**JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE**
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
*J. Int. Math. Virtual Inst.*, **11**(1)(2021), 137-159
DOI: 10.7251/JIMVI2101137B

Formerly
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

# FIXED POINTS OF $(\psi, \varphi)$-WEAKLY CYCLIC COUPLED CONTRACTIONS IN $S$-METRIC SPACES

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Abstract. Let $X$ be an $S$-metric space and $F : X \times X \to X$ be a mapping. We introduce $(\psi, \varphi)$-weakly cyclic coupled contraction mapping and Kannan type $(\psi, \varphi)$-weakly cyclic coupled contraction mapping in $S$-metric spaces. If $F : X \times X \to X$ then we prove the existence and uniqueness of strong coupled fixed point of $F$ in complete $S$-metric spaces where $F$ is of $(\psi, \varphi)$-weakly cyclic coupled contraction mapping and Kannan type $(\psi, \varphi)$-weakly cyclic coupled contraction mapping. Examples are provided to support our results.

## 1. Introduction and Preliminaries

Generalization of contraction conditions in proving the existence and uniqueness of fixed points play an important role in nonlinear analysis. In 1969, Kannan [**20**] proved the existence of fixed points of certain type of contraction mappings which are not continuous and different from contraction mappings. Later Kannan type mappings in various spaces have been considered in large number of works, some of which are in [**4**], [**7**], [**9**], [**10**], [**11**]. In 1997, Alber and Guerre-Delabriere [**2**] introduced the concept of weakly contractive mapping as a generalization of contractive mapping and proved the existence of fixed points for such mappings in Hilbert spaces. Rhoades [**30**] extended this study to metric space setting. On the other hand, in 2003, Kirk, Srinivasan and Veeramani [**22**] introduced cyclic contractions in metric spaces and proved the existence and uniqueness of cyclic

2010 *Mathematics Subject Classification.* 47H10; 54H25.
*Key words and phrases.* $S$-metric space, Cyclic mapping, Coupled fixed point, Strong coupled fixed point, $(\psi, \varphi)$-weakly cyclic coupled contraction mapping and Kannan type $(\psi, \varphi)$-weakly cyclic coupled contraction mapping.
Communicated by Daniel A. Romano.

contractions in complete metric spaces. In 2006, Gnana Bhaskar and Lakshmikantham [**15**] introduced and developed coupled fixed point theory for mixed monotone operators. For more works involving cyclic contractions and coupled fixed points, we refer [**1**], [**2**], [**3**], [**5**], [**8**],[**13**], [**16**], [**23**], [**24**], [**25**], [**26**], [**27**], [**28**], [**29**], [**33**]. In 2013, Chandok [**7**], introduced Kannan type cyclic weakly contraction mapping and proved the existence of fixed points for such mappings in complete metric spaces. For more works on cyclic weakly contraction mappings, we refer [**3**], [**8**], [**19**], [**21**], [**28**]. In 2014, Choudhury and Maity [**11**] introduced cyclic coupled Kannan type mapping and in 2017, Choudhury, Maity and Konar [**12**], introduced Banach type coupling (cyclic coupled Banach type mapping) and proved the existence of strong coupled fixed point theorems for such mappings.

We use the following propositions in proving our results.

PROPOSITION 1.1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Then*

$$\limsup_{n\to\infty} \max\{a_n, b_n\} = \max\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n\}.$$

PROPOSITION 1.2. *The following holds*

(i) *Let $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be real sequences then*

$$\max\{c_n + d_n, e_n + f_n\} \leqslant \max\{c_n, e_n\} + \max\{d_n, f_n\}.$$

(ii) *Let $\{a_n\}$, $\{b_n\}$ be two real sequences, $\{b_n\}$ be bounded. Then*

$$\liminf_{n\to\infty}(a_n + b_n) \leqslant \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

