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ON CLASSIFICATIONS OF MULTI-VALUED FUNCTIONS USING MULTI-HOMOTOPY

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Abstract. In this work, we extend properties for single-valued functions especially in homotopy theory to multi-valued functions. We present new definitions for multi-valued functions such as m-pathwise connected, m-homotopy equivalence, m-contractibility and (deformation) m-retraction. We apply some topological properties from algebraic topology to the multi-valued functions.

1. Introduction

Multi-valued functions are functions that image of at least one element in their domain set is a set. Therefore, we can say that multi-valued functions are a generalization of single-valued functions. So the algebraic properties of functions are similar to those of single-valued functions. In some researches, single-valued functions are inadequate. For this reason, the notion of the multi-valued function has emerged. Multi-valed functions have a lot of applications in many fields such as optimal control theory, calculus of variation, probability, statistics and economy.

The concept of a continuous function is a fundamental thought to the study of topological spaces and help to generalize a considerable part of the homotopy theory. The idea of continuity for multi-valued functions has put forward for special cases by [**9**], [**10**] and [**25**]. Later, more general definitions have given by [**3**] and [**13**]. Various equivalent definitions of continuity of single-valued functions are discussed separately for multi-valued functions. In this reason, Kurotowski [**13**] has described notions of semi continuity (lower semi-continuity, upper semi-continuity).

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The concept of continuity has later defined in many different ways for multi-valued functions and Strother [**24**] has studied the relationship that exists between different definitions of continuity. There are lots of papers about multi-valued functions in the literature. Kakutani's work [**11**] is one of the most important ones because he has given a generalization of Brouwer's Fixed Point Theorem in this paper. Choquet [**5**] has studied the lower semi-continuity and upper semi-continuity of multi-valued functions by the concepts of limit on set families. Michael [**14**] has described a topology on the collection of non-empty closed subsets of a topological space. Then he has applied this topology to the study of multi-valued functions and adapted some properties of single-valued functions to the multi-valued functions. Strother [**22**] introduced fixed point theory for multi-valued functions. The definition of homotopy for functions is an important notion in algebraic topology. Strother's work [**23**] is one of the main papers in this field. Strother introduced a multi-valued version of homotopy. Moreover, Kruse [**12**] and Hahn [**8**] have been extensively studied on multi-valued functions. Ponomarev in [**16**], [**17**] and [**18**] has investigated the basic properties of multi-valued functions and answered the question which properties of topological spaces are preserved under multi-valued functions. Furthermore, Rhee [**19**] worked on homotopy theorems which can be applied to the study of multi-valued functions.

In this work, we first give some background for multi-valued functions that we need to prove the theorems which are already known for single-valued functions. We apply some properties from algebraic topology to the multi-valued functions, then give multi-valued versions of some important properties such as homotopy equivalence, pathwise connectedness, contractibility, and retraction for some special classes of spaces.

2. Preliminaries

In this paper, we assume that all spaces are Hausdorff. Multi-valued functions will be denoted by uppercase letters such as F, G, H. Let $X \neq \emptyset$, Y be any topological spaces. We say that $F : X \rightrightarrows Y$ is a multi-valued function if for each $x \in X$, $F(x)$ is a subset of Y.

Remark 2.1. 1) Single-valued functions are just special cases of multi-valued functions.

2) A multi-valued function $F : X \rightrightarrows Y$ can be considered as a single-valued function from X to the set $\mathcal{P}(Y)$ of all subsets of Y.

Let $F : X \rightrightarrows Y$ be a multi-valued function. The *range* of F is

$$R(F) = \bigcup_{x \in X} F(x)$$

by [**1**]. For each $A \subset X$ we have

$$F(A) = \bigcup_{x \in A} F(x)$$

from [**2**]. The multi-valued function F is called *one-to-one* if for any $x, x^{'} \in X$, $x \neq x^{'}$, we have $F(x) \cap F(x^{'}) = \emptyset$ and F is *surjective (onto)* if $R(F) = Y$, [**1**]. The multi-valued function F is called *closed valued* if for each $x \in X$, $F(x)$ is closed in Y, by [**6**]. Let $F_1, F_2 : X \rightrightarrows Y$ be multi-valued functions. The *union* of the functions F_1 and F_2 is a function denoted by $F_1 \cup F_2 : X \rightrightarrows Y$ given by $(F_1 \cup F_2)(x) = F_1(x) \cup F_2(x)$, the *intersection* of F_1 and F_2 is a function denoted by $F_1 \cap F_2 : X \rightrightarrows Y$ given by $(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x)$, the *cartesian product* of F_1 and F_2 is a function denoted by $F_1 \times F_2 : X \rightrightarrows Y \times Y$ given by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$, the *composition* of $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ is a function denoted by $G \circ F : X \rightrightarrows Z$ given by $(G \circ F)(x) = G(F(x)) = \bigcup_{y \in F(x)} G(y)$ and the *graph* gr$F$ of a multi-valued function $F : X \rightrightarrows Y$ is the set

$$\mathrm{gr}F = \{(x, y) \in X \times Y | y \in F(x) \;\text{ and }\; x \in X\},$$

by [**1**].

Now, we have the following lemma for the continuity of multi-valued functions due to Strother:

Lemma 2.1 ([**23**]). *A closed valued function $F : X \rightrightarrows Y$ is continuous if and only if statements (1) and (2) hold:*

(1) *If $x_0 \in X$, V is open in Y and if $F(x_0) \cap V \neq \emptyset$, then there exists an open set U of X with $x_0 \in U$ such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.*

(2) *If $x_0 \in X$ and $F(x_0) \subset V$, where V is open in Y, then there exists an open set U containing x_0 such that $F(U) \subset V$.*

In the Lemma 2.1, the first statement is called lower semi-continuity (lsc) of a multi-valued function F and second is called upper semi-continuity (usc) of a multi-valued function F.

The inverse of the multi-valued function $F : X \rightrightarrows Y$ is the function denoted by $F^{-} : Y \rightrightarrows X$ and given by $F^{-}(B) = \{x \in X | F(x) \subset B \neq \emptyset\}$ for each $B \subset Y$ [**2**]. Geletu has given useful properties [**6**, Proposition 5.3.5] and [**6**, Proposition 5.3.14] regarding semi-continuities of multi-valued functions which are gathered in the following proposition.

Proposition 2.1 ([**6**]). *Let $F : X \rightrightarrows Y$ be a multi-valued function. Then the following statements are hold:*

(*i*) *F is usc if and only if for each closed set $W \subset Y$, $F^{-}(W)$ is closed set in X;*

(*ii*) *F is lsc if and only if for each open set $V \subset Y$, $F^{-}(V)$ is open set in X.*

In the following definition, Borges defined a multi-valued quotient function $F : X \rightrightarrows Y$ in such a way that F is either a usc-function or lsc-function.

Definition 2.1. ([**2**]) Let X and Y be topological spaces and $F : X \rightrightarrows Y$ an onto multi-valued function. Then F is said to be a us-quotient (ls-quotient) function provided that a subset U of Y is closed (open) if and only if $F^{-}(U)$ is closed subset of X (open subset of X). F is said to be a quotient function whenever F is both a us-quotient function and ls-quotient function.

We define the following new continuous multi-valued function with the help of two continuous multi-valued functions.

LEMMA 2.2. *Let $F : X \rightrightarrows Z$ and $G : Y \rightrightarrows W$ be two continuous multi-valued functions. Then the function defined by*

$$H : X \times Y \rightrightarrows Z \times W$$
$$(x, y) \mapsto F(x) \times G(y)$$

is continuous.

PROOF. Assume that V is an open set such that

$$H(x, y) = F(x) \times G(y) \subset V \subset Z \times W$$

for all $(x, y) \in X \times Y$.

Case 1. Let the open set V be a base element of product topology on $Z \times W$. Then, there exist two open sets $V_Z \subset Z$ and $V_W \subset W$ such that $V = V_Z \times V_W$. Since F is upper semi-continuous multi-valued function, for all $F(x) \subset V_Z$, where V_Z is open in Z, there exists an open set U_x containing x such that

$$F(x^{'}) \subset V_Z \quad \text{for all} \quad x^{'} \in U_x.$$

Since G is the upper semi-continuous multi-valued function, for all $G(y) \subset V_W$, where V_W is open in W, there exists an open set U_y containing y such that

$$G(y^{'}) \subset V_W \quad \text{for all} \quad y^{'} \in U_y.$$

Thus, the set $U = U_x \times U_y$ is an base element since U_x and U_y are open sets. Consequently, there exists an open set $U = U_x \times U_y$ containing (x, y) such that

$$H(x^{'}, y^{'}) = F(x^{'}) \times G(y^{'}) \subset V_Z \times V_W = V \quad \text{for all} \quad (x^{'}, y^{'}) \in U.$$

Case 2. Assume that the open set V is not a base element. Then, V is a union of basis elements. So, for open sets $V_Z \subset Z$ and $V_W \subset W$ we can take $V_Z \times V_W \subset V$. We know from Case 1, the upper semi-continuity condition is satisfied for all open sets V in the form $V_Z \times V_W$. Thus, there exists an open set $U = U_x \times U_y$ containing (x, y) such that

$$H(x^{'}, y^{'}) = F(x^{'}) \times G(y^{'}) \subset V_Z \times V_W \subset V \quad \text{for all} \quad (x^{'}, y^{'}) \in U.$$

Consequently, H is upper semi-continuous.

Similarly, the lower semi-continuity of H can be shown. $\square$

THEOREM 2.1 ([**1**]). *The cartesian product $F = \prod_{i=1}^{n} F_i$ of a finite family of the lower semi-continuous (l.s.c.) functions $F_i : X \rightrightarrows Y_i$ is a l.s.c. function $F : X \rightrightarrows \prod_{i=1}^{n} Y_i$ given by $F(x) = (F_1(x), \cdots, F_n(x))$ for any $x \in X$.*

THEOREM 2.2 ([**1**]). *The cartesian product $F = \prod_{i=1}^{n} F_i$ of a finite family of the upper semi-continuous (u.s.c.) and compact valued functions $F_i : X \rightrightarrows Y_i$ is a u.s.c. and compact valued function $F : X \rightrightarrows \prod_{i=1}^{n} Y_i$, where $F_i(x_i)$ is a compact set for any $x_i \in X$, for all i.*

THEOREM 2.3 ([**1**]). *The cartesian product* $F = \prod_{i=1}^{n} F_i$ *of a family of closed functions* $F_i : X \rightrightarrows Y_i$ *is a closed function* $F : X \rightrightarrows \prod_{i=1}^{n} Y_i$.

