which satisfy the condition $0 \leq \mu(m) + \nu(m) \leq 1$ for all $m \in C$.

DEFINITION 2.4. ([10]) Let $I = \{(m, \mu(m), \nu(m)) | m \in C\}$ be a IFS of a group (C, *). Then I is said to be an intuitionistic fuzzy subgroup (IFSG) of C if the following conditions hold:

(i) $\mu(m*n) \ge \mu(m) \land \mu(n)$ and $\nu(m*n) \le \nu(m) \lor \nu(n)$ for all $m, n \in C$,

(ii) $\mu(m^{-1}) \ge \mu(m)$ and $\nu(m^{-1}) \le \nu(m)$ for all $m \in C$.

In 2013, Yager [**37**] defined Pythagorean fuzzy subset (PFS) as a generalization of IFS.

DEFINITION 2.5. Let C be a crisp set. A Pythagorean fuzzy set (IFS) ψ on C is defined by $\psi = \{(m, \mu(m), \nu(m)) | m \in C\}$ where $\mu(m) \in [0, 1]$ and $\nu(m) \in [0, 1]$ are the degree of membership and non membership of $m \in C$ respectively, which satisfy the condition $0 \leq \mu^2(m) + \nu^2(m) \leq 1$ for all $m \in C$.

In 2020, Senapti and Yager [29] defined Fermatean fuzzy subset (PFS) as a generalization of IFS.

DEFINITION 2.6. Let C be a crisp set. A Fermatean fuzzy set (IFS) ξ on C is defined by $\xi = \{(m, \mu(m), \nu(m)) | m \in C\}$ where $\mu(m) \in [0, 1]$ and $\nu(m) \in [0, 1]$ are the degree of membership and non membership of $m \in C$ respectively, which satisfy the condition $0 \leq \mu^3(m) + \nu^3(m) \leq 1$ for all $m \in C$.

Some operations on FFSs [29] are stated below. Let

 $\xi_1 = \{(m, \mu_1(m), \nu_1(m)) | m \in C\} \text{ and } \xi_2 = \{(m, \mu_2(m), \nu_2(m)) | m \in C\}$

- be two FFSs of C. Then the following holds:
 - $\xi_1 \cup \xi_2 = \{(m, \mu_1(m) \lor \mu_2(m), \nu_1(m) \land \nu_2(m)) | m \in C\}$
 - $\xi_1 \cap \xi_2 = \{(m, \mu_1(m) \land \mu_2(m), \nu_1(m) \lor \nu_2(m)) | m \in C\}$
 - $\xi_1^C = \{(m, \nu_1(m), \mu_1(m)) | m \in C\}$
 - $\xi_1 \subseteq \xi_2$ if $\mu_1(m) \leq \mu_2(m)$ and $\nu_1(m) = \nu_2(m)$ for all $m \in C$
 - $\xi_1 = \xi_2$ if $\mu_1(m) = \mu_2(m)$ and $\nu_1(m) = \nu_2(m)$ for all $m \in C$

Throughout this paper, we will write Fermatean fuzzy subset as FFS and we will write $\xi = (\mu, \nu)$ instead of $\xi = \{(m, \mu(m), \nu(m)) | m \in C\}$.

3. Fermatean fuzzy subgroup

In this section, we define Fermatean fuzzy subgroup (FFSG) as an extension of intuitionistic fuzzy subgroup (IFSG) and Pythagorean fuzzy subgroup (PFSG).

DEFINITION 3.1. Let $\xi = \{(m, \mu(m), \nu(m)) | m \in C\}$ be a FFS of a group (C, *). Then xi is said to be a Fermatean fuzzy subgroup (FFSG) of C if the following conditions hold:

(i) $\mu^3(m*n) \ge \mu^3(m) \land \mu^3(n)$ and $\nu^3(m*n) \le \nu^3(m) \land \nu^3(n)$ for all $m, n \in C$, (ii) $\mu^3(m^{-1}) \ge \mu^3(m)$ and $\nu^3(m^{-1}) \le \nu^3(m)$ for all $m \in C$.

Here, $\mu^{3}(m) = {\mu(m)}^{3}$ and $\nu^{3}(m) = {\nu(m)}^{3}$ for all $m \in C$.

EXAMPLE 3.1. Let us take the set $C = \{1, -1, i, -i\}$. Then (C, .) is a group, where '.' s the usual multiplication. Define a FFS $\xi = (\mu, \nu)$ on C by

$$\mu(1) = 0.9, \ \mu(-1) = 0.6, \ \mu(i) = 0.5, \ \mu(-i) = 0.5, \ \nu(1) = 0.6, \ \nu(-1) = 0.2, \ \nu(i) = 0.4, \ \nu(-i) = 0.4,$$

Here, $\mu^{3}(i,-i) = \mu^{3}(1) = (0.9)^{3} = 0.729$ and $\nu^{3}(i,-i) = \nu^{3}(1) = (0.6)^{3} = 0.216$. Now, $\mu^{3}(i) \wedge \mu^{3}(-i) = \min\{0.125, 0.125\} = 0.125$ and $\mu^{3}(i) \vee \mu^{3}(-i) = \max\{0.064, 0.064\} = 0.064$. So, $\mu^{3}(i,-i) \supset \mu^{3}(i) \wedge \mu^{3}(-i)$ and $\nu^{3}(i,-i) \subset \nu^{3}(i) \vee \nu^{3}(-i)$. Also, $\mu^{3}(i) = \mu^{3}(-i)$ and $\nu^{3}(i) = \nu^{3}(-i)$. In the same manner it can be shown that $\mu^{3}(m * n) \ge \mu^{3}(m) \wedge \mu^{3}(n)$ and $\nu^{3}(m * n) \le \nu^{3}(m) \wedge \nu^{3}(n)$ for all $m, n \in C$, and $\mu^{3}(m^{-1}) \ge \mu^{3}(m)$ and $\nu^{3}(m^{-1}) \le \nu^{3}(m)$ for all $m \in C$. Hence, ξ is a FFSG of the group (C, .).

PROPOSITION 3.1. Let $\xi = (\mu, \nu)$ be a PFSG of a group (C, *). Then the following holds:

(i) $\mu^3(e) \ge \mu^3(m)$ and $\nu^3(e) \le \nu^3(m) \forall m \in C$ and

(ii) $\mu^{3}(m^{-1}) = \mu^{3}(m)$ and $\nu^{3}(m^{-1}) \leq \nu^{3}(m)$ for all $m \in C$,

where e is the identity element in C.