PROPOSITION 1.3. *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be nonnegative sequences satisfying*

$$\max\{a_n, b_n\} \leqslant \max\{c_n + d_n, e_n + f_n\} \text{ with } \limsup_{n\to\infty} c_n = 0 \text{ and } \limsup_{n\to\infty} e_n = 0.$$

*Then*

$$\liminf_{n\to\infty} \max\{a_n, b_n\} \leqslant \liminf_{n\to\infty} \max\{d_n, f_n\}.$$

DEFINITION 1.1. ([**22**]) Let $A$ and $B$ be two nonempty subsets of $X$. A mapping $f : X \to X$ is *cyclic* with respect to $A$ and $B$ if $f(A) \subseteq B$ and $f(B) \subseteq A$.

Choudhury and Maity [**11**] extended the above notion of cyclic mapping to the case of mappings defined on $X \times X$ in the following.

DEFINITION 1.2. [**11**] Let $A$ and $B$ be two nonempty subsets of $X$. A mapping $F : X \times X \to X$ is said to be *cyclic* with respect to $A$ and $B$ if $F(A, B) \subseteq B$ and $F(B, A) \subseteq A$.

DEFINITION 1.3. ([**15**]) Let $X$ be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of $F$ if $F(x, y) = x$ and $F(y, x) = y$.

DEFINITION 1.4. ([**11**]) Let $X$ be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, x) \in X \times X$ is said to be a *strong coupled fixed point* of $F$ if $F(x, x) = x$.

DEFINITION 1.5. ([**20**]) Let $(X,d)$ be a metric space. A mapping $f : X \to X$ is called a *Kannan type mapping* if $f$ satisfies the following inequality: there exists $k \in (0, \frac{1}{2})$ such that

$$d(fx, fy) \leqslant k[d(x, fx) + d(y, fy)] \tag{1.1}$$

for all $x, y \in X$.

DEFINITION 1.6. ([**12**]) Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$. A mapping $F : X \times X \to X$ is called a *cyclic coupled Banach type mapping* with respect to $A$ and $B$ if $F$ is cyclic with respect to $A$ and $B$ satisfying the inequality

$$d(F(x,y), F(u,v)) \leqslant \frac{k}{2}[d(x,u) + d(y,v)], \tag{1.2}$$

where $x, v \in A$ and $y, u \in B$ for some $k \in (0,1)$.

DEFINITION 1.7. ([**11**]) Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$. We call a mapping $F : X \times X \to X$ a *cyclic coupled Kannan type mapping* with respect to $A$ and $B$ if $F$ is cyclic with respect to $A$ and $B$ satisfying the inequality

$$d(F(x,y), F(u,v)) \leqslant k[d(x, F(x,y)) + d(u, F(u,v))], \tag{1.3}$$

where $x, v \in A$ and $y, u \in B$ for some $k \in (0, \frac{1}{2})$.

THEOREM 1.1 ([**12**]). *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X,d)$. Let $F : X \times X \to X$ be a cyclic coupled Banach type mapping with respect to $A$ and $B$ and $A \cap B \neq \emptyset$. Then $F$ has a unique strong coupled fixed point in $A \cap B$.*

THEOREM 1.2 ([**11**]). *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X,d)$. Let $F : X \times X \to X$ be a cyclic coupled Kannan type mapping with respect to $A$ and $B$ and $A \cap B \neq \emptyset$. Then $F$ has a strong coupled fixed point in $A \cap B$.*

In this paper, we denote

$\Psi = \{\psi : [0,\infty) \to [0,\infty)$ such that: (i) $\psi$ is continuous (ii) $\psi$ is non-decreasing, and (iii) $\psi(t) = 0$ if and only if $t = 0\}$.

DEFINITION 1.8. ([**22**]) Let $X$ be a nonempty set and $T : X \to X$ be a mapping. If $X_i$, $i = 1, 2, ...m$ are nonempty subsets of $X$ with $X = \bigcup_{i=1}^{m} X_i$ satisfying

$$T(X_1) \subset X_2,\ \dots,\ T(X_{m-1}) \subset X_m,\ T(X_m) \subset X_1$$

is called a cyclic representation of $X$ with respect to $T$.