Let X and Y be compact and let $F : X \rightrightarrows Y$ be continuous. Then F is closed i.e. $F(A)$ is closed in Y whenever A is closed in X [**21**].

PROPOSITION 2.2 ([**7**]). *Let* $F : X \rightrightarrows Y$ *and* $G : Y \rightrightarrows Z$ *be two closed value multi-valued functions. Then for any* $B \subset Z$ *we obtain*

$$(G \circ F)^{-}(B) = F^{-} \circ G^{-}(B).$$

LEMMA 2.3 ([**21**]). *If* $F_1 : X \rightrightarrows Y$ *and* $F_2 : Y \rightrightarrows Z$ *are continuous and if* X*,* Y *and* Z *are compact, then* $F = F_2 \circ F_1$ *is continuous.*

The restriction of a multi-valued function is defined in [**21**]. Let $F : X \rightrightarrows Y$ and let A be a subspace of X. Then the restriction of F to A is defined by $F|_A(x) = F(x)$ for all $x \in A$.

Let $F : X \rightrightarrows Y$ be continuous and let $A \subset X$. Then the restriction of F to A is continuous [**21**]. The multi-valued function $\mathrm{id}_X : X \rightrightarrows X$, $\mathrm{id}_X(x) = \{x\}$ is called the *identical function* of the set X [**1**]. A function $F : X \rightrightarrows Y$ is called *constant* if $F(x) = C$, for all $x \in X$, where C is a fixed subset of Y [**1**].

DEFINITION 2.2. Let A be a subset of X. Then the m-inclusion function is defined by

$$\begin{aligned} \mathcal{I} : A &\rightrightarrows X \\ a &\mapsto \{a\}. \end{aligned}$$

Moreover, $\mathcal{I}(a) \cap \mathcal{I}(a^{'}) = \emptyset$ for all $a \neq a^{'} \in A$. Then the m-inclusion function is one to one. Furthermore, the continuity of the m-inclusion function can be shown easily by Lemma 2.1.

LEMMA 2.4 ([**23**]). *Let* Y *be a topological space,* A *and* B *both open (or both closed) subsets of a topological space* X *such that* $X = A \cup B$*. Assume that* $F : A \rightrightarrows Y$ *and* $G : B \rightrightarrows Y$ *be continuous multi-valued functions such that for all* $x \in A \cap B$*,* $F(x) = G(x)$*. Then the multi-valued function defined by*

$$\begin{aligned} H : X &\rightrightarrows Y \\ x &\mapsto \begin{cases} F(x), & x \in A \\ G(x), & x \in B \end{cases} \end{aligned}$$

is continuous.

PROOF. Suppose that $F : A \rightrightarrows Y$ is the upper semi-continuous multi-valued function. If $x \in A$ and $F(x) \subset V$, where V is open in Y, then there exists an open set U containing x such that

$$F(x^{'}) \subset V \quad \text{for all} \quad x^{'} \in U.$$

From the definition of a multi-valued function H, we have $F(x) = H(x)$. Since U is open in A and A is open in X, U is open in X. So, we can consider U as an open neighborhood.

Then for $H(x) \subset V$, where V is open in Y there exists an open set U of x such that

$$H(x^{'}) \subset V \quad \text{for all} \quad x^{'} \in U.$$

If $x \in B$, then it can be shown that the multi-valued function H is upper semi-continuous by taking multi-valued function G instead of F and following the same way.

Similarly, the lower semi-continuity of H can be shown. As a result, the multi-valued function H is continuous. □

3. m-Homeomorphisms

Definition 3.1. Let X and Y be topological spaces. Let $F : X \rightrightarrows Y$ be one-to-one and onto multi-valued function. If both F and its inverse $F^{-} : Y \rightrightarrows X$ are continuous, then F is called an m-homeomorphism.

Lemma 3.1. *Let X, Y and Z be compact topological spaces. If $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are m-homeomorphisms, then $G \circ F : X \rightrightarrows Z$ is an m-homeomorphism.*

Proof. Let F and G be m-homeomorphisms. Then $G \circ F$ is continuous. For all $x \neq x^{'} \in X$, we have

$$G \circ F(x) = G(F(x)) = \bigcup_{y \in F(x)} G(y)$$
$$G \circ F(x^{'}) = G(F(x^{'})) = \bigcup_{y^{'} \in F(x^{'})} G(y^{'}).$$

As F is an m-homeomorphism, it is one-to-one i.e. $F(x) \cap F(x^{'}) = \emptyset$ for $x \neq x^{'}$. So, $y \in F(x)$ and $y^{'} \in F(x^{'})$ are different from each other. At the same time, as G is one-to-one, if $y \neq y^{'}$, then $G(y) \cap G(y^{'}) = \emptyset$. So, we have $G \circ F(x) \cap G \circ F(x^{'}) = \emptyset$. It means that $G \circ F$ is one-to-one.

Since the range of the multi-valued function $G \circ F$ is

$$\begin{aligned} R(G \circ F) &= \bigcup_{x \in X} G \circ F(x) = \bigcup_{x \in X} G(F(x)) \\ &= G\Big(\bigcup_{x \in X} F(x)\Big) = G(Y) && \because \ F \text{ is onto } F(X) = Y \\ &= \bigcup_{y \in Y} G(y) = Z && \because \ G \text{ is onto } G(Y) = Z, \end{aligned}$$

$G \circ F$ is onto.

From [**7**, Proposition 13.3.2] we have $(G \circ F)^{-} = F^{-} \circ G^{-}$. Since F^{-} and G^{-} are continuous, their composite is also continuous.

Thus, $G \circ F$ is an m-homeomorphism. □

4. m-Pathwise Connectedness

DEFINITION 4.1. ([**23**]) A space Y is said to be m-pathwise connected if for closed subsets Y_0 and Y_1 of Y, there exists a continuous multi-valued function F from the unit interval I to Y such that $F(0) = Y_0$ and $F(1) = Y_1$.

LEMMA 4.1. *Let X be a topological space. Asuume that A, B and C are closed subsets of X. Let $A \sim_m B$ denote the relation*

$$A \sim_m B :\Leftrightarrow \textit{there is a } m\textit{-path from } A \textit{ to } B \textit{ in } X,$$

i.e. there is a continuous multi-valued function $F : I \rightrightarrows X$ such that $F(0) = A$ and $F(1) = B$.

PROOF. For a closed subset A, let $F : I \rightrightarrows X$ be a constant multi-valued function such that $F(t) = A$, for all $t \in I$. Since F is constant, it is continuous. So, there exists a m-path from A to A.

Assume that $A \sim_m B$. Then there exists a continuous multi-valued function $F : I \rightrightarrows X$ such that $F(0) = A$ and $F(1) = B$. So, define a continuous multi-valued function $G : I \rightrightarrows X$ such that $G(t) = F(1 - t)$. We get $B \sim_m A$ by the multi-valued function G.

Suppose now that $A \sim_m B$ and $B \sim_m C$. Then there exist continuous multi-valued function $F, G : I \rightrightarrows X$ such that $F(0) = A$, $F(1) = B$ and $G(0) = B$, $G(1) = C$. Define a multi-valued function such that

$$H : I \rightrightarrows X$$
$$t \mapsto H(t) = \begin{cases} F(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ G(2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is continuous from Lemma 2.4. On the other hand, we have

$$H(0) = F(0) = A \text{ and}$$
$$H(1) = G(1) = C.$$

So, we get $A \sim_m C$. □

THEOREM 4.1. *Let X and Y be compact topological spaces. X and Y are m-pathwise connected if and only if $X \times Y$ be a m-pathwise connected space.*

PROOF. Suppose that X and Y are m-pathwise connected spaces. Since X is m-pathwise connected, there exists a continuous multi-valued function $F : I \rightrightarrows X$ such that $F(0) = X_0$ and $F(1) = X_1$, where X_0 and X_1 are closed subsets of X. Since Y is m-pathwise connected, there exists a continuous multi-valued function $G : I \rightrightarrows Y$ such that $G(0) = Y_0$ and $G(1) = Y_1$, where Y_0 and Y_1 are closed subsets of Y. So, we have

$$F \times G : I \rightrightarrows X \times Y$$

such that $F \times G(0) = F(0) \times G(0) = X_0 \times Y_0$ and $F \times G(1) = F(1) \times G(1) = X_1 \times Y_1$. Since X_0, X_1 are closed in X and Y_0, Y_1 are closed in Y, $X_0 \times Y_0$ and $X_1 \times Y_1$ are closed in $X \times Y$. Moreover, we know that the multi-valued function $F \times G$

is continuous from Theorem 2.1 and 2.2. Consequently, $X \times Y$ is m-pathwise connected.

Since the space $X \times Y$ is m-pathwise connected, there exists a continuous multi-valued function $F : I \rightrightarrows X \times Y$ such that $F(0) = (X_0, Y_0)$ and $F(1) = (X_1, Y_1)$, where X_0, X_1 are closed subsets of X and Y_0, Y_1 are closed subsets of Y. Define a continuous multi-valued function

$$\begin{aligned} G : I &\rightrightarrows X \\ t &\mapsto G(t) = \pi \circ F(t), \end{aligned}$$

where $\pi : X \times Y \rightrightarrows X$, $(x, y) \mapsto \{x\}$ is continuous multi-valued function. Since

$$\pi(X_0, Y_0) = \bigcup_{x \in X_0, y \in Y_0} \pi(x, y) = \bigcup_{x \in X_0} \{x\} = X_0,$$

we have

$$\begin{aligned} G(0) = \pi \circ F(0) = \pi(X_0, Y_0) = X_0 \subset X \\ G(1) = \pi \circ F(1) = \pi(X_1, Y_1) = X_1 \subset X. \end{aligned}$$

Therefore, X is m-pathwise connected. Similarly, it can be shown that Y is m-pathwise connected. □

EXAMPLE 4.1. A unit interval I is an m-pathwise connected space.

