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UP–ALGEBRA WITH APARTNESS

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Abstract. The environment of this article is the Bishop's constructive mathematics - a mathematics based on the Intuitionisic logic. In this paper, in leaning on published article: A. Iampan. A new branch the logical algebra: UP-Algebras. *J. Algebra Rel. Topics*, **5**(1)(2017), 35–54, we introduce the concept of a new algebraic structure, called an 'UP-algebra with apartness' and concepts of UP-ideals, UP-coideals, UP-filters and UP-cofilters, co-congruences and strongly extensional UP- homomorphisms in UP-algebras with apartness. In addition, we investigated some related properties of them.

1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras, BCI-algebras, BCH-algebras, SU-algebras, KU-algebras and others. In 2017, the notion of a UP-algebra was first introduced by A. Iampan in his paper [**4**]. In the aforementioned article, the author introduced and analyzed the concepts of UP-ideal, UP-congruence and UP-homomorphism, also. This logical-algebraic concept has been the subject of considerable research (See, for example [**5, 6, 10, 11, 12, 14, 21, 22, 23, 27, 28, 29, 30, 31, 34, 36**]). The concept of UP-filters in this class of logical algebras was introduced by Somjanta et al. et al. in [**35**]. After that, Iampan and Jun introduced classes of comparative, implicative and shift UP-filters in UP-algebras ([**10, 11, 12**]). This author took part in the analysis of filters in UP-algebrs: he introduced the concept of proper UP-filters and (together with Y. B. Jun) the concept of weak implicative UP-filters ([**29**]). The concept of meet-commutative UP-algebras was introduced in article [**33**]. In article [**15**], a number of important

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properties of meet-commutative UP-algebras are given. In the last mentioned paper, the concepts of prime UP-filters and irreducible UP-filters in such UP-algebras are introduced and analyzed. In addition, in such UP-algebras, the concept of prime of the second kind ([**32**]) and prime of the third kind ([**28**]) and weakly irreducible UP-filters ([**31**]) was introduced and analyzed.

In this paper, we introduce (Definition 3.3) a new algebraic structure, called a UP- algebra with apartness, in analogy with the (classical) concept of UP-algebra and based on the already published articles [**25, 26**] on some other classes of logical algebras with apartness. In additional, we introduce and analyze the concept of UP-coideal, UP-cofilter, UP-cocongruence and UP-homomorphisms in UP-algebras with apartness. Given a UP-coideal K of a UP-algebra with apartness A allows us (Theorem 5.1) to design a co-congruence q_K on A which allows us (Theorem 5.3) to construct a UP-algebra $[A : q_K]$ which has no counterpart in the classical theory of UP-algebras.

The environment in which this research was realized is the Intuitionistic logic [**37**]) and the principled-philosophical orientation of the Bishop's Constructive mathematics ([**1, 2, 7**]). Since the principle of TNT (the logical principle of exclusion of the third) is not valid in the Intuitionistic logic, the concept of the set is treated as a relational system with two associate relations, an equality and a diversity, on a carrier. Therefore, the properties of algebraic structures in a constructive algebra can be determined not only by equality, but also by diversity.

2. Preliminaries

Our setting is Bishop's constructive mathematics ([**1, 2, 7, 8, 13**] and [**37**]), mathematics developed with Constructive logic (or Intuitionistic logic [**37**]) - logic without the Law of Excluded Middle $P \vee \neg P$ [TND]. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in the Minimal logic. In Constructive logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold as well. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'.

Dual of the equality relations '=' in a set A is diversity relation '$\neq$'. This last relation is extensive in terms of equality in the following sense:

$$= \circ \neq \subseteq \neq \quad \text{and} \quad \neq \circ = \subseteq \neq .$$

It is obvious that the following connection between these relations is valid:

$$= \subseteq \neg \neq$$

. In this case for relations $=$ and $\neq$ we say that they are associate. So, it's quite natural to ask the question: Is there the maximal relation '$\neq$' such that it is associated with equality '='?

Generally speaking: Let S be a subset of set $(A, =, \neq)$ determined by a predicate $\mathfrak{P}$. The first task is to construct a dual T of the set S so that the subsets

$\neg T = \{a \in A : \neg(a \in T)\}$ and its strong compliment $T^{\lhd} = \{a \in A : a \lhd T\}$ have property $\mathfrak{P}$. In addition, $T^{\lhd} \subseteq \neg T$ holds.

Let ρ be an equality relation on the set A. For the relation q we say that it is a *co-equality* relation to A if and only if the following is valid

$$q \subseteq \neq,\ q^{-1} = q,\ q \subseteq q * q.$$

Here, '$*$' is the filed product between relations defined by the following way: If α and β are relations on set A, then filed product $\beta * \alpha$ of relation α and β is the relation given by $\{(x,z) \in A \times A : (\forall y \in A)((x,y) \in \alpha \vee (y,z) \in \beta)\}$. Of course, the strong compliment $q^{\lhd}$ of the relation q is an equivalence in A and the following

$$q^{\lhd} \subseteq \neg q, \quad q \circ q^{\lhd} \subseteq q \text{ and } q^{\lhd} \circ q \subseteq q$$

are valid (see, for example, Proposition 1.1 in [**24**]). In addition, the relation q called *co-congruence* on a grupoid $(A, =, \neq, \cdot)$ if it is strongly extensional (or, it is cancellative with respect to apartness).

For couple ρ and q, of a relation of equivalence ρ and a relation of co-equivalence q, we say it is associated if the following inclusions

$$\rho \circ q \subseteq q \text{ and } q \circ \rho \subseteq q$$

are valid.

This investigation is in Bishop's constructive algebra in sense of papers [**3, 9, 17, 18, 19, 20**] and books [**13**], [**37**] (Chapter 8: Algebra). In this section, we will describe with more detail of basic concepts in constructive algebra and their properties.