DEFINITION 1.9. ([**7**]) Let $(X,d)$ be a metric space, $m$ be a natural number, $A_1, A_2, \dots, A_m$ be nonempty subsets of $X$ and $Y = \bigcup_{i=1}^{m} A_i$. A mapping $T : Y \to Y$ is called a *Kannan type cyclic weakly contraction* if $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of

$Y$ with respect to $T$ and if there exist $\psi \in \Psi$ and a function $\varphi : [0,\infty)^2 \to [0,\infty)$ with $\varphi$ is lower semi continuous, $\varphi(t,t) > 0$ for $t \in (0,\infty)$ and $\varphi(0,0) = 0$ such that

$$\psi(d(Tx,Ty)) \leqslant \psi(\frac{1}{2}[d(x,Tx) + d(y,Ty)]) - \varphi(d(x,Tx), d(y,Ty)), \tag{1.4}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, ..., m$, where $A_{m+1} = A_1$.

THEOREM 1.3 ([**7**]). *Let $(X,d)$ be a metric space, $m$ be a natural number, $A_1$, $A_2$, ..., $A_m$ be nonempty subsets of $X$ and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T$ is a Kannan type cyclic weakly contraction. Then, $T$ has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.*

DEFINITION 1.10. ([**31**]) Let $X$ be a nonempty set. An $S$-metric on $X$ is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions:
for each $x, y, z, a \in X$

(S1) $S(x,y,z) \geqslant 0$,

(S2) $S(x,y,z) = 0$ if and only if $x = y = z$ and

(S3) $S(x,y,z) \leqslant S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair $(X,S)$ is called an $S$-metric space.

EXAMPLE 1.1. ([**31**]) Let $(X,d)$ be a metric space. Define $S : X^3 \to [0,\infty)$ by $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ for all $x, y, z \in X$. Then $S$ is an $S$-metric on $X$ and $S$ is called the $S$-metric induced by the metric d.

EXAMPLE 1.2. ([**14**]) Let $X = \mathbb{R}$, the set of all real numbers and let $S(x,y,z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. Then $(X,S)$ is an $S$-metric space.

EXAMPLE 1.3. ([**32**]) Let $\mathbb{R}$ be the real line. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an $S$-metric on $\mathbb{R}$. This $S$-metric is called the usual $S$-metric.

LEMMA 1.1 ([**31**]). *In an $S$-metric space, we have $S(x,x,y) = S(y,y,x)$.*

LEMMA 1.2 ([**14**]). *Let $(X,S)$ be an $S$-metric space. Then*

$$S(x,x,z) \leqslant 2S(x,x,y) + S(y,y,z).$$

DEFINITION 1.11. ([**31**]) Let $(X,S)$ be an $S$-metric space.

(i) A sequence $\{x_n\} \in X$ converges to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\} \in X$ is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geqslant n_0$.

(iii) An $S$-metric space $(X,S)$ is said to be complete if each Cauchy sequence in $X$ is convergent in $X$.

LEMMA 1.3 ([**31**]). *Let $(X,S)$ be an $S$-metric space. If the sequence $\{x_n\}$ in $X$ converges to $x$, then $x$ is unique.*

LEMMA 1.4 ([**31**]). *Let $(X,S)$ be an $S$-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

LEMMA 1.5 ([**6**]). *Let $(X,S)$ be an $S$-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that*

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0 \text{ and } \lim_{n\to\infty} S(y_n, y_n, y_{n+1}) = 0.$$

*If either $\{x_n\}$ or $\{y_n\}$ is not Cauchy, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that*

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geqslant \epsilon. \tag{1.5}$$

*We choose $m_k$ as the smallest integer with $m_k > n_k$ satisfying (1.5) i.e.,*

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geqslant \epsilon \text{ with}$$
$$\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} < \epsilon.$$