PROOF. Assume that A and B are any closed subsets of I. If there exists a continuous multi-valued function $F : I \rightrightarrows I$ such that $F(0) = A$ and $F(1) = B$, then I is m-pathwise connected. So let define a multi-valued function F such that

$$\begin{aligned} F : I &\rightrightarrows I \\ t &\mapsto \begin{cases} A, & 0 \leqslant t < \frac{1}{2} \\ I, & t = \frac{1}{2} \\ B, & \frac{1}{2} < t \leqslant 1. \end{cases} \end{aligned}$$

First, we shall show that the multi-valued function is upper semi-continuous:

For $0 \leqslant t < \frac{1}{2}$, let V be an open set such that $F(t) = A \subset V$. If we take the open set $[0, \frac{1}{2}) = U$, then for all $t^{'} \in U$, $F(t^{'}) = A \subset V$. For $\frac{1}{2} < t \leqslant 1$, suppose that V be an open set such that $F(t) = B \subset V$. If we take the open set $(\frac{1}{2}, 1] = U$, then for all $t^{'} \in U$, $F(t^{'}) = B \subset V$. For $t = \frac{1}{2}$, let V be an open set such that $F(t) = I \subseteq I = V$. If we take the open set such that $t = \frac{1}{2} \in U$, then for all $t^{'} \in U$, $F(t^{'}) \subset I$. So, F is upper semi continuous. Similarly, the lower semi-continuity of F can be shown. Therefore, I is an m-pathwise connected space. □

From Example 4.1 and Theorem 4.1, we conclude that I^n is m-pathwise connected.

THEOREM 4.2. *Let X and Y be compact spaces. If $F : X \rightrightarrows Y$ is a continuous closed multi-valued function and X is m-pathwise connected, then $F(X)$ is m-pathwise connected.*

PROOF. Assume that X is m-pathwise connected. Then there exists a continuous multi-valued function $G : I \rightrightarrows X$ such that $G(0) = A$ and $G(1) = B$, where A and B are closed subsets of X [**23**]. Since I, X and Y are compact and F and G are continuous, the composite function $G \circ F$ is continuous. On the other hand, we have

$$F \circ G(0) = \bigcup_{x \in G(0)} F(x) = \bigcup_{x \in A} F(x) = F(A) \text{ and}$$
$$F \circ G(1) = \bigcup_{x \in G(1)} F(x) = \bigcup_{x \in B} F(x) = F(B).$$

By the definition of a closed multi-valued function, $F(A)$ and $F(B)$ are closed subsets of $F(X)$ whenever A and B are closed subsets of X.

Thus, $F(X)$ is m-pathwise connected. □

THEOREM 4.3. *Let X and Y be compact spaces and $F : X \rightrightarrows Y$ be a multi-valued quotient function. If X is a m-pathwise connected space, then Y is m-pathwise connected.*

PROOF. Assume that $F : X \rightrightarrows Y$ is a m-quotient function. Then it is both upper semi-quotient and lower semi-quotient by Definition 2.1. Since F is upper semi-quotient, for a closed subset V of Y, $F^{-}(V)$ is closed in X. So, it is upper semi-continuous by Proposition 2.1. Similarly, since F is upper semi-quotient, we have for an open subset U of Y, $F^{-}(U)$ is open in X. So, it is upper semi-continuous. By Theorem 4.2, $F(X)$ is m-pathwise connected. Since F is the multi-valued quotient function, it is onto i.e. $F(X) = Y$. Thus, Y is m-pathwise connected. □

LEMMA 4.2. *Let X be a finite topological space and C be a subset of X. If X_1 and X_2 be m-pathwise connected subsets of X such that $C \subset X_1 \cap X_2$, then $X_1 \cup X_2$ is m-pathwise connected.*

PROOF. Let A and B be subsets of $X_1 \cup X_2$. Since X is finite recall that all spaces are assumed to be Hausdorff, $X_1 \cup X_2$ is Hausdorff and finite. So, A and B are closed subsets of $X_1 \cup X_2$. There exists three cases for the subsets A and B:

Case 1: If A and B are subsets of X_1, then it is obvious from m-pathwise connectedness of X_1.

Case 2: Similarly, if A and B are subsets of X_2, then it is obvious for the same reason.

Case 3: Let A be a subset of X_1 and B be a subset of X_2. The set C is closed since X is finite. Since X_1 is m-pathwise connected, there exists a continuous multi-valued function $F : I \rightrightarrows X_1$ such that $F(0) = A \subset X_1$ and $F(1) = C \subset X_1$. As X_2 is m-pathwise connected, there exists a continuous multi-valued function $G : I \rightrightarrows X_2$ such that $G(0) = B \subset X_2$ and $G(1) = C \subset X_2$. Since m-pathwise connectedness is an equivalence relation, there exists a m-path between A and B. Thus, $X_1 \cup X_2$ is m-pathwise connected. □

5. Some Properties of the Multi-valued Function in Homotopy Theory

In this section, some properties of a multi-valued function will be given with respect to the definition of multi-homotopy [**23**].

Definition 5.1. ([**23**]) Let F and G be multi-valued functions from X to Y. Then F is said to be m-homotopic (multi-homotopic) to G if there exists a continuous multi-valued function $H : X \times I \rightrightarrows Y$ such that $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$. We write $F \simeq_m G$.

Lemma 5.1. *The m-homotopy relation is an equivalence relation.*

Proof. Let $F : X \rightrightarrows Y$ be the continuous multi-valued function and the m-homotopy function defined by

$$\begin{aligned} H &: X \times I \rightrightarrows Y \\ &(x, t) \mapsto H(x, t) = F(x). \end{aligned}$$

Then it is continuous as F is. Furthermore, because of the equalities $H(x, 0) = F(x)$ and $H(x, 1) = F(x)$ we get $F \simeq_m F$.

Assume that $F \simeq_m G$. Then there exists an m-homotopy function such that $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$. If we define an m-homotopy function by

$$\begin{aligned} K &: X \times I \rightrightarrows Y \\ &(x, t) \mapsto K(x, t) = H(x, 1 - t), \end{aligned}$$

then it is continuous since H is continuous. Since

$$K(x, 0) = H(x, 1) = G(x) \quad \text{and} \quad K(x, 1) = H(x, 0) = F(x)$$

we have $G \simeq_m F$.

Suppose that $F \simeq_m G$ and $G \simeq_m H$. Then there exists an m-homotopy functions defined by $K, W : X \times I \rightrightarrows Y$ such that $K(x, 0) = F(x)$, $K(x, 1) = G(x)$ and $W(x, 0) = G(x)$, $W(x, 1) = H(x)$. So, in order to show that $F \simeq_m H$ we define an m-homotopy function M as follows:

$$\begin{aligned} M &: X \times I \rightrightarrows Y \\ &(x, t) \mapsto M(x, t) = \begin{cases} K(x, 2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ W(x, 2t - 1), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \end{aligned}$$

and it is continuous from Lemma 2.4. Since $M(x, 0) = K(x, 0) = F(x)$ and $M(x, 1) = W(x, 1) = H(x)$, we get $F \simeq_m H$. □

Theorem 5.1. *Let X, Y and Z be compact spaces and $F, G : X \rightrightarrows Y$ be two continuous multi-valued functions. If $F \simeq_m G$ and $H : Y \rightrightarrows Z$ are continuous multi-valued function, then $H \circ F \simeq_m H \circ G$.*

Proof. If $F \simeq_m G$, then there exists a multi-homotopy function $K : X \times I \rightrightarrows Y$ such that $K(x, 0) = F(x)$ and $K(x, 1) = G(x)$. Define a multi-valued function $W = H \circ K$, where $H : Y \rightrightarrows Z$ is a continuous multi-valued function. Since H and K are continuous, W is continuous. On the other hand, we have

$$W(x,0) = H \circ K(x,0) = \bigcup_{y \in K(x,0)} H(y) = \bigcup_{y \in F(x)} H(y) = H \circ F(x) \text{ and}$$
$$W(x,1) = H \circ K(x,1) = \bigcup_{y \in K(x,1)} H(y) = \bigcup_{y \in G(x)} H(y) = H \circ G(x).$$

So, $H \circ F \simeq_m H \circ G$ is obtained. □

THEOREM 5.2. *Let X, Y and Z be compact spaces and $F, G : X \rightrightarrows Y$ be two continuous multi-valued functions. If $F \simeq_m G$ and $H : Z \rightrightarrows X$ is a continuous multi-valued function, then $F \circ H \simeq_m G \circ H$.*

PROOF. If $F \simeq_m G$, then there exists a multi-homotopy function $K : X \times I \rightrightarrows Y$ between F and G. Define a multi-valued function $W = K \circ (H \times \mathrm{id}_I)$, where $H : Z \rightrightarrows X$ is a continuous multi-valued function and id_I is the identical function on I. Since K, H and $H \times \mathrm{id}_I$ are continuous, W is continuous. Moreover, we have

$$W(z,0) = K(H(z),0) = F(H(z)) = F \circ H(z) \text{ and}$$
$$W(z,1) = K(H(z),1) = G(H(z)) = G \circ H(z).$$

So, $F \circ H \simeq_m G \circ H$ is obtained. □

THEOREM 5.3. *Let F_i be constant multi-valued functions such that for closed subsets of X, X_0 and X_1, $F_i : X \rightrightarrows X$, $x \mapsto X_i$, where $i = 0, 1$. Then $F_0 \simeq_m F_1$ if and only if there exists a continuous multi-valued function $F : I \rightrightarrows X$ such that $F(0) = X_0$ and $F(1) = X_1$.*

PROOF. Let

$$\begin{array}{ll} F_0 : X \rightrightarrows X & \text{and} \quad F_1 : X \rightrightarrows X \\ \quad x \mapsto X_0 & \qquad\quad x \mapsto X_1 \end{array}$$

be constant multi-valued functions and $F_0 \simeq_m F_1$. Then there exists a multi-homotopy function $H : X \times I \rightrightarrows X$ such that $H(x,0) = F_0(x) = X_0$ and $H(x,1) = F_1(x) = X_1$. In this case, if the function F is defined by

$$\begin{array}{l} F : I \rightrightarrows X \\ \quad t \mapsto F(t) = H(x,t), \end{array}$$

we get a continuous multi-valued function F such that $F(0) = H(x,0) = X_0$ and $F(1) = H(x,1) = X_1$.