PROOF. Since $\xi = (\mu, \nu)$ is a FFSG of a group (C, *), then

$$\mu^{3}(m * n) \ge \mu^{3}(m) \land \mu^{3}(n), \nu^{3}(m * n) \le \nu^{3}(m) \lor \nu^{3}(n) \text{ and}$$
$$\mu^{3}(m^{-1}) \ge \mu^{3}(m), \nu^{3}(m^{-1}) \le \nu^{3}(m) \text{ for all } m, n \in C.$$

(i) Now,

$$\mu^{3}(e) = \mu^{3}(m * m^{-1}) \ge \mu^{3}(m) \land \mu^{3}(m^{-1}) = \mu^{3}(m)$$

Also,

$$\nu^{3}(e) = \nu^{3}(m \ast m^{-1}) \leqslant \nu^{3}(m) \lor \nu^{3}(m^{-1}) = \nu^{3}(m) \text{ for all } m \in C.$$

(ii) We have

$$\mu^3(m^{-1}) \geqslant \mu^3(m), \nu^3(m^{-1}) \leqslant \nu^3(m) \text{ for all } m, n \in C.$$

Putting m^{-1} in place of m, we obtain that $\mu^3((m^{-1})^{-1}) \ge \mu^3(m^{-1})$ implies $\mu^3(m) \ge \mu^3(m^{-1})$ for all $m \in C$. Again, from $\nu^3((m^{-1})^{-1}) \le \nu^3(m^{-1})$ it follows $\nu^3(m) \le \nu^3(m^{-1})$ for all $m \in C$. Hence, combining all the results we get $\mu^3(m^{-1}) \ge \mu^3(m), \nu^3(m^{-1}) \le \nu^3(m)$ for all $m, n \in C$.

Now, we will show that every intuitionistic fuzzy subgroup (IFSG) and Pythagorean fuzzy subgroup of a group (C, *) is also a Fermatean fuzzy subgroup (PFSG) of the group (C, *). But the converse is not true.

THEOREM 3.1. If $\xi = (\mu, \nu)$ is a IFSG and PFSG of a group (C, *), then ξ is a FFSG of the group (C, *).

PROOF. Since $\xi = (\mu, \nu)$ is a IFSG and PFSG of a group (C, *), then

$$\mu(m * n) \ge \mu(m) \land \mu(n) \text{ and } \nu(m * n) \le \nu(m) \lor \nu(n) \text{ for all } m, n \in C.$$

Thus

$$\mu(m^{-1}) \ge \mu(m)$$
 and $\nu(m^{-1}) \le \nu(m)$ for all $m \in C$.

Here $\mu(m) \in [0, 1]$ and $\nu(m) \in [0, 1] \forall m \in C$. Many cases arise.

Case 1: Let $\mu(m) > \mu(n)$ and $\nu(m) > \nu(n)$ for all $m, n \in C$. So, $\mu^3(m) > \mu^3(n)$ and $\nu^3(m) > \nu^3(n)$ for all $m, n \in C$. Now, $\mu(m * n) \ge \mu(m) \land \mu(n) = \mu(n)$ implies $\mu^3(m*n) \ge \mu^3(n) = \mu^3(m) \land \mu^3(n)$, i.e. $\mu^3(m*n) \ge \mu^3(m) \land \mu^3(n)$. Again, $\nu(m*n) \le \nu(m) \lor \nu(n) = \nu(m)$ implies $\nu^3(m*n) \le \nu^3(n) = \nu^3(m) \lor \nu^3(n)$, i.e. $\nu^3(m*n) \le \nu^3(m) \lor \nu^3(n)$.

Case 2: Let $\mu(m) < \mu(n)$ and $\nu(m) < \nu(n)$ for all $m, n \in C$. So, $\mu^3(m) < \mu^3(n)$ and $\nu^3(m) < \nu^3(n)$ for all $m, n \in C$. Now, $\mu(m * n) \ge \mu(m) \land \mu(n) = \mu(m)$ implies $\mu^3(m*n) \ge \mu^3(m) = \mu^3(m) \land \mu^3(n)$, i.e. $\mu^3(m*n) \ge \mu^3(m) \land \mu^3(n)$. Again, $\nu(m*n) \le \nu(m) \lor \nu(n) = \nu(n)$ implies $\nu^3(m*n) \le \nu^3(n) = \nu^3(m) \lor \nu^3(n)$, i.e. $\nu^3(m*n) \le \nu^3(m) \lor \nu^3(n)$, i.e.

Case 3: Let $\mu(m) = \mu(n)$ and $\nu(m) = \nu(n) \forall m, n \in C$. So, $\mu^3(m) = \mu^3(n)$ and $\nu^3(m) = \nu^3(n)$ for all $m, n \in C$. Now, $\mu(m * n) = \mu(m) = \mu(n) = \mu(m)$ implies $\mu^3(m * n) = \mu^3(m) = \mu^3(m) = \mu^3(n)$, i.e. $\mu^3(m * n) = \mu^3(m) \land \mu^3(n)$. Again, $\nu(m * n) = \nu(m) = \nu(n) = \nu(n)$ implies $\nu^3(m * n) = \nu^3(m) = \nu^3(n)$, i.e. $\nu^3(m * n) = \nu^3(m) \lor \nu^3(n)$.

In this way, considering all the cases we can easily show that

$$\mu^3(m*n) \ge \mu^3(m) \land \mu^3(n) \text{ and } \nu^3(m*n) \le \nu^3(m) \lor \nu^3(n) \text{ for all } m, n \in C.$$

Again, $\mu(m^{-1}) \ge \mu(m)$ and $\nu(m^{-1}) \le \mu(m)$ for all $m, n \in C$. Since $\mu(m) \in [0, 1]$ and $\nu(m) \in [0, 1]$, we have $\mu^3(m^{-1}) \ge \mu^3(m)$ and $\nu^3(m^{-1}) \le \mu^3(m)$ for all $m, n \in C$. Hence, $\xi = (\mu, \nu)$ is a FFSG of the group (C, *).

EXAMPLE 3.2. Let us consider the Kleins 4-group $C = \{e, a, b, c\}$, where $a^3 = b^3 = c^3 = e$ and ab = c, bc = a, ca = b. Define a FFS $\xi = (\mu, \nu)$ on C by

$$\mu(e) = 0.9, \ \mu(c) = 0.8, \ \mu(a) = 0.6, \ \mu(b) = 0.6 \\ \nu(e) = 0.2, \ \nu(c) = 0.3, \ \nu(a) = 0.4, \ \nu(b) = 0.4.$$

We can easily verify that $\xi = (\mu, \nu)$ is a PFSG of C. But here $\mu(e) + \nu(e) = 1.1$, which is greater than one. So ξ is not a IFS of C. Therefore ξ is not a IFSG of C.

This example shows that FFSG may not be an IFSG and PFSG.