Let $(A, =, \neq)$ be a constructive set. The diversity relation "$\neq$" is a binary relation on A, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ (x \neq y \wedge y = z) \Longrightarrow x \neq z.$$

If it satisfies the following condition

$$x \neq z \Longrightarrow (\forall y \in A)(x \neq y \vee y \neq z),$$

it is called *apartness* (A. Heyting). The apartness relation in a set A should not be regarded as a negation of the equality relation in the set A. It needs to be accept as one extensive relation on the set A. This relation on the set A is a dual to the equality relation in A.

For subsets X and Y of A we say that set X is set-set apartness from Y, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. We set $x \lhd Y$ instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $X^{\lhd} = \{x \in A : x \bowtie X\}$ we denote apartness complement of X in A. So, "$\bowtie$" is a relation between pairs of subsets of A. and the relation "$\lhd$" is a relation between elements and a set.

For a function $f : (X, =, \neq) \longrightarrow (Y, =, \neq)$ we say that it is a strongly extensional if and only if $(\forall a, b \in X)(f(a) \neq f(b) \Longrightarrow a \neq b)$. A total strongly extensional function $w : X \times X \longrightarrow X$ is an internal operation in X and the couple (X, w) is a grupoid. It is understood that the following implications are valid

$$(\forall x, y, u, v \in X)((x,y) = (u,v) \implies w(x,y) = w(u,v)),$$
$$(\forall x, y, u, v \in X)(w(x,y) \neq w(u,v) \implies (x,y) \neq (u,v)).$$

The second implication can be written in the following way

$$(\forall x, y, z \in X)((w(x,z) \neq w(y,z) \vee w(z,x) \neq w(z,y)) \Longrightarrow x \neq y).$$

Speaking of the classical algebraic language, in this case we talking that the operation w is left and right cancellative with respect to apartness.

Since in the Constructive logic the logical principe 'Law of Excluded Middle' is not valid, in Bishop's constructive algebra the following relation is also interesting - a relation symmetric to order relation $\leqslant$. A relation θ on A is *co-order* ([**19, 20**]) on set A if and only if

$$\theta \subseteq \neq \text{ (consistency)}, \theta \subseteq \theta * \theta \text{ (co-transitivity)}, \neq \subseteq \theta \cup \theta^{-1} \text{ (linearity)}.$$

The relation θ is left strongly extensional with respect to the internal operation or left cancellative if

$$(\forall x, y, z \in A)((x \cdot z, y \cdot z) \in \theta \Longrightarrow (x, y) \in \theta))$$

and right strongly extensional with respect to the internal operation or right cancellative if

$$(\forall x, y, z \in A)(z \cdot x, z \cdot y) \in \theta \Longrightarrow (x, y) \in \theta)).$$

holds.

A system $(A, =, \neq, w, \theta)$ is an ordered grupoid under co-order θ if $(A, =, \neq, w)$ is a grupoid where the internal operation w is strongly extensional and the relation θ is a co-order relation on $(A, =, \neq)$.

At the end of this section we remind readers about the notion 'co-quasiorder relation'. For a relation determined in A we say ([**9, 16, 17, 18, 19, 20**]) that it is a *co-quasiorder relation* in A if it is a consistent and co-transitive. If $\not\leqslant$ is a co-quasiorder relation in set A, then the relation $q = \not\leqslant \cup \not\leqslant^{-1}$ is a co-equality relation in A ([**16**], Lemma 0).

3. UP-Algebra with Apaertness

We begin this section with the indication of the UP-algebra definition taken from article [**4**]

3.1. UP-algebra in the classical sense.

Definition 3.1. ([**4**], Definition 1.3) An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a (classical) *UP- algebra* if it satisfies the following axioms:
(UP - 1): $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP - 2): $(\forall x \in A)(0 \cdot x = x)$,
(UP - 3): $(\forall x \in A)(x \cdot 0 = 0)$,
(UP - 4): $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

A reader can founds several examples of this algebraic structure in the aforementioned article [**4**]: Examples 1.4-1.6.

The following propositions are very important for the study of UP-algebras.

Proposition 3.1 ([**4**], Proposition 1.7). *In a UP-algebra A, the following properties hold:*
(1) $(\forall x \in A)(x \cdot x = 0)$,
(2) $(\forall x, y, z \in A)((x \cdot y = 0 \wedge y \cdot z = 0) \Longrightarrow x \cdot z = 0)$,
(3) $(\forall x, y, z \in A)(x \cdot y = 0 \Longrightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,
(4) $(\forall x, y, z \in A)(x \cdot y = 0 \Longrightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,
(5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,
(6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \iff x = y \cdot x)$,
(7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$.

Remark 3.1. Here one should pay reader's attention that assertions (1) - (5) and (7) are proved without reference to the axiom (UP - 4).

The order relation $\leqslant$ in an UP-algebra $(A, \cdot, 0)$ is introduced by the following definition

Definition 3.2. ([**4**]) $(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0)$.

The features of this relationship are given by Proposition 1.8 in the article [**4**].

3.2. UP-Algebra with Apartness. In this subsection, we will introduce the concept of 'UP-algebras with apartness'.

Definition 3.3. An algebra $A = ((A, =, \neq), \cdot, 0)$ of type $(2, 0)$, where the internal operation $\cdot$ is strongly extensional, is called a (constructive) *UP- algebra with apartness* if it satisfies the following axioms:
(UP - 1): $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP - 2): $(\forall x \in A)(0 \cdot x = 0)$,
(UP - 3): $(\forall x \in A)(x \cdot 0 = x)$,
(UP - 4a): $(\forall x, y \in A)(x \neq y \Longrightarrow (x \cdot y \neq 0 \vee y \cdot x \neq 0))$.

Without difficulties, the following statement can be proven

Proposition 3.2. *If the apartness relation in the set A is tight, then* (UP - 4a) *implies* (UP - 4).