Let $F : I \rightrightarrows X$ be a continuous multi-valued function such that $F(0) = X_0$ and $F(1) = X_1$, where $X_0, X_1 \subset X$. If we define a multi-homotopy function such that

$$\begin{array}{l} H : X \times I \rightrightarrows X \\ \quad (x,t) \mapsto H(x,t) = F(t), \end{array}$$

then H is continuous and for a constant multi-valued function defined by

$$\begin{aligned} F_0 : X &\rightrightarrows X \quad \text{and} \quad & F_1 : X &\rightrightarrows X \\ x &\mapsto X_0 & x &\mapsto X_1, \end{aligned}$$

we have $H(x,0) = F(0) = X_0 = F_0(x)$ and $H(x,1) = F(1) = X_1 = F_1(x)$. Thus, we obtain $F_0 \simeq_m F_1$. $\square$

Corollary 5.1. *Let X be a topological space. X is m-pathwise connected if and only if any two constant multi-valued function from X to X are m-homotopic to each other.*

Proof. By Teorem 5.3, proof is completed. $\square$

Definition 5.2. ([**23**]) Two continuous functions $F, G : X \rightrightarrows Y$ are said to be m-homotopic relative to A contained in X and B contained in Y if there exists an m-homotopy H connecting F and G such that $x \in A$ and $t \in I$ imply that $H(x,t) = B$

Theorem 5.4. *Let X,Y and Z be compact topological spaces and A, B and C be closed subsets of X, Y and Z respectively. Let $F_0, F_1 : X \rightrightarrows Y$ be continuous multi-valued functions such that $F_0|_A = F_1|_A$ and for $i = 0,1$ $F_i(A) \subset B$. Let $G_0, G_1 : Y \rightrightarrows Z$ be a continuous multi-valued function such that $G_0|_B = G_1|_B$. If $F_0 \simeq_m F_1$ relative to (A,B) and $G_0 \simeq_m G_1$ relative to (B,C), then $G_0 \circ F_0$ and $G_1 \circ F_1$ are m-homotopic relative to (A,C).*

Proof. If $F_0 \simeq_m F_1$ relative to (A,B), then there exists an m-homotopy function H connecting F_0 and F_1 such that for $x \in A$ and $t \in I$, $H(x,t) = B$.

Moreover, if $G_0 \simeq_m G_1$ relative to (B,C), then there exists an m-homotopy function K connecting G_0 and G_1 such that for $x \in B$ and $t \in I$, $K(x,t) = C$.

Thus, define an m-homotopy function

$$\begin{aligned} H \times 1 : X \times I &\rightrightarrows Y \times I \\ (x,t) &\mapsto H(x,t) \times 1(x,t) \end{aligned}$$

using a continuous multi-valued function

$$\begin{aligned} 1 : X \times I &\rightrightarrows I \\ (x,t) &\mapsto \{t\}. \end{aligned}$$

First, the continuity of $H \times 1$ and K implies the continuity of $K \circ (H \times 1)$. Furthermore, we have

$$\begin{aligned} K \circ (H \times 1)(x,0) &= K(H \times 1(x,0)) = K(H(x,0) \times 1(x,0)) \\ &= K(F_0(x) \times \{0\}) \\ &= G_0(F_0(x)) = G_0 \circ F_0(x), \end{aligned}$$

where $F_0(x) \times \{0\} = \{(m,n)|m \in F_0(x), n = 0\}$ and

$$\begin{aligned} K \circ (H \times 1)(x,1) &= K(H \times 1(x,1)) = K(H(x,1) \times 1(x,1)) \\ &= K(F_1(x) \times \{1\}) \\ &= G_1(F_1(x)) = G_1 \circ F_1(x), \end{aligned}$$

where $F_1(x) \times \{1\} = \{(m,n)|m \in F_1(x), n = 0\}$.
For $x \in A$ and $t \in I$, we know that $H(x,t) = B$ and $K(x,t) = C$. So, we also have

$$\begin{aligned} K \circ (H \times 1)(x,t) &= K(H(x,t) \times 1(x,t)) \\ &= K(B \times \{t\}) = C, \end{aligned}$$

where $B \times \{t\} = \{(m,n)|m \in B, n = t\}$.

Consequently, $G_0 \circ F_0$ and $G_1 \circ F_1$ are m-homotopic relative to (A, C). $\square$

Strother [**23**] has given a multi-valued version of null homotopy for single-valued functions.

Definition 5.3. ([**23**]) A multi-valued function F is said to be constant if, for some fixed C_0 contained in Y, $x \in X$ implies that $F(x) = C_0$. If G is m-homotopic to a constant function we say that G is null m-homotopic and denote it by $G \simeq_m 0$.

Lemma 5.2. *Let X, Y and Z be compact topological spaces. If $F : X \rightrightarrows Y$ is null m-homotopic and $G : Y \rightrightarrows Z$ is a continuous multi-valued function, then $G \circ F$ is null m-homotopic.*

Proof. Since F is null m-homotopic, there exists a constant multi-valued function $C : X \rightrightarrows Y$ such that $F \simeq_m C$. Then we have the m-homotopy function $H : X \times I \rightrightarrows Y$, $H(x,0) = F(x)$ and $H(x,1) = C(x)$. Consider the multi-valued function $K = G \circ H$. Since G and H are continuous so K is. Let $G(Y_0) = Z_0$, where $Z_0 \subset Z$. Then we have

$$\begin{aligned} K(x,0) &= G \circ H(x,0) = G(H(x,0)) = G(F(x)) = G \circ F(x) \quad \text{and} \\ K(x,1) &= G \circ H(x,1) = G(H(x,1)) = G(C(x)) = G(Y_0) \\ &= Z_0 = C^{'}(x), \end{aligned}$$

where $C^{'} : X \rightrightarrows Z$, $x \mapsto Z_0$ is a constant multi-valued function. Thus, K can be considered as an m-homotopy function between $G \circ F$ and the constant multi-valued function. So, $G \circ F$ is null m-homotopic. $\square$

Lemma 5.3. *If for any topological space Y the multi-valued function $F : S^1 \rightrightarrows Y$ is null m-homotopic, then it can be extended to a continuous multi-valued function $G : D^2 \rightrightarrows Y$.*

Proof. Assume that $F : S^1 \rightrightarrows Y$ is null m-homotopic. Then for a constant multi-valued function $C : S^1 \rightrightarrows Y$, $x \mapsto Y_0$, where Y_0 is a subset of Y, there exists an m-homotopy function $H : S^1 \times I \rightrightarrows Y$ such that $H(x,0) = F(x)$ and

$H(x, 1) = C(x)$. Define a function G by

$$G : D^2 \rightrightarrows Y$$
$$x \mapsto \begin{cases} Y_0, & 0 \leqslant ||x|| \leqslant \frac{1}{2} \\ H(\frac{x}{||x||}, 2 - 2||x||), & \frac{1}{2} \leqslant ||x|| \leqslant 1. \end{cases}$$

It is continuous from Lemma 2.4. Furthermore, for $x \in S^1 \subset D^2$, we get $G(x) = H(x, 0) = F(x)$. This means that the multi-valued function G is an extension of a multi-valued function F. $\square$

THEOREM 5.5. *Let Y be a topological space. If Y is m-pathwise connected, then an m-homotopy class of continuous multi-valued functions from a unit interval I to Y has only one element.*

PROOF. Let $F : I \rightrightarrows Y$ be a continuous multi-valued function such that $F(0) = Y_1$, $F(1) = Y_2$, where Y_1, Y_2 are closed in Y and $C_0 : I \rightrightarrows Y$, $t \mapsto Y_0$ be a constant multi-valued function, where Y_0 closed in Y. Since Y is m-pathwise connected, there exists a continuous multi-valued function $G : I \rightrightarrows Y$ such that $G(0) = Y_1$ and $G(1) = C_0(t) = Y_0$. If we define a multi-valued function H such that

$$H : I \times I \rightrightarrows Y$$
$$(t, s) \mapsto H(t, s) = \begin{cases} F((1 - 2s)t) & 0 \leqslant s \leqslant \frac{1}{2} \\ G(2s - 1) & \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$

then it is continuous from Lemma 2.4. As

$$H(t, 0) = F(t)$$
$$H(t, 1) = G(1) = Y_0 = C_0(t),$$

we get $F \simeq_m C_0$. Consequently, any contiuous multi-valued function from I to Y is m-homotopic to the constant multi-valued function. So, the m-homotopy class has only one element. $\square$

Strother [**23**] defined the product $F * G$ of F and G.

DEFINITION 5.4. ([**23**]) For each space Y, a closed subset Y_0 of Y, and a positive integer n we define

$$MQ(n, Y, Y_0) = \{F : I^n \rightrightarrows Y | F \text{ is continuous and } F(x) = Y_0 \text{ for all } x \in B^{n-1}\}.$$

The subset B^{n-1} of I^n (product unit interval) consisting of points $(x_1, \cdots, x_n)$ for which some coordinate is zero or one, and is called the boundary of I^n.

DEFINITION 5.5. ([**23**]) Let F and G be elements of $MQ(n, Y, Y_0)$. Define $H = F * G$ by

$$H(x_1, \cdots, x_n) = \begin{cases} F(2x_1, \cdots, x_n) & \text{if } 0 \leqslant x_1 \leqslant \frac{1}{2} \\ G(2x_1 - 1, \cdots, x_n) & \text{if } \frac{1}{2} \leqslant x_1 \leqslant 1. \end{cases}$$

Denote the class of functions m-homotopic to F relative to (B^{n-1}, Y_0) by $[F]$. Define $[F] * [G]$ to be $[F * G]$

Moreover, the function H is well-defined and continuous by Lemma 2.4.