REMARK 3.1. Every IFSG of a group (C, *) is a PFSG of (C, *), but the converse need not be true.

PROPOSITION 3.2. Let $\xi = (\mu, \nu)$ be a FFS of a group (C, *). Then ξ is a FFSG of (C, *) iff $\mu^3(m * n^{-1}) \ge \mu^3(m) \land \mu^3(n)$ and $\nu^3(m * n^{-1}) \le \nu^3(m) \lor \nu^3(n)$ for all $m, n \in C$.

PROOF. Let $\xi = (\mu, \nu)$ be a FFSG of a group (C, *). So, $\mu^3(m*n^{-1}) \ge \mu^3(m) \land \mu^3(n^{-1}) = \mu^3(m) \land \mu^3(n)$ and $\nu^3(m*n^{-1}) \le \nu^3(m) \lor \nu^3(n^{-1}) = \nu^3(m) \lor \nu^3(n)$ for all $m, n \in C$.

Conversely, let us assume that $\mu^3(m*n^{-1}) \ge \mu^3(m) \land \mu^3(n)$ and $\nu^3(m*n^{-1}) \le \nu^3(m) \lor \nu^3(n)$ for all $m, n \in C$. Now, $\mu^3(e) = \mu^3(m*m^{-1}) \ge \mu^3(m) \land \mu^3(m^{-1}) = \mu^3(m)$ and $\nu^3(e) = \nu^3(m*m^{-1}) \le \nu^3(m) \lor \nu^3(m^{-1}) = \nu^3(m)$ for all $m \in C$, where e is the identity element of C.

Again, $\mu^{3}(m^{-1}) = \mu^{3}(e * m^{-1}) \ge \mu^{3}(e) \land \mu^{3}(m^{-1}) = \mu^{3}(m)$ and $\nu^{3}(m^{-1}) = \nu^{3}(e * m^{-1}) \le \nu^{3}(e) \lor \nu^{3}(m^{-1}) = \nu^{3}(m)$ for all $m \in C$. Therefore, $\mu^{3}(m * n) = \mu^{3}(m * (n^{-1})^{-1}) \ge \mu^{3}(m) \land \mu^{3}(n^{-1}) \ge \mu^{3}(m) \land \mu^{3}(n)$ and $\nu^{3}(m * n) = \nu^{3}(m * (n^{-1})^{-1}) \le \nu^{3}(m) \lor \nu^{3}(n^{-1}) \le \nu^{3}(m) \lor \nu^{3}(n)$ for all $m, n \in C$. Hence $\xi = (\mu, \nu)$ be a FFSG of a group (C, *).

Now we will check whether the union and intersection of two FFSGs of a group (C, *) is a FFSG of C.

THEOREM 3.2. Intersection of two FFSGs of a group (C, *) is a FFSG of the group (C, *).

PROOF. Let $\xi_1 = (\mu_1, \nu_1)$ and $\xi_2 = (\mu_2, \nu_2)$ be two FFSGs of a group (C, *). Then $\xi = \xi_1 \cap \xi_2 = (\mu, \nu)$, where $\mu(m) = \mu_1(m) \wedge \mu_2(m)$ and $\nu(m) = \nu_1(m) \vee \nu_2(m)$

for all $m \in C$. Now for all $m, n \in C$

$$\begin{split} \mu^{3}(m*n^{-1}) &= & \mu_{1}^{3}(m*n^{-1}) \wedge \mu_{2}^{3}(m*n^{-1}) \\ &\geqslant & \left(\mu_{1}^{3}(m) \wedge \mu_{1}^{3}(n)\right) \wedge \left(\mu_{2}^{3}(m) \wedge \mu_{2}^{3}(n)\right) \\ &= & \left(\mu_{1}^{3}(m) \wedge \mu_{2}^{3}(m)\right) \wedge \left(\mu_{1}^{3}(n) \wedge \mu_{2}^{3}(n)\right) \\ &= & \mu^{3}(m) \wedge \mu^{3}(n) \end{split}$$

and

$$\begin{array}{lll} \nu^{3}(m*n^{-1}) & = & \nu_{1}^{3}(m*n^{-1}) \lor \nu_{2}^{3}(m*n^{-1}) \\ & \leqslant & \left(\nu_{1}^{3}(m) \lor \nu_{1}^{3}(n)\right) \lor \left(\nu_{2}^{3}(m) \lor \nu_{2}^{3}(n)\right) \\ & = & \left(\nu_{1}^{3}(m) \lor \nu_{2}^{3}(m)\right) \lor \left(\nu_{1}^{3}(n) \lor \mu_{2}^{3}(n)\right) \\ & = & \nu^{3}(m) \lor \nu^{3}(n) \end{array}$$

Therefore $\xi = \xi_1 \cap \xi_2$ is a FFSG of (C, *). Hence intersection of two FFSGs of a group is also a FFSG of the group.

COROLLARY 3.1. Intersection of a family of FFSGs of a group (C, *) is also a FFSG of the group (C, *).

PROOF. Let $W = \{\xi_1, \xi_2, ..., \xi_f\}$ be a family of FFSG of (C, *). We have show that $\xi = \bigcap_{i=1}^{f} \xi_i$ is a FFSG of (C, *). Then $\xi = (\mu, \nu)$ is given by

$$\mu(m) = \mu_1(m) \land \mu_2(m) \land \ldots \land \mu_f(m) \text{ and } \nu(m) = \nu_1(m) \lor \nu_2(m) \lor \ldots \lor \nu_f(m)$$

for all $m \in C$. Now for all $m, n \in C$

$$\begin{split} \mu^{3}(m*n^{-1}) &= \mu_{1}^{3}(m*n^{-1}) \wedge \mu_{2}^{3}(m*n^{-1}) \wedge \dots \wedge \mu_{f}^{3}(m*n^{-1}) \\ &\geqslant (\mu_{1}^{3}(m) \wedge \mu_{1}^{3}(n)) \wedge (\mu_{2}^{3}(m) \wedge \mu_{2}^{3}(n)) \wedge \dots (\mu_{f}^{3}(m) \wedge \mu_{f}^{3}(n)) \\ &= (\mu_{1}^{3}(m) \wedge \mu_{2}^{3}(m) \wedge \dots \wedge \mu_{f}^{3}(m)) \wedge (\mu_{1}^{3}(n) \wedge \mu_{2}^{3}(n) \wedge \dots \wedge \mu_{f}^{3}(n)) \\ &= \mu^{3}(m) \wedge \mu^{3}(n) \end{split}$$

and

$$\begin{array}{ll} \nu^{3}(m*n^{-1}) &=& \nu_{1}^{3}(m*n^{-1}) \lor \nu_{2}^{3}(m*n^{-1}) \lor \ldots \lor \nu_{f}^{3}(m*n^{-1}) \\ &\leqslant& \left(\nu_{1}^{3}(m) \lor \nu_{1}^{3}(n)\right) \lor \left(\nu_{2}^{3}(m) \lor \nu_{2}^{3}(n)\right) \lor \ldots \left(\nu_{f}^{3}(m) \lor \nu_{f}^{3}(n)\right) \\ &=& \left(\nu_{1}^{3}(m) \lor \nu_{2}^{3}(m) \lor \ldots \lor \nu_{f}^{3}(m)\right) \lor \left(\nu_{1}^{3}(n) \lor \nu_{2}^{3}(n) \lor \ldots \lor \nu_{f}^{3}(n)\right) \\ &=& \nu^{3}(m) \lor \nu^{3}(n) \end{array}$$

Therefore $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). Hence intersection of a family of FFSGs of a group is also a FFSG of that group.