Proof. Let $x \cdot y = 0 \wedge y \cdot x = 0$ be holds and suppose that $x \neq y$. Thus, by (UP - 4a) we have $x \cdot y \neq 0 \vee y \cdot x \neq 0$. It is in a contradiction with the hypothesis. So, $\neg(x \neq y)$ and $x = y$ since the apartness is tight. Therefore, the formula (UP - 4) is valid. □

Remark 3.2. The logical environment in which these algebraic structures are analyzed is the Intuitionistic Logic. That's why the opposition implication of the previous implication is not valid in the general case.

Remark 3.3. We remind readers that they should accept that the assertions (1) - (5) and (7) presented in Proposition 3.1 also are valid in a UP algebra with apartness. This statement should be linked to our comment in the Remark 3.1.

In the following definition we introduce a co-order in a UP-algebra with apartness.

Definition 3.4. $(\forall x, y \in A)(x \nleqslant y \iff x \cdot y \neq 0)$.

In the following three claims, we will show two elementary characteristics of this relation.

Statement 1. *For relations* $\leqslant$ *and* $\nleqslant$*, defined as above, holds*

$$(\forall x, y \in A)\neg(x \leqslant y \wedge x \nleqslant y).$$

Statement 2. *For relations* $\leqslant$ *and* $\nleqslant$*, defined as above, holds*

$$(\forall x, y \in A)((y \leqslant x \wedge y \nleqslant z) \implies x \nleqslant z).$$

Proof. Let $x, y, z \in A$ be elements such that $y \leqslant x$ and $y \nleqslant z$. Then $y \nleqslant x$ or $x \nleqslant z$ by co-transitivity of $\nleqslant$. Thus $x \nleqslant z$ becuase the option $y \nleqslant x$ is impossible. □

Statement 3. *For relation* $\nleqslant$*, defined as above, holds*

$$(\forall x, y \in A)(x \nleqslant y \implies x \neq y).$$

Proof. Let for elements x and y holds $x \nleqslant y$. Thus $x \cdot y \neq 0$. Then, from $x \cdot y \neq 0 = y \cdot y$ immediately follows $x \neq y$ by cancelativity of the internal operation in A with respect to the apartness. □

In the following theorem we describe some special properties of this relation.

Theorem 3.1. *In a UP-algebra with apartness* $((A, =, \neq), \cdot, 0)$ *the following properties hold*

(1a) $(\forall x \in A)\neg(x \nleqslant x)$,
(2a) $(\forall x, y \in A)(x \neq y \implies x \nleqslant y \vee y \nleqslant x)$,
(3a) $(\forall x, y, z \in A)(x \nleqslant z \implies x \nleqslant y \vee y \nleqslant z)$,
(4a) $(\forall x, y, z \in A)(z \cdot x \nleqslant z \cdot y \implies x \nleqslant y)$,
(5a) $(\forall x, y, z \in A)(y \cdot z \nleqslant x \cdot z \implies x \nleqslant y)$,
(6a) $(\forall x, y \in A)(x \nleqslant^{\triangleleft} y \cdot x)$,
(7a) $(\forall x, y \in A)(x \nleqslant^{\triangleleft} y \cdot y)$.

Proof. (1a) Let x be an arbitrary element of UP-algebra with apartness. Because $x \cdot x = 0$ holds, we have $\neg(x \cdot x \neq 0)$. So, $\neg(x \nleqslant x)$ is valid.

(2a) The statement (2a) is the (UP-4a) axiom only written by the co-order relation $\nleqslant$.

(3a) Let $x, yz \in A$ be arbotrary elements of A such that $x \nleqslant z$. It means $x \cdot z \neq 0$. Thus $x \cdot z \neq (y \cdot z) \cdot (x \cdot z) \vee (y \cdot z) \cdot (x \cdot z) \neq 0$.
The first case $0 \cdot (x \cdot z) \neq (y \cdot z) \cdot (x \cdot z)$ gives $0 \neq y \cdot z$.
The second case $(y \cdot z) \cdot (x \cdot z) \neq 0$ gives

$$0 \neq (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \ \vee \ (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \neq (y \cdot z) \cdot (x \cdot z).$$

Since the first option is impossible by (UP - 1), from the second option we got $(x \cdot y) \cdot (x \cdot z) \neq (x \cdot z) = 0 \cdot (x \cdot z)$ and $x \cdot y \neq 0$. Therefore, we are proved the implication $x \nleqslant z \implies x \nleqslant y \vee y \nleqslant z$.

(4a) Suppose $z \cdot x \nleqslant z \cdot y$. Thus $(z \cdot x)(z \cdot y) \neq 0$. From this, by co-transitivity, we have

$$0 \cdot ((z \cdot x)(z \cdot y)) \neq (x \cdot y) \cdot ((z \cdot x)(z \cdot y)) \ \vee \ (x \cdot y) \cdot ((z \cdot x)(z \cdot y)) \neq 0.$$

Since the second option is impossible, from the first case we give $0 \neq x \cdot y$.

(5a) Let $y \cdot z \nleqslant x \cdot z$ be holds. This means $(y \cdot z) \cdot (x \cdot z) \neq 0$. Thus

$$(y \cdot z) \cdot (x \cdot z) \neq (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \ \vee \ (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \neq 0.$$

Since the second option in previous disjunction is impossible, from the first option we have $0 \cdot (x \cdot z) \neq (x \cdot z) \cdot (x \cdot z)$ and finally $0 \neq x \cdot y$.

(6a) Let u, v, x, y be arbitrary elements of A such that $u \nleqslant v$. Thus $u \nleqslant x \vee x \nleqslant y \cdot x \ \vee \ y \cdot x \nleqslant v$. Since the second option is impossible by the statement (5) in Proposition 1.8 in [**4**], we have $u \neq x$ or $y \cdot x \neq v$. Then $(x, y \cdot x) \neq (u, v)$. Therefore $x \nleqslant^{\lhd} y \cdot x$ holds.