THEOREM 5.6 ([**23**]). *Let Y be a compact Hausdorff space. Then the m-homotopy classes of the continuous functions in $MQ(n, Y, Y_0)$ form a group $M\Pi_n(Y, Y_0)$ called the n-th m-homotopy group of the space Y with base set Y_0. For $x \in B^{n-1}$, the zero element in $M\Pi_n(Y, Y_0)$ is the constant function $F(x) = Y_0$ and the inverse of a function F is defined by*

$$F^{-1}(x_1, \cdots, x_n) = F(1 - x_1, \cdots, x_n),$$

for $(x_1, \cdots, x_n) \in B^{n-1}$.

From the definition of a product operation and Theorem 5.6, we will give the following property about product operation. Suppose that $F, G \in MQ(n, Y, Y_0)$. Then for $x = (x_1, \cdots, x_n)$,

$$\begin{aligned} G * F^{-1}(x) &= \begin{cases} G(2x_1, x_2, \cdots x_n) & 0 \leqslant x \leqslant \frac{1}{2} \\ F^{-1}(2x_1 - 1, x_2, \cdots, x_n) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \\ &= \begin{cases} G(2x_1, x_2, \cdots, x_n) & 0 \leqslant x \leqslant \frac{1}{2} \\ F(1 - (2x_1 - 1), x_2, \cdots, x_n) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \\ &= \begin{cases} G(2x_1, x_2, \cdots, x_n) & 0 \leqslant x \leqslant \frac{1}{2} \\ F(2 - 2x_1, x_2, \cdots, x_n) & \frac{1}{2} \leqslant x \leqslant 1. \end{cases} \end{aligned}$$

Moreover, when $x_1 = \frac{1}{2}$, then $G_1 = Y_0 = F_1$. Since

$$\begin{aligned} (F * G)^{-1}(x) &= F * G(1 - x_1, x_2, \cdots, x_n) \\ &= \begin{cases} F(2(1 - x_1), x_2, \cdots, x_n) & 0 \leqslant 1 - x \leqslant \frac{1}{2} \\ G(2(1 - x_1) - 1, x_2, \cdots, x_n) & \frac{1}{2} \leqslant 1 - x \leqslant 1 \end{cases} \\ &= \begin{cases} F(2 - 2x_1, x_2, \cdots, x_n) & \frac{1}{2} \leqslant x \leqslant 1 \\ G(1 - 2x_1, x_2, \cdots, x_n) & 0 \leqslant x \leqslant \frac{1}{2}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} G^{-1} * F^{-1}(x) &= \begin{cases} G^{-1}(2x_1, x_2, \cdots, x_n) & 0 \leqslant x \leqslant \frac{1}{2} \\ F^{-1}(2x_1 - 1, x_2, \cdots, x_n) & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \\ &= \begin{cases} G(1 - 2x_1, x_2, \cdots, x_n) & \frac{1}{2} \leqslant x \leqslant 1 \\ F(1 - (2x_1 - 1), x_2, \cdots, x_n) & 0 \leqslant x \leqslant \frac{1}{2} \end{cases} \\ &= \begin{cases} G(1 - 2x_1, x_2, \cdots, x_n) & \frac{1}{2} \leqslant x \leqslant 1 \\ F(2 - 2x_1, x_2, \cdots, x_n) & 0 \leqslant x \leqslant \frac{1}{2}. \end{cases} \end{aligned}$$

Then $(F * G)^{-1} = G^{-1} * F^{-1}$ is obtained.

THEOREM 5.7. *Let $F_0, F_1, G_0, G_1 \in MQ(n, Y, Y_0)$. If F_0 and F_1 are m-homotopic relative to (B^{n-1}, Y_0) and G_0, G_1 are m-homotopic relative to (B^{n-1}, Y_0), then $F_0 * G_0$ and $F_1 * G_1$ are m-homotopic relative to (B^{n-1}, Y_0).*

PROOF. If F_0 and F_1 are m-homotopic relative to B^{n-1} contained in I^n and Y_0 contained in Y, then there exists an m-homotopy $H : I^n \times I \to Y$ connecting F_0 and F_1 such that for $x \in B^{n-1}$ and $t \in I$, $H(x,t) = Y_0$.

If G_0 and G_1 are m-homotopic relative to B^{n-1} contained in I^n and Y_0 contained in Y, then there exists an m-homotopy $K : I^n \times I \to Y$ connecting G_0 and G_1 such that for $x \in B^{n-1}$ and $t \in I$, $K(x,t) = Y_0$.

Therefore, we can define an m-homotopy function between $F_0 * G_0$ and $F_1 * G_1$ as follows:

$$M : I^n \times I \to Y$$

$$(x_1, x_2, \cdots x_n, t) \mapsto \begin{cases} H(2x_1, x_2, \cdots x_n, t) & 0 \leqslant x_1 \leqslant \frac{1}{2} \\ K(2x_1 - 1, x_2, \cdots x_n, t) & \frac{1}{2} \leqslant x_1 \leqslant 1, \end{cases}$$

$$\begin{aligned} M(x_1, x_2, \cdots x_n, 0) &= \begin{cases} H(2x_1, x_2, \cdots x_n, 0) & 0 \leqslant x_1 \leqslant \frac{1}{2} \\ K(2x_1 - 1, x_2, \cdots x_n, 0) & \frac{1}{2} \leqslant x_1 \leqslant 1 \end{cases} \\ &= \begin{cases} F_0(2x_1, x_2, \cdots x_n) & 0 \leqslant x_1 \leqslant \frac{1}{2} \\ G_0(2x_1 - 1, x_2, \cdots x_n) & \frac{1}{2} \leqslant x_1 \leqslant 1 \end{cases} \\ &= F_0 * G_0, \end{aligned}$$

and

$$\begin{aligned} M(x_1, x_2, \cdots x_n, 1) &= \begin{cases} H(2x_1, x_2, \cdots x_n, 1) & 0 \leqslant x_1 \leqslant \frac{1}{2} \\ K(2x_1 - 1, x_2, \cdots x_n, 1) & \frac{1}{2} \leqslant x_1 \leqslant 1 \end{cases} \\ &= \begin{cases} F_1(2x_1, x_2, \cdots x_n) & 0 \leqslant x_1 \leqslant \frac{1}{2} \\ G_1(2x_1 - 1, x_2, \cdots x_n) & \frac{1}{2} \leqslant x_1 \leqslant 1 \end{cases} \\ &= F_1 * G_1 \end{aligned}$$

and it is continuous from Lemma 2.4. Moreover, for all $x \in B^{n-1}$ and $t \in I$, $H(x,t) = Y_0$ and $K(x,t) = Y_0$ because H and K are m-homotopy functions.
If $x = (x_1, \cdots, x_n) \in B^{n-1}$, then $x_1 = 0$ or $x_1 = 1$. If $x_1 = 0$, then $M(x_1, x_2, \cdots x_n, t) = H(2x_1, x_2, \cdots x_n, t) = Y_0$ and if $x_1 = 1$, then $M(x_1, x_2, \cdots x_n, t) = K(2x_1, x_2, \cdots x_n, t) = Y_0$.

Thus, $F_0 * G_0$ and $F_1 * G_1$ are m-homotopic relative to (B^{n-1}, Y_0). □

THEOREM 5.8. *Let $F : I \rightrightarrows Y$ and $G : I \rightrightarrows Y$ be continuous multi-valued functions such that $F(0) = Y_0$, $F(1) = Y_1$, $G(0) = Y_1$ and $G(1) = Y_2$, where Y_0, Y_1 and Y_2 are closed subsets of Y. If $H = F * G$, then for the multi-valued function $\hat{H}$ defined by*

$$\begin{aligned} \hat{H} : M\Pi_n(Y, Y_0) &\to M\Pi_n(Y, Y_2) \\ [Z] &\mapsto [H^{-1}] * [Z] * [H], \end{aligned}$$

we have $\hat{H} = \hat{G} \circ \hat{F}$.

PROOF. We have the following equality;

$$\begin{aligned}\hat{H}([Z]) &= [H^{-1}] * [Z] * [H] = [(F * G)^{-1}] * [Z] * [F * G]\\ &= [G^{-1} * F^{-1}] * [Z] * [F * G]\\ &= [G^{-1}] * [F^{-1}] * [Z] * [F] * [G]\\ &= [G^{-1}] * \hat{F}[Z] * [G]\\ &= \hat{G}((\hat{F}[Z])) = \hat{G} \circ \hat{F}([Z]).\end{aligned}$$

So, we obtain the desired result. □

DEFINITION 5.6. Let $K : (Y, Y_0) \rightrightarrows (X, X_0)$ be a continuous multi-valued function and $F \in M\Pi_n(Y, Y_0)$. Define a function as follows:

$$\begin{aligned}K_* : M\Pi_n(Y, Y_0) &\to M\Pi_n(X, X_0)\\ [F] &\mapsto K_*[F] = [K \circ F].\end{aligned}$$

The function K_* is called the m-homomorphism induced by the multi-valued function H relative to the base set Y_0.

Moreover, if H is an m-homotopy between F and $F^{'}$, then $K \circ H$ is an m-homotopy between $K \circ F$ and $K \circ F^{'}$. So, K_* is well-defined.

The fact that K_* is a homomorphism follows from the following equality

$$\begin{aligned}K_*([F] * [H]) &= K_*([F * H]) = [K \circ (F * H)]\\ &= [(K \circ F) * (K \circ H)] = [(K \circ F)] * [(K \circ H)]\\ &= K_*([F]) * K_*([H]).\end{aligned}$$

PROPOSITION 5.1. *Let X, Y and Z be compact spaces. If $F : (X, X_0) \rightrightarrows (Y, Y_0)$ and $G : (Y, Y_0) \rightrightarrows (Z, Z_0)$ are the continuous multi-valued functions, then*

$$(G \circ F)_* = G_* \circ F_*$$

holds.