REMARK 3.2. Union of two FFSGs of a group may not be a FFSG of that group.

EXAMPLE 3.3. Let us take the group C = (Z, +), the group of integers under usual addition and let $\xi_1 = (\mu_1, \nu_1), \xi_2 = (\mu_2, \nu_2)$ be two FFSGs of C defined by

$$\mu_1(a) = \begin{cases} 0.3, when \ a \in 5Z \\ 0, elsewhere \end{cases}$$
$$\nu_1(a) = \begin{cases} 0, when \ a \in 5Z \\ 0.5, elsewhere \end{cases}$$
$$\mu_2(a) = \begin{cases} 0.15, when \ a \in 3Z \\ 0, elsewhere \end{cases}$$
$$\nu_2(a) = \begin{cases} 0.2, when \ a \in 3Z \\ 0.3, elsewhere \end{cases}$$

Let $\xi = \xi_1 \cup \xi_2 = (\mu, \nu)$, where

$$\mu(a) = \begin{cases} 0.3, when \ a \in 5Z \\ 0.15, when \ a \in 3Z - 5Z \\ 0, elsewhere \end{cases}$$
$$\nu(a) = \begin{cases} 0, when \ a \in 5Z \\ 0.2, when \ a \in 3Z - 5Z \\ 0.3, elsewhere \end{cases}$$

Here, $\mu^3(5 + (-3)) = \mu^3(2) = 0$, but $\mu^3(5) \wedge \mu^3(-3) = \min\{0.3^3, 0.15^3\} = 0.15^3$. So, $\mu^3(5 + (-3)) \not\geq \mu^3(5) \wedge \mu^3(-3)$ Again, $\nu^3(5 + (-3)) = \nu^3(2) = 0.027$, but $\nu^3(5) \vee \nu^3(-3) = \min\{0, 0.008\} = 0.008$. So, $\nu^3(5 + (-3)) \not\leq \nu^3(5) \vee \nu^3(-3)$ Hence, $\xi = \xi_1 \cup \xi_2 = (\mu, \nu)$ is not a FFSG of C = (Z, +).

PROPOSITION 3.3. If $\xi = (\mu, \nu)$ is a FFSG of a (C, *). Then $\mu^3(m^k) \ge \mu^3(m)$ and $\nu^3(m^k) \le \nu^3(m)$ for all $m \in C$ and $k \in N$. Here $m^k = m * m * ... * m(k \text{ times})$.

PROOF. Since $\xi = (\mu, \nu)$ is a FFSG of a (C, *), then $\mu^3(m * n) \ge \mu^3(m) \land \mu^(n)$ and $\nu^3(m * n) \le \nu^3(m) \lor \nu^(n)$ for all $m, n \in C$. So, $\mu^3(m^3) = \mu^3(m * n) \ge \mu^3(m) \land \mu^3(m) = \mu^3(m)$ and $\nu^3(m^3) = \nu^3(m * n) \le \nu^3(m) \lor \nu^3(m) = \nu^3(m)$ for all $m, n \in C$. Thus by induction, we can show that $\mu^3(m^k) \ge \mu^3(m)$ and $\nu^3(m^k) \le \nu^3(m)$ for all $m \in C$ and $k \in N$.

The next result produces the condition when equality occurs in the definition of FFSG.

PROPOSITION 3.4. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). If $\mu(m) \neq \mu(n)$ and $\nu(m) \neq \nu(n)$, then $\mu^3(m * n) = \mu^3(m) \wedge \mu^3(n)$ and $\nu^3(m * n) = \nu^3(m) \vee \nu^3(n)$ for all $m, n \in C$.

PROOF. Let us assume that $\mu(m) > \mu(n)$ and $\nu(m) < \nu(n)$. So, $\mu^3(m) > \mu^3(n)$ and $\nu^3(m) < \nu^3(n)$ for all $m, n \in C$. Now

$$\begin{split} \mu^{3}(n) &= \mu^{3}(m^{-1}*m*n) \\ &\geqslant \mu^{3}(m^{-1}) \wedge \mu^{3}(m*n) \\ &= \mu^{3}(m) \wedge \mu^{3}(m*n) \\ &\geqslant \mu^{3}(m*n), \text{otherwise } \mu^{3}(n) \geqslant \mu^{3}(m), \text{ a contradiction.} \end{split}$$

This show that $\mu^3(n) \ge \mu^3(m*n)$. Again $\mu^3(m*n) \ge \mu^3(m) \land \mu^3(n) = \mu^3(n)$. Therefore $\mu^3(m*n) \ge \mu^3(n)$. So, $\mu^3(m*n) = \mu^3(n) = \mu^3(m) \land \mu^3(n)$ for all $m, n \in C$. Similarly, when $\mu(m) < \mu(n)$ this result also holds. Also,

$$\begin{split} \nu^{3}(n) &= \nu^{3}(m^{-1}*m*n) \\ &\leqslant \nu^{3}(m^{-1}) \lor \mu^{3}(m*n) \\ &= \nu^{3}(m) \lor \nu^{3}(m*n) \\ &\leqslant \nu^{3}(m*n), \text{otherwise } \nu^{3}(n) \leqslant \nu^{3}(m), \text{ a contradiction.} \end{split}$$

Thus $\nu^3(n) \leq \nu^3(m*n)$. Again $\nu^3(m*n) \leq \nu^3(m) \vee \nu^3(n) = \nu^3(n)$. Therefore $\nu^3(m*n) \leq \mu^3(n)$. So, $\nu^3(m*n) = \nu^3(n) = \nu^3(m) \vee \nu^3(n)$ for all $m, n \in C$. Similarly, when $\nu(m) > \nu(n)$ this result also holds. Hence $\mu^3(m*n) = \mu^3(n) = \mu^3(m) \wedge \mu^3(n)$ and $\nu^3(m*n) = \nu^3(n) = \nu^3(m) \vee \nu^3(n)$, when $\mu(m) \neq \mu(n)$ and $\nu(m) \neq \nu(n)$ for all $m, n \in C$.