(7a) Let u, v, x, y be arbitrary elements of A such that $u \nleqslant v$. Thus $u \nleqslant x \vee x \nleqslant y \cdot y \vee y \cdot y \nleqslant u$. Since the option $x \nleqslant y \cdot y$ is impossible because $x \cdot (y \cdot y) = x \cdot 0 = 0$ we have $u \neq x$ or $y \cdot y \neq v$. Therefore, we have $(x, y \cdot y) \neq (u, v) \in \nleqslant$. □

So, the relation $\nleqslant$, introduced in Definition 3.4, is a co-order on the set $(A, =, \neq)$ left cancellative (the formula (4a)) and right anti-cancellative (the formula (5a)) with respect to the internal operation in the UP-algebra $((A, =, \neq), \cdot, 0)$ with apartness.

4. UP-substructues

4.1. UP-idels and UP-coideals. Firstly, let's recall how the concept of UP-ideals was introduced into UP-algebra in the initials text.

Definition 4.1. ([**4**], Definition 2.1) Let A be a UP-algebra. A subset J of A is called a *UP-ideal* of A if it satisfies the following properties:
(1) $0 \in J$, and
(2) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in J \wedge y \in J \implies x \cdot z \in J)$.

An example of UP-ideal in an UP-algebra readers can seen in Example 2.2 of the article [**4**].

In the following definition we introduce the concept of UP-coideals in an UP-algebra with apartness.

Definition 4.2. Let $((A, =, \neq), \cdot, 0)$ be a UP-algebra with apparntess. A subset K of A is called a *UP-coideal* of A if it satisfies the following properties:
(1) $0 \lhd K$, and
(2) $(\forall x, y, z \in A)(x \cdot z \in K \implies (x \cdot (y \cdot z) \in K \vee y \in K))$.

Clearly, A and $\emptyset$ are so-call trivial UP-coideals of A.

By the following theorem we describe some fundamental properties of UP-coideals in a UP-algebra.

THEOREM 4.1. *Let $A = ((A, =, \neq), \cdot, 0)$ be a UP-algebra with apartness and K an UP-coideal of A. Then the following statements hold:*
(1) $(\forall x, y \in A)(x \in K \implies (y \in K \vee y \cdot x \in K))$,
(2) $(\forall x, y \in A)(x \cdot y \in K \implies y \in K)$,

PROOF. Let x, y be arbitrary elements of A such that $x \in K$. Then $0 \cdot x = x \in K$. Thus, by definition of the UP-coideal K of UP-algebra A, we have $0 \cdot (y \cdot x) \in K$ or $y \in K$. Finally, we have $y \cdot x \in K \vee y \in K$.

Let x, y be arbitrary elements of A such that $x \cdot y \in K$. Thus $x \cdot (y \cdot y) \in K \vee y \in K$ by definition of UP-coideal. Since the first option is impossible because $x \cdot (y \cdot y) = 0 \lhd K$ we have the option to $y \in K$. □

COROLLARY 4.1. *A UP-coidea K of UP-algebra A with apartness is a strongly extensional subset of A.*

PROOF. Let $x, y \in A$ arbitrary elements such that $y \in K$. Thus follows $x \in K$ or $x \cdot y \in K$ by Theorem 4.1 (1). From the second case follows $x \cdot y \neq 0 = y \cdot y$ and $x \neq y$. Therefore, the implication $y \in K \implies (x \in K \vee x \neq y)$ is valid. So, the subset K is a strongly extensional subset in A. □

COROLLARY 4.2. *Let K be a UP-coideal of a UP-with apartness. Then the set $K^{\lhd}$ is a UP-ideal of A.*

PROOF. Let $x, y, z, u \in A$ arbitrary elements such that $y \in K^{\lhd}$, $x \cdot (y \cdot z) \in K^{\lhd}$ and $u \in K$. Since the UP-coideal K is a strongly extensional subset of A, thus $x \cdot z \neq u \vee x \cdot z \in K$. From the second option, by definition of UP-coideal K, we conclude $x \cdot (y \cdot z) \in K \vee y \in K$. Both cases are impossible by the assumptions. So, we have $x \cdot z \neq u \in K$. Since u was arbitrary element we got $x \cdot z \in K^{\lhd}$. Since it is obvious that $0 \in K^{\lhd}$ holds, we have proven that set $K^{\lhd}$ are a UP-ideal in A. □

COROLLARY 4.3. *Let A be an UP-algebra with apartness and K an UP-coideal of A. Then*
(3) $(\forall x, y \in A)(x \in K \implies (y \in K \vee y \nleqslant x))$.

PROOF. Let x, y be arbitrary elements of A such that $x \in K$. Thus, by Theorem 4.1 (1), we have $y \in K \vee y \cdot x \in K$ and $y \in K \vee y \cdot x \neq 0$. Then we have $y \in K \vee y \nleqslant x$. □

COROLLARY 4.4. *Let A be an UP-algebra with apartness and K an UP-coideal of A. Then*
(4) $(\forall x, a, b \in A)(x \in K \implies (a \in K \vee b \in K \vee b \nleqslant a \cdot x))$.