If $id_{(X,X_0)} : (X, X_0) \rightrightarrows (X, X_0)$ is the identical function, then id_* is the identical homomorphism.

PROOF. Let $H : I \rightrightarrows X$ be a continuous multi-valued function such that $H(0) = H(1) = X_0$, where $X_0 \subset X$ is closed set. By the definition, we have

$$\begin{aligned}G_* \circ F_*([H]) &= G_*([F \circ H]) = [G \circ (F \circ H)]\\ &= [(G \circ F) \circ H] = (G \circ F)_*([H]).\end{aligned}$$

So, $(G \circ F)_* = G_* \circ F_*$ is obtained.

Similarly, we get

$$\mathrm{id}_*([H]) = [\mathrm{id} \circ H] = [H].$$

□

LEMMA 5.4. *Let X and Y be compact topological spaces. If C is the constant multi-valued function defined by*

$$\begin{aligned} C : X &\rightrightarrows Y \\ x &\mapsto Y_0, \end{aligned}$$

where Y_0 is closed subset of Y, then the m-homomorphism induced by C is a trivial homomorphism.

PROOF. Let F be the zero element in $M\Pi(Y, Y_0)$. Then $F : I^n \rightrightarrows Y$ is a continuous multi-valued function such that $F(x) = Y_0$, for all $x \in B^{n-1}$, where Y_0 is closed subset of Y. Since the m-homomorphism induced by C is

$$\begin{aligned} C_* : M\Pi_n(X, x_0) &\to M\Pi_n(Y, Y_0) \\ [F] &\mapsto [C \circ F] \end{aligned}$$

and $C \circ F : I^n \rightrightarrows Y$ here is a constant multi-valued function, $C \circ F$ is a zero element in $M\Pi_n(Y, Y_0)$. Thus, the m-homomorphism induced by C is a trivial homomorphism. □

LEMMA 5.5. *Let X and Y be compact topological spaces. If $H : X \rightrightarrows Y$ and $K : X \rightrightarrows Y$ are continuous multi-valued function such that they are m-homotopic, then the m-homomorphism induced by these functions are equal.*

PROOF. Assume that $H : X \rightrightarrows Y$ and $K : X \rightrightarrows Y$ are m-homotopic. Since the m-homotopy relation is reflexive, for any continuous multi-valued function $F : I^n \rightrightarrows X$, $F \simeq_m F$. So, we have

$$H \circ F \simeq_m K \circ F.$$

Hence, for the induced homomorphisms $H_*, K_* : M\Pi_n(X, X_0) \rightrightarrows M\Pi_n(Y, Y_0)$, we get $[H \circ F] = [K \circ F]$. So, the equality $H_*([F]) = K_*([F])$ holds. □

COROLLARY 5.2. *Let X and Y be compact topological spaces. If $H : X \rightrightarrows Y$ is a null m-homotopic, then the induced m-homomorphism H_* is trivial.*

PROOF. Since $H : X \rightrightarrows Y$ is null m-homotopic, it is m-homotopic to a constant multi-valued function C i.e. $H \simeq_m C$, where $C : X \rightrightarrows Y$ is a constant function. From Lemma 5.5, their induced homomorphisms are equal, i.e. $H_* = C_*$. Since the m-homomorphism induced by constant function is a trivial homomorphism, H_* is trivial. □

In this part, we start with giving the multi-valued version of the homotopy equivalence in algebraic topology called m-homotopy equivalence and m-homotopy type.

DEFINITION 5.7. Let X and Y be compact spaces. A continuous multi-valued function $F : X \rightrightarrows Y$ is an m-homotopy equivalance if there is a continuous multi-valued function $G : Y \rightrightarrows X$ with $G \circ F \simeq_m \mathrm{id}_X$ and $F \circ G \simeq_m \mathrm{id}_Y$.

Two spaces X and Y have the same m-homotopy type if there is an m-homotopy equivalence and written as $X \simeq_m Y$.

LEMMA 5.6. *Let X, Y and Z be compact topological spaces. If $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are m-homotopy equivalences, then $G \circ F$ is also an m-homotopy equivalence.*

PROOF. Assume that $F : X \rightrightarrows Y$ is an m-homotopy equivalence. Then there exists a continuous multi-valued function $H : Y \rightrightarrows X$ such that $F \circ H \simeq_m \mathrm{id}_Y$ and $H \circ F \simeq \mathrm{id}_X$.

Since $G : Y \rightrightarrows Z$ is an m-homotopy equivalence, there exists a continuous multi-valued function $K : Z \rightrightarrows Y$ such that $G \circ K \simeq_m \mathrm{id}_Z$ and $K \circ G \simeq \mathrm{id}_Y$. We obtain

$$(G \circ F) \circ (H \circ K) = G \circ (F \circ H) \circ K \simeq_m G \circ \mathrm{id}_Y \circ K = G \circ K \simeq_m \mathrm{id}_Z$$

and

$$(H \circ K) \circ (G \circ F) = H \circ (K \circ G) \circ F \simeq_m H \circ \mathrm{id}_Y \circ F = H \circ F \simeq_m \mathrm{id}_X.$$

Therefore, $G \circ F$ is an m-homotopy equivalence. □

LEMMA 5.7. *The same m-homotopy type relation on compact spaces is an equivalence relation.*

PROOF. For any topological space X since $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X \simeq_m \mathrm{id}_X$, the space X has the same m-homotopy type with itself. Suppose that topological spaces X and Y has the relation $X \simeq_m Y$. So, there exists an m-homotopy equivalence $F : X \rightrightarrows Y$, this means that there exists a continuous multi-valued function $G : Y \rightrightarrows X$ such that $G \circ F \simeq_m \mathrm{id}_X$ and $F \circ G \simeq_m \mathrm{id}_Y$. Thus, G is an m-homotopy equivalence and Y has the same m-homotopy type with X i.e. $Y \simeq_m X$. Assume that $X \simeq Y$ and $Y \simeq Z$. So, there exists m-homotopy equivalences $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$. Then from the Lemma 5.6 $G \circ F : X \rightrightarrows Z$ is an m-homotopy equivalence. Thus, we get $X \simeq_m Z$. □

PROPOSITION 5.2. *Let X, Y be compact spaces. Assume that $F : X \rightrightarrows Y$ is an m-homotopy equivalence and $G : X \rightrightarrows Y$ is continuous. If G is m-homotopic to F, then G is an m-homotopy equivalence.*

PROOF. Let $F : X \rightrightarrows Y$ be an m-homotopy equivalence. Then there exists a continuous multi-valued function $H : Y \rightrightarrows X$ such that $F \circ H \simeq_m \mathrm{id}_Y$ and $H \circ F \simeq \mathrm{id}_X$. Let $G : X \rightrightarrows Y$ be a continuous multi-valued function which is m-homotopic to F. So

$$\begin{aligned} F \simeq_m G &\Rightarrow H \circ F \simeq_m H \circ G \\ &\Rightarrow \mathrm{id}_X \simeq_m H \circ G \end{aligned}$$

and

$$\begin{aligned} F \simeq_m G &\Rightarrow F \circ H \simeq_m G \circ H \\ &\Rightarrow \mathrm{id}_Y \simeq_m G \circ H. \end{aligned}$$

Hence, G is an m-homotopy equivalence. □

THEOREM 5.9. *Let X and Y be compact topological spaces. If $F : X \rightrightarrows Y$ is an m-homotopy equivalence, then m-homomorphism induced by F is an isomorphism.*

PROOF. The m-homomorphism induced by $F : X \rightrightarrows Y$ is defined by

$$\begin{aligned} F_* : M\Pi_n(X, X_0) &\to M\Pi_n(Y, Y_0) \\ [G] &\mapsto [F \circ G], \end{aligned}$$

where X_0 is closed in X and Y_0 is closed in Y. The multi-valued function maps one equivalence class to another. So, it can be considered as a single-valued function. Thus, we will continue the proof by considering F_* as a single-valued function. Since F is an m-homotopy equivalence, there exists a continuous multi-valued function $H : Y \rightrightarrows X$ such that $H \circ F \simeq_m \mathrm{id}_X$ and $F \circ H \simeq_m \mathrm{id}_Y$. Then we have

$$H \circ F \simeq_m \mathrm{id}_X \Rightarrow (H \circ F)_* = H_* \circ F_* = (\mathrm{id}_X)_*.$$

On the other hand, we know that an m-homomorphism induced by identical function is an identical homomorphism i.e.

$$\begin{aligned} (\mathrm{id}_X)_* : M\Pi_n(X, X_0) &\to M\Pi_n(X, X_0) \\ [G] &\mapsto [\mathrm{id}_X \circ G] = [G]. \end{aligned}$$

So, F_* has a left m-homotopy inverse. Similarly, it is shown that the right m-homotopy inverse exists.

Hence, F_* is an m-isomorphism. □

COROLLARY 5.3. *Let X, Y and Z be compact topological spaces. If $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are m-homotopy equivalences, then the m-homomorphism induced by $G \circ F$ is an isomorphism.*

PROOF. By Lemma 5.6, $G \circ F$ is an m-homotopy equivalence. From Theorem 5.9, the m-homomorphism $(G \circ F)_*$ induced by $G \circ F$ is an isomorphism. □

Now, we introduce the notions of contractibility and retraction that are adapted to multi-valued functions.

DEFINITION 5.8. A compact topological space X is m-contractible if the identical function id_X is null m-homotopic.