PROPOSITION 3.5. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *) with e as the identity element and $m \in C$. If $\mu^3(m) = \mu^3(e)$ then $\mu^3(m * n) = \mu^3(n)$ for all $m, n \in C$ and if $\nu^3(m) = \nu^3(e)$, then $\nu^3(m * n) = \nu^3(n)$ for all $m, n \in C$.

PROOF. Let us assume that $\mu^3(m) = \mu^3(e)$ and $\nu^3(m) = \nu(e)$. So, $\mu^3(m * n) \ge \mu^3(m) \land \mu^3(n) = \mu^3(e) \land \mu^3(n) = \mu^3(n)$, sine $\mu^3(e) \ge \mu^3(n)$ for all $n \in C$. Also,

$$\begin{split} \mu^{3}(n) &= \mu^{3}(m^{-1}*m*n) \\ &\geqslant \mu^{3}(m^{-1}) \wedge \mu^{3}(m*n) \\ &= \mu^{3}(m) \wedge \mu^{3}(n*n) \\ &= \mu^{3}(e) \wedge \mu^{3}(m*n) \\ &= \mu^{3}(m*n), \text{ since } \mu^{3}(e) \geqslant \mu^{3}(n) \text{ for all } m, n \in C. \end{split}$$

Therefore $\mu^3(m * n) = \mu^3(n)$ for all $n \in C$, where $\mu^3(m) = \mu^3(e)$

Again, $\nu^3(m*n) \leq \nu^3(m) \vee \nu^3(n) = \nu^3(e) \vee \nu^3(n) = \nu^3(n)$, since $\nu^3(e) \leq \nu^3(n)$ for all $n \in C$. In the same way we can show that $\nu^3(n) \leq \nu^3(m*n)$. Hence, $\nu^3(m*n) = \nu^3(n)$ for all $n \in C$, when $\nu^3(m) = \mu^3(e)$.

THEOREM 3.3. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). Then the set $N = \{m \in C | \mu^3(e) = \mu^3(m), \nu^3(e) = \nu^3(m)\}$ forms a subgroup of the group (C, *), where e is the identity element in C.

PROOF. Here $N = \{m \in C | \mu^3(e) = \mu^3(m), \nu^3(e) = \nu^3(m)\}$, clearly N is non empty, as $e \in N$. To show that (N, *) is a subgroup of (C, *), we have to show that $m * n^{-1} \in N$ for all $m, n \in C$. Let $m, n \in N$. Then $\mu^3(m) = \mu^3(e) = \mu^3(n)$, $\nu^3(m) = \nu^3(e) = \nu^3(n)$. Since $\xi = (\mu, \nu)$ is a FFSG of the group (C, *), then

$$\mu^{3}(m * n^{-1}) \geq \mu^{3}(m) \wedge \mu^{3}(n^{-1}) \\ = \mu^{3}(m) \wedge \mu^{3}(n) \\ = \mu^{3}(e) \wedge \mu^{3}(e) \\ = \mu^{3}(e).$$

Similarly, we can show that, $\nu^3(m*n^{-1}) \leq \nu^3(e)$. We have $\mu^3(e) \geq \mu^3(m*n^{-1})$ and $nu^3(e) \leq nu^3(m*n^{-1})$. Therefore $\mu^3(m*n^{-1}) = \mu^3(e)$ and $\nu^3(m*n^{-1}) = \nu^3(e)$, so $m*n^{-1} \in N$. Hence (N,*) is a subgroup of (C,*).

4. Fermatean fuzzy coset and Fermatean fuzzy normal subgroup

In this section, we will define Fermatean fuzzy coset and Fermatean fuzzy normal subgroup. Further, we will describe properties related to Fermatean fuzzy normal subgroup.

DEFINITION 4.1. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). Then for $m \in C$, the Fermatean fuzzy left coset of ξ is the FFS $m\xi = (m\mu, m\nu)$, defined by

$$(m\mu)^3(u) = \mu^3(m^{-1} * u), (m\nu)^3(u) = \nu^3(m^{-1} * u)$$

and the Fermatean fuzzy right coset of ξ is the FFS $m\xi = (\mu m, \nu m)$, defined by

$$(\mu m)^3(u) = \mu^3(u * m^{-1}), (\nu m)^3(u) = \nu^3(u * m^{-1})$$
 for all $u \in C$.

DEFINITION 4.2. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). Then ξ is a Fermatean fuzzy normal subgroup (FFNSG) of the group (C, *) if every Fermatean fuzzy left coset of ξ is also a Fermatean fuzzy right coset of ξ in C. Equivalently, $m\xi = \xi m$ for all $m \in C$.

EXAMPLE 4.1. Let us take the group $C = (Z_3, +_3)$, where $+_3$ is addition of integers modulo 3. Define a FFS $\xi = (\mu, \nu)$ on C by

$$\mu(0) = 0.9, \mu(1) = 0.7, \mu(2) = 0.7, \nu(0) = 0.1, \nu(1) = 0.2, \nu(2) = 0.2.$$

We can easily verify that $\xi = (\mu, \nu)$ is a FFSG of *C*. For $m = 1 \in C$, the Fermatean fuzzy left coset of ξ is the FFS $1\xi = (1\mu, 1\nu)$, defined by $(1\mu)^3(u) = \mu^3(1^{-1} + u), (1\nu)^3(u) = \nu^3(1^{-1} + u)$ and the Fermatean fuzzy right coset of ξ is the FFS $1\xi = (\mu 1, \nu 1)$, defined by

$$(\mu 1)^3(u) = \mu^3(u+_3 1^{-1})$$
 and $(\nu 1)^3(u) = \nu^3(u+_3 1^{-1})$ for all $u \in C$.

When u = 0,

$$(1\mu)^3(0) = \mu^3(1^{-1} + {}_3 0) = \mu^3(2 + {}_3 0) = \mu^3(2) = 0.04$$
$$(\mu 1)^3(0) = \mu^3(0 + {}_3 2) = \mu^3(2) = 0.04.$$

Also,

$$\begin{array}{l} (1\nu)^3(0) = \nu^3(1^{-1}+_3 0) = \nu^3(2+_3 0) = \nu^3(2) = 0.04 \\ (1\mu)^3(0) = \nu^3(0+_3 1^{-1}) = \mu^3(0+_3 2) = \mu^3(2) = 0.04. \end{array}$$

So, $(1\mu)^3(0) = (\mu 1)^3(0)$ and $(1\nu)^3(0) = (\nu 1)^3(0)$. Similarly, we can check that the result holds when u = 1 and 2. Therefore $(1\mu)^3(u) = (\mu 1)^3(u)$ and $(1\nu)^3(u) = (\nu 1)^3(u)$. In the same manner it can be shown that $m\xi = \xi m$ for all $m \in C$. Hence $\xi = (\mu, \nu)$ is a FFNSG of the group $(Z_3, +_3)$.