PROOF. Let $x, a, b \in A$ be arbitrary elements such that $x \in K$. Thus $a \in K$ or $a \cdot x \in K$ be theorem 4.1(1). Again, by Theorem 4.1(1), we have

$$b \in K \vee a \in K \vee b \cdot (a \cdot x) \in K.$$

Since $0 \lhd K$, finally we have $b \in K \vee a \in K \vee b \nleqslant a \cdot x$. □

Theorem 4.2. *Let A be an UP-algebra with apartness and $\{K_i\}_{i\in I}$ a family of UP- coideals of A. Then $\bigcup_{i\in I} K_i$ is a UP-coideal of A.*

Proof. Let $x, y, z \in A$ be arbitrary elements such that $x \cdot z \in \bigcup_{i\in I} K_i$. Thus, there exists an index $i \in I$ such that $x \cdot z \in K_i$. Since K_i is a UP-coideal of A we have $x \cdot (y \cdot z) \in K_i$ or $y \in K_i$. Then $x \cdot (y \cdot z) \in \bigcup_{i\in I} K_i$ or $y \in \bigcup_{i\in I} K_i$. Hence, $\bigcup_{i\in I} K_i$ is a UP-coideal of A. □

Corollary 4.5. *The family of all UP-coideals of UP-algebra A forms join semi-lattice.*

Remark 4.1. Let $\{K_i\}_{i\in I}$ a family of UP- coideals of A and let $\mathfrak{K}$ be family of all UP-coideals included in the intersection $\bigcap_{i\in I} K_i$. By the previous theorem, $\bigcup\{K : K \in \mathfrak{K}\}$ is the maximal UP-coideal included in the intersection $\bigcap_{i\in I} K_i$. The problem we are faced with is related to the principle-philosophical orientations of Bishop's constructive mathematics. To be we able to deal with this set as an UP coideal, it is necessary to have at least one its constructive algorithm with a finitely many steps.

In the following theorem we prove that besides the order relation $\nleqslant$ in a UP-algebra A with the apartness given by definition 3.4, there is a family of order relations determined by the family of all co-ideals.

Theorem 4.3. *Let K be a UP-coideal of UP-algebra A. The the relation $\not\prec_K$, defined by $x \not\prec_K y \iff x \cdot y \in K$, is a co-quasiorder in A left cancellative and right anti-cancellative with respect to the internal operation in A.*

Proof. (i) Suppose $x \not\prec_K y$ for elements $x, y \in A$. Then $x \cdot y \in K$. Thus $x \cdot y \neq 0 = y \cdot y$. So, $x \neq y$. Therefore, $\not\prec_K \subseteq \neq$ holds and the relation $\not\prec_K$ is a consistent relation in A.
(ii) Let $x, y, z \in A$ be arbitrary elements of A such that $x \not\prec_K z$. Thus $x \cdot z \in K$. Then $x \cdot y \in K \ \vee \ (x \cdot y) \cdot (x \cdot z) \in K$ and

$$x \cdot y \in K \ \vee \ y \cdot z \in K \ \vee \ (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \in K.$$

Since the third option is impossible, we have $x \not\prec_K y$ $y \not\prec_K z$. So, the relation $\not\prec_K$ is co-transitive.
(iii) Let $x, y, z \in A$ be arbitrary element such that $z{\cdot}x \not\prec_K z{\cdot}y$. Thus $(z{\cdot}x){\cdot}(z{\cdot}y) \in K$. Then $(x \cdot y) \cdot ((z \cdot x) \cdot (z \cdot y)) \in K \ \vee \ x \cdot y \in K$. Since, the first option is impossible, we have $x \cdot y \in K$. So, we have $x \not\prec_K y$. Therefore, the relation $\not\prec_K$ is left cancellative.
(iv) Let $x, y, z \in K$ be arbitrary elements such that $y{\cdot}z \not\prec_K x{\cdot}z$. Thus $(y{\cdot}z){\cdot}(x{\cdot}z) \in K$. Then $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) \in K \ \vee \ x \cdot y \in K$ by definition of UP-coideal K. So $x \not\prec_K y$ because the second option is impossible by (UP - 1). Therefore, the relation $\not\prec_K$ is right anti-cancellative in A. □

Remark 4.2. For any UP-coideal K in UP-algebra A the following implication $x \not\prec_K y \implies x \nleqslant y$ holds. Indeed. From $x \not\prec_K y$ we have $x \cdot y \in K$ and $x \cdot y \neq 0$. So, $x \nleqslant y$.

4.2. UP-filters and UP-cofilters. Recall that the creators of the concept of UP-filters in an UP-algebra have defined this notion.

Definition 4.3. ([**35**], Definition 1.11) Let A be a UP-algebra. A subset F of A is called a *UP-filter* of A, if it satisfies the following properties:
(i) $0 \in F$,
(ii) $(\forall x, y \in A)((x \in F \wedge x \cdot y \in F) \implies y \in F)$.

Relying on the understanding of the concept of duality of substructures in algebraic structures with apartness, we can now introduce the concept of UP-cofilters.

Definition 4.4. Let A be a UP-algebra with apartness. A nonempty subset G of A is called a *UP-cofilter* of A, if it satisfies the following properties:
(i) $0 \lhd G$,
(ii) $(\forall x, y \in A)(y \in G \implies (x \in G \vee x \cdot y \in G))$.

Lemma 4.1. *Let G be a UP-cofilter in a UP-algebra with apartness A. Then*

$$(\forall x, y \in A)(y \in G \implies (x \in G \vee x \nleqslant y).$$

Proof. Let $x, y \in A$ ne elements such that $y \in G$. Then $x \in G \vee x \cdot y \in G$. Thus $x \in G \vee x \nleqslant y$ since $0 \neq x \cdot y \in G$ holds. □

Our first statement about UP-cofilters is:

Statement 4. *Any UP-cofiltet G in UP-algebra A with apartness is a strongly extensional subset of A.*

Proof. Let x, y be arbitrary elements of UP-algebra A such that $y \in G$. By definition of UP-cofilter, we have $x \in G \vee x \cdot y \in G$. From the second option $x \cdot y \in G$, by $0 \lhd G$ we have $x \cdot y \neq 0 = y \cdot y$. So, finally we have $x \in G \vee x \neq y$. Therefore, the UP-cofilter G is a strongly extensional subset of A. □

The following theorem shows that the concept of UP-cofilter in UP-algebra with apartness is correctly determined in the sense that a strong compliment $G^{\lhd}$ of a UP-cofilter G in a UP-algebra with apartness A is a UP-filter in A.