*Also, the following limits hold.*

(i) $\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} = \epsilon$

(ii) $\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} = \epsilon$ *and*

(iii) $\lim_{k\to\infty} \max\{S(x_{n_k}, x_{n_k}, x_{m_k-1}), S(y_{n_k}, y_{n_k}, y_{m_k-1})\} = \epsilon.$

In this paper, we introduce $(\psi,\varphi)$-weakly cyclic coupled contraction mapping and Kannan type $(\psi,\varphi)$-weakly cyclic coupled contraction mapping in S-metric space setting. We prove the existence and uniqueness of strong coupled fixed points of such mappings in complete $S$-metric spaces. Also, we present illustrative examples to examine the validity of our results.

## 2. $(\psi,\varphi)$-weakly cyclic coupled contractions

We denote

$\Phi = \{\varphi : [0,\infty)^2 \to [0,\infty)$ such that (i) $\varphi$ is continuous in each of its variables and (ii) $\varphi(t_1,t_2) = 0$ if and only if $t_1 = 0$ and $t_2 = 0\}$.

In the following, we define $(\psi,\varphi)$-weakly cyclic coupled contraction mapping.

DEFINITION 2.1. Let $(X,S)$ be an $S$-metric space. Let $A$ and $B$ be two nonempty subsets of $X$. Let $F : X\times X \to X$ be a mapping. If (i) $F$ is cyclic with respect to $A$ and $B$ and (ii) there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\begin{aligned} \psi(S(F(x,y), F(u,v), F(w,z))) \leqslant \psi(&\tfrac{1}{2}[\max\{S(x,x,w), S(u,u,w)\} \\ &+ \max\{S(y,y,z), S(v,v,z)\}]) \\ &-\varphi(\max\{S(x,x,w), S(u,u,w)\}, \\ &\max\{S(y,y,z), S(v,v,z)\}) \end{aligned} \tag{2.1}$$

where $x,u,z \in A$ and $y,v,w \in B$, then we say that $F$ is a $(\psi,\varphi)$-weakly cyclic coupled contraction mapping with respect to $A$ and $B$.

THEOREM 2.1. *Let $(X, S)$ be a complete $S$-metric space. Let $A$ and $B$ be two nonempty closed subsets of $X$. Let $F : X \times X \to X$ be a $(\psi, \varphi)$-weakly cyclic coupled contraction mapping with respect to $A$ and $B$. Then $A \cap B \neq \emptyset$ and $F$ has a unique strong coupled fixed point in $A \cap B$.*

PROOF. Let $x_0 \in A$ and $y_0 \in B$ be arbitrary. We define sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = F(y_n, x_n),\ y_{n+1} = F(x_n, y_n),\ n = 0, 1, 2, \dots . \tag{2.2}$$

If $S(x_{n+1}, x_{n+1}, y_n) = 0$ and $S(y_{n+1}, y_{n+1}, x_n) = 0$ for some $n$, then $x_{n+1} = y_n$ and $y_{n+1} = x_n$ for some $n$ so that $(x_n, y_n)$ is a strong coupled fixed point of $F$. Therefore we assume that $S(x_{n+1}, x_{n+1}, y_n) \neq 0$ or $S(y_{n+1}, y_{n+1}, x_n) \neq 0$ for all $n$.