THEOREM 5.10. *Let X and Y be compact topological spaces. If X and Y are m-contractible spaces, then $X \times Y$ is m-contractible.*

PROOF. Assume that X is m-contractible. Then there exists a constant multi-valued function $C_X : X \rightrightarrows X$, $x \mapsto X_0$ such that $\mathrm{id}_X \simeq_m C_X$, where X_0 is closed in X. Because Y is m-contractible, there exists a constant multi-valued function $C_Y : Y \rightrightarrows Y$, $y \mapsto Y_0$ such that $\mathrm{id}_Y \simeq_m C_Y$, where Y_0 is closed in Y. As $\mathrm{id}_X \simeq_m C_X$, we have a multi-homotopy function $H : X \times I \rightrightarrows X$ such that $H(x, 0) = \mathrm{id}_X(x)$ and $H(x, 1) = C_X(x)$. Since $\mathrm{id}_Y \simeq_m C_Y$, we have a multi-homotopy function $K : Y \times I \rightrightarrows Y$ such that $K(y, 0) = \mathrm{id}_Y(y)$ and

$K(y, 1) = C_Y(y)$. So, if the multi-valued function F is defined by

$$\begin{aligned} F : (X \times Y) \times I &\rightrightarrows X \times Y \\ (x, y, t) &\mapsto \Big\{ \Big(H(x, t), K(y, t) \Big) \Big\}, \end{aligned}$$

then we have

$$\begin{aligned} F(x, y, 0) &= \big\{ \big(H(x, 0), K(y, 0) \big) \big\} = \{ (\mathrm{id}_X(x), \mathrm{id}_Y(y)) \} \\ &= \{ (x, y) \} = \mathrm{id}_{X \times Y}(x, y) \end{aligned}$$

and

$$\begin{aligned} F(x, y, 1) &= \big\{ \big(H(x, 1), K(y, 1) \big) \big\} = \big\{ \big(C_X(x), C_Y(y) \big) \big\} = \{ (X_0, Y_0) \} \\ &= C_0 = C_{X \times Y}(x, y), \end{aligned}$$

where $C_{X \times Y}$ is the constant multi-valued function defined by

$$\begin{aligned} C_{X \times Y} : X \times Y &\rightrightarrows X \times Y \\ (x, y) &\mapsto C_0. \end{aligned}$$

The set C_0 is defined by $C_0 = \{(X_0, Y_0)\} = \{(x, y) \in X \times Y | x \in X_0, y \in Y_0\}$. By the upper-semi continuity and the lower-semi continuity of the functions H and K, F is continuous. Consequently, $X \times Y$ is m-contractible. □

Lemma 5.8. *If a compact space Y is m-contractible, then any two multi-valued function from any compact topological space X to Y are m-homotopic.*

Proof. Suppose that the space Y is an m-contractible space. Then id_Y is an m-homotopic to a constant multi-valued function $C : Y \rightrightarrows Y$ on the closed subset Y_0 of Y.

For any continuous multi-valued function $F : X \rightrightarrows Y$, we get

$$F = id_Y \circ F \simeq_m C \circ F,$$

where $C \circ F$ is a constant multi-valued function.

Since m-homotopy relation is an equivalence relation, any two continuous multi-valued function from X to Y are m-homotopic. □

Theorem 5.11. *Let X and Y be compact topological spaces. If Y is m-contractible, then an m-homotopy class of a continuous multi-valued function from X to Y has only one element.*

Proof. Let Y be m-contractible. Then an identical function $\mathrm{id}_Y : Y \rightrightarrows Y$ is m-homotopic to a constant multi-valued function $C : Y \rightrightarrows Y$, $y \mapsto Y_0$, where $Y_0 \subset Y$. So, there exists a multi-homotopy function $H : Y \times I \rightrightarrows Y$ between id_Y and C. Let $F : X \rightrightarrows Y$ be any continuous multi-valued function. So, we know that a multi-valued function $F \times \mathrm{id}_I : X \times I \rightrightarrows Y \times I$ is continuous, where id_I is a identical function on unit interval I by Lemma 2.2. Then a multi-valued function G defined by

$$\begin{aligned} G : X \times I &\rightrightarrows Y \\ (x, t) &\mapsto G(x, t) = H \circ (F \times \mathrm{id}_I)(x, t) \end{aligned}$$

is continuous. Moreover, we have

$$G(x,0) = H \circ (F \times \mathrm{id}_I)(x,0) = H(F(x),0) = \mathrm{id}_Y(F(x)) = F(x),$$
$$G(x,1) = H \circ (F \times \mathrm{id}_I)(x,1) = H(F(x),1) = C(F(x)) = Y_0.$$

Thus, F is m-homotopic to a constant multi-valued function. Since F is arbitrarly chosen, it means that an m-homotopy class of a continuous multi-valued function from A to B has only one element. □

Theorem 5.12. *Let X be a compact topological space. The space X is m-contractible if and only if for any compact topological space Y, every continuous multi-valued function $F : Y \rightrightarrows X$ is null m-homotopic.*

Proof. If the space X is m-contractible, then every continuous multi-valued function $F : Y \rightrightarrows X$ is null m-homotopic by Theorem 5.11. Conversely, let continuous multi-valued function $F : Y \rightrightarrows X$ is null m-homotopic. Since the space Y is any space, we can take $Y = X$. Thus, the identical function on X is null m-homotopic. So, X is m-contractible. □

Theorem 5.13. *Let X be a finite space. If X is m-contractible, then it is m-pathwise connected.*

Proof. Since X is m-contractible, the identical function on X is null m-homotopic. For any subset X_0 of X, define the constant multi-valued function C by

$$\begin{aligned} C : X &\rightrightarrows X \\ x &\mapsto X_0. \end{aligned}$$

Thus, there exists a continuous multi-valued function $H : X \times I \rightrightarrows X$ such that $H(x,0) = \mathrm{id}_X(x) = \{x\}$ and $H(x,1) = C(x) = X_0$. Define a multi-valued function F such that

$$\begin{aligned} F : I &\rightrightarrows X \\ t &\mapsto F(t) = H(x,t). \end{aligned}$$

A continuity of H implies that F is continuous. Moreover, we have

$$F(0) = H(x,0) = \mathrm{id}_X(x) = \{x\} \quad \text{and}$$
$$F(1) = H(x,1) = C(x) = X_0,$$

where $\{x\}$ and X_0 are closed subsets of X. Therefore, we obtain that X is m-pathwise connected. □

Definition 5.9. A closed subspace A of a compact topological space X is m-retract of X if there is a continuous multi-valued function $R : X \rightrightarrows A$ with $R(a) = \{a\}$, for all $a \in A$ such a multi-valued function is called an m-retraction.

Equivalently, a continuous multi-valued function R such that $R \circ \mathcal{I} = id_A$ is called an m-retraction, where $\mathcal{I} : A \rightrightarrows X$ is an m-inclusion function.

THEOREM 5.14. *Let X, Y and Z be compact topological spaces and $X \subset Y$. If X is an m-retract of Y, then every continuous multi-valued function $F : X \rightrightarrows Z$ can be extended to a continuous multi-valued function $\tilde{F} : Y \rightrightarrows Z$. So, if X is the m-retract of Y and $F_0, F_1 : X \rightrightarrows Z$ are the m-homotopy equivalent functions, then $\tilde{F_0} \simeq_m \tilde{F_1}$.*

PROOF. Assume that X is the m-retract of Y. Then there exists a continuous multi-valued function $R : Y \rightrightarrows X$ with $R(x) = \{x\}$, for all $x \in X$. If $F_0, F_1 : X \rightrightarrows Z$ are m-homotopy equivalent functions, then there exists a continuous multi-valued function $H : X \times I \rightrightarrows Z$ such that $H(x, 0) = F_0(x)$ and $H(x, 1) = F_1(x)$. Assume that $\mathrm{id}_I : I \rightrightarrows I$ is an identical function on a unit interval. The continuities of id_I, R and H imply that $H \circ (R \times I)$ is continuous. Thus, the multi-valued function defined by

$$\begin{aligned} K : Y \times I &\rightrightarrows Z \\ (y, t) &\mapsto K(y, t) = H \circ (R \times I)(y, t) \end{aligned}$$

is continuous. Since we have

$$\begin{aligned} K(y, 0) &= H \circ (R \times I)(y, 0) = H(R(y), 0) = F_0(R(y)) = F_0 \circ R(y) = \tilde{F_0} \quad \text{and} \\ K(y, 1) &= H \circ (R \times I)(y, 1) = H(R(y), 1) = F_1(R(y)) = F_1 \circ R(y) = \tilde{F_1}, \end{aligned}$$

$\tilde{F_0} \simeq_m \tilde{F_1}$ is obtained. □

LEMMA 5.9. *Let X be a compact topological space, A be a closed in X and $R : X \rightrightarrows A$ be an m-retraction function. If X is m-contractible, then A is also m-contractible.*

PROOF. Assume that X is m-contractible. Then there exists a constant multi-valued function

$$\begin{aligned} C_X : X &\rightrightarrows X \\ x &\mapsto X_0 \end{aligned}$$

such that $\mathrm{id}_X \simeq_m C_X$, where $X_0 \subset X$. Thus, there exists a multi-homotopy function $H : X \times I \rightrightarrows X$ such that $H(x, 0) = \mathrm{id}_X$ and $H(x, 1) = C_X$. If $R : X \rightrightarrows A$ is the m-retraction, then for all $a \in A$, $R(a) = \{a\}$. Let $A_0 = R(X_0)$ where A_0 is subset of A and let K be the composition $R \circ H$. Then this function K is continuous. On the other hand, we have

$$\begin{aligned} K(a, 0) &= R \circ H(a, 0) = R(\mathrm{id}_X(a)) \\ &= R(a) = \{a\} = \mathrm{id}_A(a) \end{aligned}$$

and

$$\begin{aligned} K(a, 1) &= R \circ H(a, 1) = R(C_X(a)) \\ &= R(X_0) = A_0 = C_A(x), \end{aligned}$$

where

$$\begin{aligned} C_A : A &\rightrightarrows A \\ a &\mapsto A_0 \end{aligned}$$

is the constant function on A. Thus, A is m-contractible. □

DEFINITION 5.10. Let X be a compact topological space and A be a closed subspace of X. If there exists a multi-homotopy function $H : X \times I \rightrightarrows X$ such that for all $x \in X$ and $a \in A$,

$$\begin{aligned} &H(x,0) = \{x\}, \\ &H(x,1) \in A \ \text{ and} \\ &H(a,1) = \{a\}, \end{aligned}$$

then a subspace A of X is called a deformation m-retract of X.