PROPOSITION 4.1. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). Then ξ is a FFNSG of C iff $\mu^3(m*n) = \mu^3(n*m)$ and $\nu^3(m*n) = \nu^3(n*m)$ for all $m, n \in C$.

PROOF. Let $\xi = (\mu, \nu)$ is a FFNSG of the group (C, *). Then $m\xi = \xi m$ for all $m \in C$. That is $(m\mu)^3(u) = (\mu m)^3(u)$ and $(m\nu)^3(u) = (\nu m)^3(u)$ for all $m \in C$. Therefore $\mu^3(m^{-1} * u) = \mu^3(u * m^{-1})$ and $\nu^3(m^{-1} * u) = \nu^3(u * m^{-1})$ for all $m, u \in C$. So, $\mu^3(m * n) = \mu^3(m * (n^{-1})^{-1}) = \mu^3((n^{-1})^{-1} * m) = \mu^3(n * m)$ and $\nu^3(m * n) = \nu^3(m * (n^{-1})^{-1}) = \nu^3((n^{-1})^{-1} * m) = \mu^3(n * m)$.

Conversely, $\mu^3(m*n) = \mu^3(n*m)$ and $\nu^3(m*n) = \nu^3(n*m)$ for all $m, n \in C$. This gives $\mu^3(m*(n^{-1})^{-1}) = \mu^3((n^{-1})^{-1}*m)$ and $\nu^3(m*(n^{-1})^{-1}) = \nu^3((n^{-1})^{-1}*m)$ for all $m, n \in C$. Put $n^{-1} = d$, $\mu^3(m*d^{-1}) = \mu^3(d^{-1}*m)$ and $\nu^3(m*d^{-1}) = \nu^3(d^{-1}*m)$ for all $m, n \in C$. So, $(\mu d)^3(m) = (d\mu)^3(m)$ and $(\nu d)^3(m) = (d\nu)^3(m)$ for all $m, d \in C$. This implies that $\mu d = d\mu$ and $\nu d = d\nu$ for all $d \in C$. Therefore $\xi d = d\xi$ for all $d \in C$. Hence ξ is a FFNSG of the group (C, *).

PROPOSITION 4.2. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). Then ξ is a FFNSG of C iff $\mu^3(k * u * k^{-1}) = \mu^3$ and $\nu^3(k * u * k^{-1}) = \nu^3$ for all $u, k \in C$.

PROOF. Let $\xi = (\mu, \nu)$ is a FFNSG of the group (C, *). Then $\mu^3(m * n) = \mu^3(n * m)$ and $\nu^3(m * n) = \nu^3(n * m)$ for all $m, n \in C$. Now for all $u, k \in C$,

$$\mu^{3}(k * u * k^{-1}) = \mu^{3} ((k * u) * k^{-1})$$

= $\mu^{3} (k^{-1} * (k * u))$ (using the above condition)
= $\mu^{3}(k^{-1} * k * u)$
= $\mu^{3}(e * u)$
= $\mu^{3}(u)$

Similarly, we get $\nu^3(k * u * k^{-1}) = \nu^3(u)$. Therefore $\mu^3(k * u * k^{-1}) = \mu^3(u)$ and $\nu^3(k * u * k^{-1}) = \nu^3(u)$ for all $u, k \in C$.

Conversely, let $\mu^3(k * u * k^{-1}) = \mu^3(u)$ and $\nu^3(k * u * k^{-1}) = \nu^3(u)$ for all $u, k \in C$. Now for all $m, n \in C$,

$$\mu^{3}(m*n) = \mu^{3}(n^{-1}*n*m*n)$$

$$= \mu^{3}((n^{-1})*(n*m)*(n^{-1})^{-1})$$

$$= \mu^{3}(n*m)$$
(using the above condition)

Similarly, we can get $\nu^3(m * n) = \nu^3(n * m)$ Therefore $\xi = (\mu, \nu)$ is a FFNSG of the group (C, *).

THEOREM 4.1. Let $\xi = (\mu, \nu)$ is a FFNSG of the group (C, *). Then the set $N = \{m \in C | \mu^3(e) = \mu^3(m), \nu^3(e) = \nu^3(m)\}$ forms a subgroup of the group (C, *), where e is the identity element in C.

PROOF. Clearly N is non empty, as $e \in N$. By Proposition 3.11 we have N is a subgroup of (C, *). Let $k \in C$ and $u \in N$. As $u \in N, \mu^3(e) = \mu^3(u)$ and $\mu^3(e) = \mu^3(u)$. Since $\xi = (\mu, \nu)$ is a FFNSG of the group (G, *), by above Proposition 4.5 we have

$$\mu^{3}(k * u * k^{-1}) = \mu^{3}(u)$$
 and $\nu^{3}(k * u * k^{-1}) = \nu^{3}(u)$ for all $u, k \in C$.

Consequently, $\mu^3(k * u * k^{-1}) = \mu^3(e)$ and $\nu^3(k * u * k^{-1}) = \nu^3(e)$ for all $u, k \in C$. Therefore $k * u * k^{-1} \in N$. Hence (N, *) is a normal subgroup of the group (C, *).

5. Fermatean fuzzy level subgroup

DEFINITION 5.1. Let C be a crisp set. Let $\xi = (\mu, \nu)$ be a FFS of the set C. For $\theta, \tau \in [0, 1]$, the set

$$\xi_{(\theta,\tau)} = \left\{ m \in C | \mu^3(m) \ge \theta, \nu^3 \le \tau \right\}$$

is called a Fermatean fuzzy level subset (FFLS) of the FFS ξ of C, where $0 \leq \theta^3 + \tau^3 \leq 1$.