We consider

$$\begin{aligned}\psi(S(x_{n+1}, x_{n+1}, y_{n+2})) &= \psi(y_{n+2}, y_{n+2}, x_{n+1})\\ &= \psi(F(x_{n+1}, y_{n+1}), F(x_{n+1}, y_{n+1}), F(y_n, x_n))\\ &\leqslant \psi(\frac{1}{2}[S(x_{n+1}, x_{n+1}, y_n) + S(y_{n+1}, y_{n+1}, x_n)])\\ &\quad -\varphi(S(x_{n+1}, x_{n+1}, y_n), S(y_{n+1}, y_{n+1}, x_n))\\ &< \psi(\tfrac{1}{2}[S(x_{n+1}, x_{n+1}, y_n) + S(y_{n+1}, y_{n+1}, x_n)])\\ &\leqslant \psi(\max\{S(x_{n+1}, x_{n+1}, y_n), S(y_{n+1}, y_{n+1}, x_n)\}).\end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned}\psi(S(y_{n+1}, y_{n+1}, x_{n+2})) &= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_{n+1}, x_{n+1})))\\ &\leqslant \psi(\frac{1}{2}[S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1})])\\ &\quad -\varphi(S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1}))\\ &< \psi(\tfrac{1}{2}[S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1})])\\ &\leqslant \psi(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\}).\end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned}&\max\{\psi(S(x_{n+1}, x_{n+1}, y_{n+2})), \psi(S(y_{n+1}, y_{n+1}, x_{n+2}))\}\\ &\qquad < \psi(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\}).\end{aligned}$$

That is

$$\begin{aligned}&\psi(\max\{S(x_{n+1}, x_{n+1}, y_{n+2}), S(y_{n+1}, y_{n+1}, x_{n+2})\})\\ &\qquad < \psi(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\}).\end{aligned}$$

That is $\psi(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\})$ is a decreasing sequence and therefore there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} \max\{\psi(S(x_n, x_n, y_{n+1})), \psi(S(y_n, y_n, x_{n+1}))\} = r.$$

We now show that $r = 0$. Suppose, if possible $r > 0$. Then from (2.3) and (2.4), we have

$$\begin{aligned}&\max\{\psi(S(x_{n+1}, x_{n+1}, y_{n+2}), S(y_{n+1}, y_{n+1}, x_{n+2}))\})\\ &\qquad \leqslant \psi(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\})\end{aligned}$$

$$- \min\{\varphi(S(x_n,x_n,y_{n+1}),S(y_n,y_n,x_{n+1})),$$
$$\varphi(S(y_n,y_n,x_{n+1}),S(x_n,x_n,y_{n+1}))\}.$$

On letting $n\to\infty$, we have

$$r \leqslant r - \min\{\lim_{n\to\infty}\varphi(S(x_n,x_n,y_{n+1})S(y_n,y_n,x_{n+1})),$$
$$\lim_{n\to\infty}\varphi(S(y_n,y_n,x_{n+1}),S(x_n,x_n,y_{n+1}))\}$$

which implies that $\lim_{n\to\infty}\varphi(S(x_n,x_n,y_{n+1}),S(y_n,y_n,x_{n+1}))=0$. That is

$$\varphi(\lim_{n\to\infty}S(x_n,x_n,y_{n+1}),\lim_{n\to\infty}S(y_n,y_n,x_{n+1}))=0$$

so that $\lim_{n\to\infty}S(x_n,x_n,y_{n+1})=0$ and $\lim_{n\to\infty}S(y_n,y_n,x_{n+1})=0$. Then we have

$$\psi(\lim_{n\to\infty}S(x_n,x_n,y_{n+1}))=0 \text{ and } \psi(\lim_{n\to\infty}S(y_n,y_n,x_{n+1}))=0$$

which gives that

$$\max\{\psi(\lim_{n\to\infty}S(x_n,x_n,y_{n+1})),\psi(\lim_{n\to\infty}S(y_n,y_n,x_{n+1}))\}=0.$$

That is $r=0$, a contradiction. Thus $r=0$. Therefore

$$\lim_{n\to\infty}\max\{\psi(S(x_n,x_n,y_{n+1})),\psi(S(y_n,y_n,x_{n+1}))\}=0.$$