Theorem 4.4. *Let G ne a UP-cofilter in a UP-algebra with apartness A. Then the set $G^{\lhd}$ is a UP-ideal in A.*

Proof. It is clear that $0 \in G$ is valid according to part (i) of the Definition 4.4.

Let $x, y, t \in A$ be elements of A such that $x \in G^{\lhd}$, $x \cdot y \in G^{\lhd}$ and $t \in G$. Then $y \in G$ or $y \cdot t \in G$ by (ii) of Definition 4.4. Suppose $y \in G$. Then we would have $x \in G \vee x \cdot y \in G$ which is in contradiction with the chosen hypotheses. The resulting contradiction disables this choice. The second option $y \cdot t \in G$ gives $y \cdot t = y \cdot y \vee 0 = y \cdot y \in G$ since G is a strongly extensional subset in A by the previous statement. The option $0 = y \cdot y \in G$ contradicts to (i) in the Definition 4.4. From the option $y \cdot t \neq y \cdot y$ it follows $y \neq t \in G$. Thus, $y \lhd G$ since t was an arbitrary element in G. This shows that $y \in G^{\lhd}$ is valid. □

THEOREM 4.5. *Let $\{G_j\}_{j\in L}$ be a family of UP-coideals in a UP-algebra with apartness A. Then the set $\bigcup_{j\in L} G_j$ is a UP-cofilter in A.*

PROOF. It is clear that $0 \lhd \bigcup_{j\in L} G_j$ holds.

Let $x, y \in A$ be elements such that $y \in \bigcup_{j\in L} G_j$. Then $y \in G_j$ for all $j \in L$. Thus $x \in G_j \subseteq \bigcup_{j\in L} G_j$ or $x \cdot y \in H_j \subseteq \bigcup_{j\in L} G_j$. □

COROLLARY 4.6. *The family $\mathfrak{G}(A)$ of all UP-cofilters in a UP-algebra with apartness A forms a complete upper semi-lattice.*

5. Co-congruence in UP-algebra with apartness

First, let's look at how the concept of a congruence in the UP algebra (= equivalence relations associated with the internal operation in the UP algebra) is introduced.

DEFINITION 5.1. ([**4**], Definition 3.1) Let A be a UP-algebra and J an UP-ideal of A. Define the binary relation $\sim_J$ on A as follows

$$(\forall x, y \in A)(x \sim_J y \iff (x \cdot y \in J \wedge y \cdot x \in J)).$$

This relation is called an *equivalence relation* on UP-algebra A generated by the UP-ideal J.

DEFINITION 5.2. ([**4**], Definition 3.3) Let A be a UP-algebra. An equivalence relation ϱ on A is called a *congruence* if

$$(\forall x, y, z \in A)(x\varrho y \implies ((x \cdot z)\varrho(y \cdot z) \wedge (z \cdot x)\varrho(z \cdot y))).$$

The concept of UP-cocongruence on UP-algebra with apartness is introduced by the following definition.

DEFINITION 5.3. For a co-equality relation q on a UP-algebra with apartness A it is said that a UP-*cocongruence* on A if the following holds

$$(\forall x, y, u, v \in A)((x \cdot u, y \cdot v) \in q \implies ((x, y) \in q \vee (y, v) \in q)).$$

LEMMA 5.1. *The implication in the previous definition is equivalent to the following two implications*

$$(\forall x, y, z \in A)((x \cdot z, y \cdot z) \in q \implies (x, y) \in q) \text{ and}$$

$$(\forall x, y, z \in A)(z \cdot z, z \cdot y) \in q \implies (x, y) \in q).$$

A co-congruence in UP-algebras with apartness we construct in the following theorem.

THEOREM 5.1. *Let A be an UP-algebra with apartness and K a UP-coideal of A. Define the binary relation q_K on A as follows*

$$(\forall x, y \in A)(xq_K y \iff (x \cdot y \in K \vee y \cdot x \in K)).$$

This relation is a co-congruence on UP-algebra A generated by the UP-coideal K.

Proof. Let $x, y \in A$ elements such that $xq_K y$. Thus $x \cdot y \in K \vee y \cdot x \in K$. Then $x \not\leqslant_K y \vee y \not\leqslant_K x$. Since the relation $\not\leqslant_K$ is a co-quasiorder relation in UP-algebra, the the relation $q_K = \not\leqslant_K \cup \not\leqslant_K$ is a co-equality relation in set $(A, =, \neq)$.

Let $x, y, z \in A$ be arbitrary elements such that $(z \cdot x)q_K(z \cdot y)$. Thus

$$z \cdot x \not\leqslant_K z \cdot y \ \vee \ z \cdot y \not\leqslant_K z \cdot x.$$

Since by Theorem 4.3 the relation $\not\leqslant_K$ in UP-algebra A is left cancellative with respect to the internal operation in UP-algebra $((A, =, \neq), \cdot, 0)$ we have $x \not\leqslant_K y \ \vee \ y \not\leqslant_K x$. Then $xq_K y$.

Let $x, y, z \in A$ be arbitrary elements such that $(x \cdot z)q_K(y \cdot z)$. Thus

$$x \cdot z \not\leqslant_K y \cdot z \ \vee \ y \cdot z \not\leqslant_K x \cdot z.$$

Since by Theorem 4.3 the relation $\not\leqslant_K$ in UP-algebra A is right anti-cancellative with respect to the internal operation in UP-algebra $((A, =, \neq), \cdot, 0)$ we have $y \not\leqslant_K x \ \vee \ x \not\leqslant_K y$. Then $xq_K y$ again.