PROPOSITION 5.1. Let $\xi' = (\mu', \nu')$ and $\xi'' = (\mu'', \nu'')$ be two FFSs of the universal set C. Then

 $\begin{array}{l} \text{(i)} \ \xi'_{(\theta,\tau)} \subseteq \xi'_{(\phi,\sigma)} \ \text{if} \ \phi \leqslant \theta \ \text{and} \ \tau \leqslant \sigma \ \text{for} \ \theta, \tau, \phi, \sigma \in [0,1], \\ \text{(ii)} \ \xi' \subseteq \xi'' \Rightarrow \xi'_{(\theta,\tau)} \subseteq \xi''_{(\theta,\tau)} \ \text{for} \ \theta, \tau \in [0,1]. \end{array}$

PROOF. (i) Let $m \in \xi'_{(\theta,\tau)} \Rightarrow (\mu')^3 \ge \theta, (\nu')^3 \le \tau$. We have $\phi \le \theta, \tau \le \sigma$. So, $\phi \le \theta \le (\mu')^3$ and $(\nu')^3 \le \tau \le \sigma$. Therefore $m \in \xi'_{(\phi,\sigma)}$. Hence $\phi \le \theta, \tau \le \sigma$ implies $\xi'_{(\theta,\tau)} \subseteq \xi'_{(\theta,\tau)}$.

(ii) Since $\xi' \subseteq \xi''$, so $\mu'(m) \leqslant \mu''(m)$ and $\nu'(m) \geqslant \nu''(m)$ for all $m \in C$. This implies $(\mu')^3(m) \leqslant (\mu'')^3(m)$ and $(\nu')^3(m) \geqslant (\nu'')^3(m)$ for all $m \in C$. Let $m \in \xi'_{(\theta,\tau)}$. Hence, it follows $(\mu')^3(m) \geqslant \theta$ and $(\nu')^3(m) \leqslant \tau$. So, $\theta \leqslant (\mu')^3(m) \leqslant (\mu'')^3(m)$ and $(\nu'')^3(m) \leqslant (\nu')^3(m) \leqslant \tau$. This shows that $\theta \leqslant (\mu'')^3(m)$ and $(\mu'')^3(m) \leqslant \tau$. Therefore $m \in \xi'_{(\theta,\tau)}$. Hence $\xi'_{(\theta,\tau)} \subseteq \xi''_{(\theta,\tau)}$.

PROPOSITION 5.2. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). Then the Fermatean fuzzy level subset $\xi_{(\theta, \tau)}$ forms a subgroup of the group (C, *), where $\theta \leq \mu^3(e)$ and $\tau \geq \nu^3(e)$, e is the identity element in C.

DEFINITION 5.2. The subgroup $\xi_{(\theta,\tau)}$ of the group (C,*) is called Fermatean fuzzy level subgroup (FFLSG) of the FFSG $\xi = (\mu, \nu)$.

PROPOSITION 5.3. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). If the FFLS $\xi_{(\theta,\tau)}$ is a subgroup of the group (C, *), where $\theta \leq \mu^3(e)$ and $\tau \geq \nu^3(e)$ then $\xi = (\mu, \nu)$ is a FFSG of the group (C, *).

PROOF. Let $m, n \in C$. Given that $\xi = (\mu, \nu)$ is a FFS of C and $\xi_{(\theta,\tau)}$ is a subgroup of the group (C, *). Let us assume that $\mu^3(m) = \theta_1, \mu^3(n) = \theta_2$ with $\theta_1 < \theta_2$ and $\nu^3(m) = \tau_1, \nu^3(n) = \tau_2$ with $\tau_1 < \tau_2$. This implies that $m \in \xi_{(\theta_1,\tau_1)}$ and $n \in \xi_{(\theta_2,\tau_2)}$. Since, $\theta_1 < \theta_2$ and $\tau_1 > \tau_2$, then $\xi_{(\theta_2,\tau_2)} \subseteq \xi_{(\theta_1,\tau_1)}$. Now $m \in \xi_{(\theta_1,\tau_1)}$ and $n \in \xi_{(\theta_1,\tau_1)}$. So, $m * n \in \xi_{(\theta_1,\tau_1)}$, since $\xi_{(\theta_1,\tau_1)}$ is a subgroup of (C, *). Therefore $\mu^3(m * n) \ge \theta_1$ and $\nu^3(m * n) \le \tau_1$. From here, it follows $\mu^3(m * n) \ge \theta_1 \land \theta_2$ and $\nu^3(m * n) \le \tau_1 \lor \tau_2$. Also, from here, it follows $\mu^3(m * n) \ge \mu^3(m) \land \mu^3 n$ and $\nu^3(m * n) \le \nu^3(m) \lor \nu^3(n)$.

Again, $m \in \xi_{(\theta_1,\tau_1)}$ umplies $m^{-1} \in \xi_{(\theta_1,\tau_1)}$, since $\xi_{(\theta_1,\tau_1)}$ is a subgroup of (C,*). From here, it implies $\mu^2(m^{-1}) \ge \theta_1$ and $\nu^2(m^{-1}) \le \tau_1$. Now, $\mu^2(m^{-1}) \ge \mu^3(m)$ and $\nu^2(m^{-1}) \leq \nu^3(m)$. Since $m, n \in C$ are arbitrary, $\mu^3(m*n) \geq \mu^3(m) \wedge \mu^3 n$ and $\nu^3(m*n) \leq \nu^3(m) \vee \nu^3 n$ for all $m, n \in C$. $\mu^2(m^{-1}) \geq \mu^3(a), \nu^2(m^{-1}) \leq \nu^3(m)$ for all $m, n \in C$. Hence, $\xi = (\mu, \nu)$ is a FFSG of the group (C, *).

PROPOSITION 5.4. Let $\xi = (\mu, \nu)$ is a FFSG of the group (C, *). Then the Fermatean fuzzy level subset $\xi_{(\theta, \tau)}$ forms a subgroup of the group (C, *), where $\theta \leq \mu^3(e)$ and $\tau \geq \nu^3(e)$, e is the identity element in C.

6. Homomorphism on Fermatean fuzzy subgroup

In this section, we will discuss the effect of group homomorphism on Fermatean fuzzy subgroup.

THEOREM 6.1. Let $(C_1, *_1)$ and $(C_2, *_2)$ be two groups. Let g be a surjective homomorphism from $(C_1, *_1)$ to $(C_2, *_2)$ and $\xi = (\mu, \nu)$ is a FFSG of the group $(C_1, *_1)$. Then $g(\xi) = (g(\mu), g(\nu))$ is a FFSG of $(C_2, *_2)$.