Then we have

$$\lim_{n\to\infty}\psi(S(x_n,x_n,y_{n+1}))=0 \text{ and } \lim_{n\to\infty}\psi(S(y_n,y_n,x_{n+1}))=0$$

which gives

$$\psi(\lim_{n\to\infty}S(x_n,x_n,y_{n+1}))=0 \text{ and } \psi(\lim_{n\to\infty}S(y_n,y_n,x_{n+1}))=0$$

so that

$$\lim_{n\to\infty}S(x_n,x_n,y_{n+1})=0 \text{ and } \lim_{n\to\infty}S(y_n,y_n,x_{n+1})=0. \tag{2.5}$$

If $x_n=y_n$ for some $n$ then $x_k=y_k$ for all $k\geqslant n$ which implies that

$$\lim_{k\to\infty}S(x_k,x_k,y_k)=0.$$

Now suppose that $x_n\neq y_n$ for any $n$. By using the inequality (2.1), we have

$$\begin{aligned}
\psi(S(x_{n+1},x_{n+1},y_{n+1})) &= \psi(S(y_{n+1},y_{n+1},x_{n+1}))\\
&= \psi(S(F(x_n,y_n),F(x_n,y_n),F(y_n,x_n)))\\
&\leqslant \psi(\tfrac{1}{2}[S(x_n,x_n,y_n)+S(y_n,y_n,x_n)])\\
&\qquad - \varphi(S(x_n,x_n,y_n),S(y_n,y_n,x_n))\\
&= \psi(S(x_n,x_n,y_n)) - \varphi(S(x_n,x_n,y_n),S(y_n,y_n,x_n)) \qquad (2.6)\\
&< \psi(S(x_n,x_n,y_n)).
\end{aligned}$$

That is $\{S(x_n,x_n,y_n)\}$ is a decreasing sequence and therefore there exists $s>0$ such that $\lim_{n\to\infty}S(x_n,x_n,y_n)=s$.

On letting $n \to \infty$ in (2.6) we have

$$\psi(s) \leqslant \psi(s) - \varphi(s, s)$$

which implies that $s = 0$ Therefore

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = 0. \tag{2.7}$$

We now consider

$$\begin{aligned} S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) &\leqslant 2S(x_n, x_n, y_n) + S(y_n, y_n, x_{n+1}) \\ &\quad + 2S(y_n, y_n, x_n) + S(x_n, x_n, y_{n+1}) \\ &= 4S(x_n, x_n, y_n) + S(x_n, x_n, y_{n+1}) \\ &\quad + S(y_n, y_n, x_{n+1}). \end{aligned}$$

On taking limits as $n \to \infty$, we get

$$\lim_{n\to\infty} [S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1})] = 0$$

which implies that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0 \text{ and } \lim_{n\to\infty} S(y_n, y_n, y_{n+1}) = 0.$$

We now prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose that either $\{x_n\}$ or $\{y_n\}$ is not Cauchy. Then there exist $\epsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geqslant \epsilon. \tag{2.8}$$

We choose $m_k$ as the smallest integer with $m_k > n_k$ satisfying (2.8). That is

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geqslant \epsilon$$

with

$$\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} < \epsilon.$$

We now prove the following.

$$\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} = \epsilon. \tag{2.9}$$

We consider

$$\begin{aligned} S(x_{m_k-1}, x_{m_k-1}, y_{n_k-1}) &= S(y_{n_k-1}, y_{n_k-1}, x_{m_k-1}) \\ &\leqslant 2S(y_{n_k-1}, y_{n_k-1}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k-1}). \end{aligned}$$

Also, we have

$$S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1}) \leqslant 2S(x_{n_k-1}, x_{n_k-1}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k-1}).$$

Thus we have

$$\begin{aligned} &\max\{S(x_{m_k-1}, x_{m_k-1}, y_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1})\} \\ &\qquad \leqslant \max\{2S(y_{n_k-1}, y_{n_k-1}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k-1}), \\ &\qquad\qquad 2S(x_{n_k-1}, x_{n_k-1}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k-1})\}. \end{aligned}$$