Therefore, the relation q_K is a co-congruence in UP-algebra A. □

Corollary 5.1. *Let K be a UP-coideal of UP-algebra A. Then the class $0q_K$ of the co-congruence q_K generated by the element 0 is a UP-coideal in A and $0q_K = K$ holds.*

Remark 5.1. The relation $\neq$ $(= \not\leqslant \cup \not\leqslant^{-1})$ is a co-congruence in UP-algebra with apartness and for any UP-coideal K the following inclusion $q_K \subseteq \neq$ holds. Indeed. Suppose $xq_K y$. Then $x \cdot y \in K \ \vee \ y \cdot x \in K$. Thus $x \cdot y \neq 0 \ \vee \ y \cdot x \neq 0$. Therefore $x \not\leqslant y \ \vee \ y \not\leqslant x$ and $x \neq y$.

Let J and K be an associated pair of UP-ideals and UP-coideals in UP-algebra with apartness A. Then the congruence relation $\sim_J$, generated with J and the co-congruence relation q_K, generated with K, are associated in the following sense

$$\sim_J \circ q_K \subseteq q_K \text{ and } q_K \circ \sim_J \subseteq q_K.$$

We can design the set $A/(\sim_J, q_K) = \{[x] : x \in A\}$ with $[x] := \{y \in A : x \sim_J y\}$ where

$$(\forall x, y \in A)([x] =_1 [y] \iff x \sim_J y \ \wedge \ [x] \neq_1 [y] \iff (x, y) \in q_K).$$

Let us define the internal operation '$*$' in $A/(\sim_J, q_K)$ as

$$(\forall x, y \in A)([x] * [y] := [x \cdot y]).$$

In what follows we need the following lemma:

Lemma 5.2. *The operation '$*$' is correctly defined: it is a strongly extensional function.*

By direct verification it can be established that the following theorem holds

Theorem 5.2. *Let J and K be an associated pair of a UP-ideal and UP-coideal of a UP-algerba with apartness A. Then $(A/(\sim_J, q_K), *, [0])$ is a UP-algebra with apartness, too.*

Proof. We will only check the condition (UP-4a) since the proof of properties (UP-1), (UP-2) and (UP-3) completely coincides with the proof demonstrated in Theorem 3.7 in article [**4**].

Let $x, y \in A$ be elements such that $[x] \neq_1 [y]$. Then $(x, y) \in q$. This means $x \cdot y \in K \lor y \cdot x \in K$. Thus $x \cdot y \neq 0 \lor y \cdot x \neq 0$. □

On the other hand, we can design the set

$$[A : q_K] := \{xq_K : x \in A\},$$

where

$$xq_K := \{y \in A : (x, y) \in q_K\} = \{y \in A : x \cdot y \in K \; y \cdot x \in K\}$$

is the class q_K of the relation q_K generated by the element x and where the equality and co-equality determined as follows:

$$(\forall x, y \in A)(xq_K =_2 yq_K \iff (x, y) \lhd q_K \;\; \land \;\; xq_K \neq_2 yq_K \iff (x, y) \in q_K).$$

Let us define the internal operation '$\star$' in $[A : q_K]$ as

$$(\forall x, y \in A)(xq_K \star yq_K := (x \cdot y)q_K).$$

Let us proved, first, that $\star$ is correctly defined:

Lemma 5.3. *'$\star$' is a strongly extensional total function.*

Proof. It is obvious that $\star$ is total.

Let $x, y, u, v, s, t \in A$ be elements such that $xq_K =_2 yq_K$, $yq_K =_2 vq_K$ and $(s, t) \in q_K$. Then $(x, u) \lhd q_K$ and $(y, v) \lhd q_K$. On the other hand, from $(s, t) \in q_K$ follows

$$(s, x \cdot y) \in q_K \lor (x \cdot y, u \cdot v) \in q_K \lor (u \cdot v, t) \in q_K.$$

and

$$s \neq x \cdot y \lor (x, y) \in q_K \lor (u, v) \in q_K \lor u \cdot v \neq t.$$

We have $(x \cdot y, u \cdot v) \neq (s, t) \in q_K$ since the second and third options are impossible. This means $(x \cdot y, u \cdot v) \lhd q_K$. So, we have

$$xq_K \star yq_K = (x \cdot y)q_K =_2 (u \cdot v)q_K = uq_K \star vq_K$$

showing that '$\star$' is a function.

Let $x, y, y, v \in A$ be elements such that

$$xq_K \star yq_K = (x \cdot y)q_K \neq_2 (u \cdot v)q_K = uq_K \star vq_K.$$

Then $(x \cdot y, u \cdot v) \in q_K$. Thus $x, u) \in q_K$ or $(y, v) \in q_K$. Hence $xq_K \neq_2 yq_K$ or $uq_K \neq_2 vq_K$ showing that '$\star$' is a strongly extensional function. □

The following theorem shows the existence of UP-algebra with apartness generated by a UP-coideal in a UP-algebra with apartness that has no a counterpart in the classic theory of UP-algebras.

Theorem 5.3. *Let K be a IP-coideal of a UP-algebra with apartness. Then the structure $(([A : q_K], =_2, \neq_2), \star, K)$ is a UP-algebra with apartness.*

Proof. Since $(y \cdot z) \cdot (y \cdot z) \cdot (x \cdot z)) = 0 \lhd K = 0q_K$ for every $x, y, z \in A$, we have

$$((yq_K) \star (zq_K)) \star (((yq_K) \star (zq_K)) \star ((xq_K) \star (zq_K))) =_2 0q_K.$$

This shows that the condition (Up-1) for the elements of the set $[A : q_K]$ is valid formula.

Let $x, s, t \in A$ be elements such that $(s, t) \in q_K$. Then

$$(s, t) \in q_K \implies (s, 0 \cdot x) \in q_K \vee (0 \cdot x, x) \in q_K \vee (x, t) \in q_K.$$

Thus $(0 \cdot x, x) \neq (s, t) \in q_K$. this means $0q_K \star xq_K =_2 xq_K$ thus showing that the adverb (UP-2) is valid for the elements of the set $[A : q_K]$.