PROOF. Since $g: C_1 \to C_2$ is a surjective homomorphism, then $g(C_1) = C_2$. Let m_2 and n_2 be two elements of C_2 . Suppose $m_2 = g(m_1)$ and $n_2 = g(n_1)$ for some $m_1, n_1 \in C_1$. We have $g(\xi) = \{(c, g(\mu)(c), g(\nu)(c)) | c \in C_2\}$. Now

$$(g(\mu))^{3}(m_{2} *_{2} n_{2}) = \{ (g(\mu))(m_{2} *_{2} n_{2}) \}^{3}$$

$$= [\lor \{\mu(t)|t \in C_{1}, g(t) = m_{2} *_{2} n_{2} \}]^{3}$$

$$= \lor \{\mu^{3}(t)|t \in C_{1}, g(t) = m_{2} *_{2} n_{2} \}$$

$$\geqslant \lor \{\mu^{3}(m_{1} *_{1} n_{1})|m_{1}, n_{1} \in C_{1} \text{ and } g(m_{1}) = m_{2}, g(n_{1}) = n_{2} \}$$

$$\geqslant \lor \{\mu^{3}(m_{1}) \land \mu^{3}(n_{1})|m_{1}, n_{1} \in C_{1} \text{ and } g(m_{1}) = m_{2}, g(n_{1}) = n_{2} \}$$

$$= (\lor \{\mu^{3}(m_{1}|m_{1} \in C_{1} \text{ and } g(m_{1}) = m_{2})\})$$

$$\land (\lor \{\mu^{3}(n_{1}|n_{1} \in C_{1} \text{ and } g(n_{1}) = n_{2})\})$$

$$= \{(g(\mu))(m_{2})\}^{3} \land \{(g(\mu))(n_{2})\}^{3}$$

$$= (g(\mu))^{3}(m_{2}) \land (g(\mu))^{3}(n_{2}).$$

Therefore $(g(\mu))^3(m_2 *_2 n_2) \ge (g(\mu))^3(m_2) \land (g(\mu))^2(n_2)$ for all $m_2, n_2 \in C_2$. Similarly, we can prove that $(g(\nu))^3(m_2 *_2 n_2) \le (g(\nu))^3(m_2) \lor (g(\nu))^2(n_2)$ for all $m_2, n_2 \in C_2$. Again

$$(g(\mu))^{3}(m_{2}^{-1}) = \{g(\mu)(m_{2}^{-1})\}^{3}$$

= $[\lor \{\mu(m)|m \in C_{1} \text{ and } g(m) = (m_{2}^{-1})\}]^{3}$
= $[\lor \{\mu(m^{-1})|m^{-1} \in C_{1} \text{ and } g(m^{-1}) = (m_{2})\}]^{3}$
= $\{g(\mu)(m_{2})\}^{3}$
= $(g(\mu))^{3}(m_{2}).$

Therefore $(g(\mu))^3(m_2^{-1}) = (g(\mu))^3(m_2)$ for all $m_2 \in C_2$. Similarly, we can show that $(g(\nu))^3(m_2^{-1}) = (g(\nu))^3(m_2)$ for all $m_2 \in C_2$. Hence $g(\xi) = (g(\mu), g(\nu))$ is a FFSG of $(C_2, *_2)$.

THEOREM 6.2. Let $(C_1, *_1)$ and $(C_2, *_2)$ be two groups. Let g be a bijective homomorphism from $(C_1, *_1)$ to $(C_2, *_2)$ and $\xi = (\mu, \nu)$ is a FFSG of the group $(C_2, *_2)$. Then $g^{-1}(\xi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a FFSG of $(C_1, *_1)$.

THEOREM 6.3. Let $(C_1, *_1)$ and $(C_2, *_2)$ be two groups. Let g be a surjective homomorphism from $(C_1, *_1)$ to $(C_2, *_2)$ and $\xi = (\mu, \nu)$ is a FFNSG of the group $(C_1, *_1)$. Then $g(\xi) = (g(\mu), g(\nu))$ is a FFNSG of $(C_2, *_2)$.

THEOREM 6.4. Let $(C_1, *_1)$ and $(C_2, *_2)$ be two groups. Let g be a bijective homomorphism from $(C_1, *_1)$ to $(C_2, *_2)$ and $\xi = (\mu, \nu)$ is a FFNSG of the group $(C_2, *_2)$. Then $g^{-1}(\xi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a FFNSG of $(C_1, *_1)$.

PROOF. We can state that $g^{-1}(\xi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a FFNSG of $(C_1, *_1)$. Since $\xi = (\mu, \nu)$ is a FFNSG of the group $(C_2, *_2)$.

$$\mu^{3}(m_{2} *_{2} n_{2}) = \mu^{3}(n_{2} *_{2} m_{2}) \text{ and } \nu^{3}(m_{2} *_{2} n_{2}) = \nu^{3}(n_{2} *_{2} m_{2}) \text{ for all } m_{2}, n_{2} \in C_{2}.$$

Let m_1 and n_1 be two elements of C_1 . Then

$$(g^{-1}(\mu))^{3}(m_{1} *_{1} n_{1}) = \{g^{-1}(\mu)(m_{1} *_{1} n_{1})\}^{3}$$

$$= \{\mu(g(m_{1} *_{1} n_{1}))\}^{3}$$

$$= \mu^{3}(g(m_{1} *_{1} n_{1}))$$

$$= \mu^{3}((g(m_{1})) *_{2} g(n_{1})) \text{ (Since } g \text{ is a homomorphism)}$$

$$= \mu^{3}(g(n_{1}) *_{2} g(m_{1}))$$

$$= \mu^{3}(g(n_{1} *_{1} m_{1}))$$

$$= \{g^{-1}(\mu)(n_{1} *_{1} m_{1})\}^{3}$$

$$= (g^{-1}(\mu))^{3}(n_{1} *_{1} m_{1}).$$

Therefore $(g^{-1}(\mu))^3(m_1*_1n_1) = (g^{-1}(\mu))^3(n_1*_1m_1)$ for all $m_1, n_1 \in C_1$. Similarly, we can prove that $(g^{-1}(\nu))^3(m_1*_1n_1) = (g^{-1}(\nu))^3(n_1*_1m_1)$ for all $m_1, n_1 \in C_1$. Hence $g^{-1}(\xi) = (g^{-1}(\mu), g^{-1}(\nu))$ is a FFNSG of $(C_1, *_1)$.

7. Conclusion

The purpose of this paper is to initiate the study of Fermatean fuzzy subgroup. We have discussed various algebraic attributes of Fermatean fuzzy subgroup. We have proved that intuitionistic fuzzy subgroup and Pythagorean fuzzy subgroup of any group is a Fermatean fuzzy subgroup of that group. We have introduced the notion of Fermatean fuzzy coset and Fermatean fuzzy normal subgroup. We have presented the necessary and sufficient condition for Fermatean fuzzy subgroup to be a Fermatean fuzzy normal subgroup. Further, we have proved that Fermatean fuzzy level subset is a normal subgroup of the given group. Moreover, we have studied the effect of group homomorphism on Fermatean fuzzy subgroup.

In our future work, we will work on Fermatean fuzzy quotient group and order of Fermatean fuzzy subgroup. We will also work on the Lagrange theorem in Fermatean fuzzy subgroup.

FERMATEAN FUZZY SUBGROUPS

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