Equivalent to this definition, if there exists an m-retraction $R : X \rightrightarrows A$ such that $R \circ \mathcal{I} = id_A$ and $\mathcal{I} \circ R \simeq_m \mathrm{id}_X$, where $\mathcal{I} : A \rightrightarrows X$ is an m-inclusion function, then a subspace A of X is called a deformation m-retract of X.

THEOREM 5.15. *Let X be a compact topological space and A, B be closed subsets of X such that $B \subset A \subset X$. If A is a deformation m-retract of X and B is a deformation m-retract of A, then B is a deformation m-retract of X.*

PROOF. If A is a deformation m-retract of X, then there exists a multi-homotopy function $H : X \times I \rightrightarrows X$ such that for all $x \in X$ and $a \in A$,

$$\begin{aligned} &H(x,0) = \{x\}, \\ &H(x,1) \in A \ \text{ and} \\ &H(a,1) = \{a\}. \end{aligned}$$

Since B is a deformation m-retract of A, there exists a multi-homotopy function $K : A \times I \rightrightarrows A$ such that for all $x \in A$ and $b \in B$,

$$\begin{aligned} &K(a,0) = \{a\}, \\ &K(a,1) \in B \ \text{ and} \\ &K(b,1) = \{b\}. \end{aligned}$$

If we define a multi-valued function such that

$$\begin{aligned} F : X \times I &\rightrightarrows X \\ (x,t) \mapsto F(x,t) &= \begin{cases} H(x,2t) & , t \in [0, \frac{1}{2}] \\ K(R(x), 2t-1) & , t \in [\frac{1}{2}, 1], \end{cases} \end{aligned}$$

where $R : X \rightrightarrows A$ is an m-retraction then it is continuous from Lemma 2.4. As A is a deformation m-retract of X, we have $R \circ \mathcal{I} = \mathrm{id}_A$ and $\mathcal{I} \circ R \simeq_m \mathrm{id}_X$. So, for all $b \in B \subset A$, we have $R(b) = \{b\}$. Since for all $x \in X$ and $b \in B$, we have

$$\begin{aligned} F(x, 0) &= H(x, 0) = \{x\}, \\ F(x, 1) &= K(R(x), 1) \in B \quad \text{and} \\ F(b, 1) &= K(R(b), 1) = \{b\}. \end{aligned}$$

Thus, B is a deformation m-retract of X. □

LEMMA 5.10. *Let X be a compact topological space and A be a closed subset of X. If A is a deformation m-retract of X, then X and A have the same m-homotopy type.*

PROOF. If A is a deformation m-retract of X, then there exists a continuous multi-valued function $R : X \rightrightarrows A$ such that $R \circ \mathcal{I} = \mathrm{id}_A$ and $\mathcal{I} \circ R \simeq_m \mathrm{id}_A$, where $\mathcal{I} : A \rightrightarrows X$ is the m-inclusion function. If $R \circ \mathcal{I} = \mathrm{id}_A$, then $R \circ \mathcal{I} \simeq_m \mathrm{id}_A$. Since $\mathcal{I} \circ R \simeq_m \mathrm{id}_A$, X and A have the same m-homotopy type. □

DEFINITION 5.11. Let X be a compact topological space, A and $A^{'}$ be nonempty closed subsets of X such that $A^{'} \subseteq A$ and $\mathcal{I} : A \rightrightarrows X$ be an m-inclusion function. If there exists an m-retraction $R : X \rightrightarrows A$ such that $R \circ \mathcal{I} = id_A$ and $\mathcal{I} \circ R \simeq_m \mathrm{id}_X$ relative to $(A^{'}, A^{'})$, then a subspace A of X is called a strong deformation m-retract of X. In this case, the multi-valued function R is called a strong deformation m-retraction function.

LEMMA 5.11. *Let X be a compact topological space, A be a nonempty closed subset of X. If A is a strong deformation m-retract of X, then the m-inclusion function $\mathcal{I} : A \rightrightarrows X$ is m-homotopy equivalence.*

PROOF. Assume that A is a strong deformation m-retract of X. Then there exists an m-retraction $R : X \rightrightarrows A$ such that $R \circ \mathcal{I} = id_A$ and $\mathcal{I} \circ R \simeq_m \mathrm{id}_X$ relative to $(A^{'}, A^{'})$, where $\emptyset \neq A^{'} \subset A$. Thus, there exists a continuous multi-valued function R such that $R \circ \mathcal{I} \simeq_m id_A$ and $\mathcal{I} \circ R \simeq_m \mathrm{id}_X$. So, the m-inclusion function $\mathcal{I}$ is m-homotopy equivalence. □

THEOREM 5.16. *Let X, Y, Z be compact topological spaces and Z, Y be nonempty subspaces such that $Z \subset Y \subset X$. If Z is a strong deformation m-retract of Y and Y is a strong deformation m-retract of X, then Z is a strong deformation m-retract of X.*

PROOF. Assume that Z is a strong deformation m-retract of Y. Then there exists an m-retraction $R_1 : Y \rightrightarrows Z$ such that $R_1 \circ \mathcal{I}_1 = id_Z$ and $\mathcal{I}_1 \circ R_1 \simeq_m \mathrm{id}_Y$ relative to (Z, Z), where $\mathcal{I}_1 : Z \rightrightarrows Y$ be an m-inclusion function. Suppose that Y is a strong deformation m-retract of X. So, there exists an m-retraction $R_2 : X \rightrightarrows Y$ such that $R_2 \circ \mathcal{I}_2 = id_Y$ and $\mathcal{I}_2 \circ R_2 \simeq_m \mathrm{id}_X$ relative to (Z, Z), where $\mathcal{I}_2 : Y \rightrightarrows X$ be an m-inclusion function. Hence, we have continuous multi-valued functions $\mathcal{I}_2 \circ \mathcal{I}_1$ and $R_1 \circ R_2$. Furthermore, we have

$$\begin{aligned}(R_1 \circ R_2) \circ (\mathcal{I}_2 \circ \mathcal{I}_1)(z) &= \bigcup_{x \in (\mathcal{I}_2 \circ \mathcal{I}_1)(z)} R_1 \circ R_2(x) \\ &= \bigcup_{x \in (\mathcal{I}_2 \circ \mathcal{I}_1)(z)} \bigcup_{y \in R_2(x)} R_1(y) = \bigcup_{\substack{x \in (\mathcal{I}_2 \circ \mathcal{I}_1)(z) \\ y \in R_2(x)}} R_1(y) \\ &= \bigcup_{y \in R_2 \circ (\mathcal{I}_2 \circ \mathcal{I}_1)(z)} R_1(y) = \bigcup_{y \in \mathcal{I}_1(z)} R_1(y) \\ &= R_1 \circ \mathcal{I}_1(z) = \{z\} = \mathrm{id}_Z(z).\end{aligned}$$

Since $\mathcal{I}_1 \circ R_1 \simeq_m \mathrm{id}_Y$ relative to (Z, Z), there exists a multi-homotopy function $H : Y \times I \rightrightarrows Y$ such that $H(y, 0) = \mathcal{I}_1 \circ R_1(y)$, $H(y, 1) = \mathrm{id}_Y(y)$ and for all $y \in Z$, $t \in I$, $H(y, t) = Z$. Since $\mathcal{I}_2 \circ R_2 \simeq_m \mathrm{id}_X$ relative to (Z, Z), there exists a multi-homotopy function $K : X \times I \rightrightarrows X$ such that $K(x, 0) = \mathcal{I}_2 \circ R_2(x)$, $K(x, 1) = \mathrm{id}_X(x)$ and for all $x \in Z$, $t \in I$, $K(x, t) = Z$. Since $\mathcal{I}_2 \circ R_2(x) \subset Y$ by the definition of $\mathcal{I}_2$, we can define a multi-valued function F such that

$$\begin{aligned}&F : X \times I \rightrightarrows X \\ &(x, t) \mapsto \begin{cases} H(\mathcal{I}_2 \circ R_2(x), 2t), & t \in [0, \frac{1}{2}] \\ K(x, 2t - 1), & t \in [\frac{1}{2}, 1]. \end{cases}\end{aligned}$$

Therefore, it is continuous by Lemma 2.4 and we find

$$\begin{aligned}F(x, 0) &= H(\mathcal{I}_2 \circ R_2(x), 0) = \mathcal{I}_1 \circ R_1(\mathcal{I}_2 \circ R_2(x)) \\ &= \mathcal{I}_1 \circ R_1(R_2(x)) = \mathcal{I}_1(R_1 \circ R_2(x)) \\ &= \mathcal{I}_2(\mathcal{I}_1(R_1 \circ R_2(x))) = (\mathcal{I}_2 \circ \mathcal{I}_1) \circ (R_1 \circ R_2)(x) \quad \text{and} \\ F(x, 1) &= K(x, 1) = \mathrm{id}_X(x).\end{aligned}$$

So, we obtain $(\mathcal{I}_2 \circ \mathcal{I}_1) \circ (R_1 \circ R_2)(x) \simeq_m \mathrm{id}_X$. Furthermore, for all $x \in Z$, we have

$$H(\mathcal{I}_2 \circ R_2(x), 2t) = Z, \quad \text{for} \quad 0 \leqslant t \leqslant \frac{1}{2} \quad \text{and}$$
$$K(x, 2t - 1) = Z, \quad \text{for} \quad \frac{1}{2} \leqslant t \leqslant 1.$$

For all $x \in Z$ and $t \in I$, we get $F(x, t) = Z$. Consequently, Z is a strong deformation m-retract of X. □

6. Conclusion

Multi-valued functions are a generalization of single-valued functions. So, it is possible to give multi-valued versions of many notions and properties for single-valued functions. In this paper, we give multi-valued versions of properties related to the path connectivity notion in general topology and lemmas and theorems related to the homotopy in algebraic topology.

Our aim is to learn more about the class of multi-valued functions using algebraic topology methods and tools. Moreover, we contribute to the theory of homotopy of multi-valued functions. For this reason, we have determined which of the existing definitions and properties can be applied to the multi-valued functions. Then we determine whether an additional condition is required for these definitions, theorems and results. As a result, our main aim is to improve the algebraic topological aspect of multi-valued functions.

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