Let $x, s, t \in A$ be elements such that $(s, t) \in q_K$. It follows from here

$$(s, x \cdot 0) \in q_K \vee (x \cdot 0, 0) \in q_K \vee (0, t) \in q_K.$$

Second option is impossible by (UP-3). So, we have $(x \cdot 0, 0) \neq (s, t) \in q_K$. This means $(x \cdot 0)q_K =_2 0q_K$. Finally, we have $xq_K \star 0q_K =_2 0q_K$. Thus, the condition (UP-3) is a valid formula for all elements of the set $[A : q_K]$.

Let $x, y \in A$ be elements such that $xq_K \neq_2 yq_K$. Then $(x, y) \in q_K$. This means $x \cdot y \in K = 0q_K \vee y \cdot z \in K = 0q_K$. Thus $(x \cdot y, 0) \in q_K$ or $(y \cdot x, 0) \in q_K$. So, finally we have $xq_K \star yq_K \neq_2 0q_K$ or $yq_K \star xq_K \neq_2 0q_K$ thus proving that condition (UP-4a) is a valid formula in $[A : q_K]$. □

6. Strongly extensional UP-homomorphism

In this section, we introduce the concept of a strongly extensional homomorphism between UP-algebras with apartness. For this purpose we will use the definition of homomorphism given in article [**4**] with the assumption that this function is strongly extensional.

Definition 6.1. Let $((A, =, \neq), \cdot, 0)$ and $((A', =', \neq'), \cdot', 0')$ be UP-algebras with apartnesses. A strongly extensional mapping $f : A \longrightarrow A'$ is called a (strongly extensional) UP-*homomorphism* if the following formula

$$(\forall x, y \in A)(f(x \cdot y) =' f(x) \cdot' f(y)).$$

is valid.

Remark 6.1. As can be seen, the UP-homomorphisms between UP-algebras in the classical case are not different from our definition of stromgly extensive UP-homomorphism between UP algebra with apartness. The differences between these two concepts of UP homomorphism are recognized when recognizing the environment in which these objects are observed. The previous definition implies that the following implications are valid

$$(\forall x, y \in A)(x = y \implies f(x) =' f(y)),$$
$$(\forall x, y \in A)(f(x) \neq' f(y) \implies x \neq y).$$

The readers who have experience in reading texts into the Constructive Algebra can understand the following statements without major difficulty.

THEOREM 6.1. *Let* $((A, =, \neq), \cdot, 0)$ *and* $((A', =', \neq'), \cdot', 0')$ *be UP-algebras with apartness and let* $f : A \longrightarrow B$ *be a strongly extensional UP-homomorphism between them. Then the following statements hold:*

(1) f *is a reverse isotone mapping.*

(2) *If* K' *be UP-coideal of UP-algebra* A'*, then* $f^{-1}(K')$ *is a UP-coideal in UP-algebra* A*. In particular,* $Coker(f) = \{x \in A : f(x) \neq' 0'\}$ *is a UP-coideal in* A *and* $Coker(f) = \{x \in A : x \neq 0\}$ *is and only if* f *is an embedding.*

PROOF. Let $x, y \in A$ be elements such that $f(x) \nleqslant' f(y)$. This means that $f(x \cdot y) =' f(x) \cdot' f(y) \neq' 0' =' f(0)$. Then $x \cdot y \neq 0$ by strongly extensionality of the mapping f. So, $x \nleqslant y$. Therefore, the mapping f is a reverse isotone se-mapping.

It is clear that $0 \lhd f^{-1}(K')$. Indeed. from $f(0) =' 0' \lhd f^{-1}(K')$, i.e. from $(\forall u \in K')(f(0) \neq' f(u))$ it follows $(\forall v \in f^{-1}(K'))(0 \neq v)$ since the mapping f is a se-mapping. Let $x, y \in A$ be elements such that $y \in f^{-1}(K)$. Then $f(y) \in K'$. Thus $f(x) \in K' \vee f(x) \neq' f(y)$ because K' is a UP-cofilter in the UP-algebra A'. Hence $x \in f^{-1}(K') \vee x \neq y$. So, the subset $f^{-1}(K')$ is a UP-cofilter in the UP-algebra A.

Since the subset $\{u \in A' : u \neq 0'\}$ is a UP-cofilter in A', we conclude that the set $f^{-1}(\{u \in A' : u \neq 0') = \{x \in A : f(x) \neq 0'\} = Coker(f)$ is a UP-cofilter in A, according to the second part of this proof.

The last part of the statement of this theorem follows directly from the definition of the notion of embedding. □

We conclude this section with the following two theorems whose proofs can be demonstrated by direct verification:

THEOREM 6.2. *Let* A *be a UP-algebra with apartness,* J *and* K *be associate a UP-ideal and a UP-coideal of* A*. Then the mapping* $\pi_{J,K} : A \longrightarrow A/(\backsim_J, q_K)$*, defined by* $\pi_{J,K}(x) := [x]$ *for all* $x \in A$*, is a strongly extensional UP-epimorphism.*

THEOREM 6.3. *Let* A *be a UP-algerba with apartness and* K *be a UP-coideal in* A*. Then the mapping* $\vartheta_K : A \longrightarrow [A : q_K]$*, defined by* $\vartheta_K(x) := xq_K$ *for all* $x \in A$*, is a strongly extensional UP-epimorphism.*

7. Conclusion

The environment in which this research was realized is the Intuitionistic logic and the principle-philosophical orientation of the Bishop's Constructive mathematics. In the present paper, we have introduced a new algebraic structure, called a UP-algebra with apartness and the concepts of UP-coideals and UP-cofilters, co-congruences and (strongly extensional) UP-homomorphisms in UP-algebras with apartness and investigated some of its essential properties.

This text, by our opinion, enables to readers to observe the complexity of the substructures in the UP algebra with apartness and, moreover, the techniques used in this part of the Constructive Algebra.

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