On taking limit supremum as $k \to \infty$, and using Proposition 1.1,

$$\begin{aligned}\limsup_{k\to\infty}\ &\max\{S(x_{m_k-1},x_{m_k-1},y_{n_k-1}),S(y_{m_k-1},y_{m_k-1},x_{n_k-1})\}\\ &\leqslant \limsup_{k\to\infty}\ \max\{S(x_{n_k},x_{n_k},x_{m_k-1}),S(y_{n_k},y_{n_k},y_{m_k-1})\}\\ &=\epsilon \text{ (by (iii) of Lemma 1.5).}\end{aligned}$$

We consider

$$\begin{aligned}S(x_{m_k},x_{m_k},y_{n_k}) &\leqslant 2S(x_{m_k},x_{m_k},x_{m_k-1})+S(x_{m_k-1},x_{m_k-1},y_{n_k})\\ &=2S(x_{m_k},x_{m_k},x_{m_k-1})+S(y_{n_k},y_{n_k},x_{m_k-1})\\ &\leqslant 2S(x_{m_k},x_{m_k},x_{m_k-1})+2S(y_{n_k},y_{n_k},y_{n_k-1})\\ &\qquad +S(y_{n_k-1},y_{n_k-1},x_{m_k-1}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}S(y_{m_k},y_{m_k},x_{n_k}) &\leqslant 2S(y_{m_k},y_{m_k},y_{m_k-1})+S(y_{m_k-1},y_{m_k-1},x_{n_k})\\ &=2S(y_{m_k},y_{m_k},y_{m_k-1})+S(x_{n_k},x_{n_k},y_{m_k-1})\\ &\leqslant 2S(y_{m_k},y_{m_k},y_{m_k-1})+2S(x_{n_k},x_{n_k},x_{n_k-1})\\ &\qquad +S(y_{m_k-1},y_{m_k-1},x_{n_k-1}).\end{aligned}$$

We have

$$\begin{aligned}&\max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}\\ &\quad\leqslant \max\{2S(x_{m_k},x_{m_k},x_{m_k-1})+2S(y_{n_k},y_{n_k},y_{n_k-1})+S(y_{n_k-1},y_{n_k-1},x_{m_k-1}),\\ &\qquad 2S(y_{m_k},y_{m_k},y_{m_k-1})+2S(x_{n_k},x_{n_k},x_{n_k-1})+S(y_{m_k-1},y_{m_k-1},x_{n_k-1})\}.\end{aligned}$$

On taking limit infimum as $k\to\infty$, we get

$$\begin{aligned}\epsilon &\leqslant \liminf_{k\to\infty}\ \max\{S(x_{m_k},x_{m_k},x_{n_k}),S(y_{m_k},y_{m_k},y_{n_k})\}\\ &\leqslant \liminf_{k\to\infty}\ \max\{S(y_{n_k},y_{n_k},x_{m_k}),S(x_{n_k},x_{n_k},y_{m_k})\} \text{ (by Proposition 1.3).}\end{aligned}$$

From the above we have

$$\begin{aligned}\epsilon &\leqslant \liminf_{k\to\infty}\ \max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}\\ &\leqslant \limsup_{k\to\infty}\ \max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}\leqslant\epsilon.\end{aligned}$$

Hence $\liminf_{k\to\infty}\ \max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}$

$$=\limsup_{k\to\infty}\ \max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}=\epsilon.$$

Therefore $\lim_{k\to\infty}\max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}$ exists and $\lim_{k\to\infty}\max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}=\epsilon$.

Hence (2.9) is proved.

We consider

$$\begin{aligned}S(x_{m_k-1},x_{m_k-1},y_{n_k-1}) &= S(y_{n_k-1},y_{n_k-1},x_{m_k-1})\\ &\leqslant 2S(y_{n_k-1},y_{n_k-1},x_{n_k})+S(x_{n_k},x_{n_k},x_{m_k-1}).\end{aligned}$$

Also, we have

$$S(y_{m_k-1},y_{m_k-1},x_{n_k-1})\leqslant 2S(x_{n_k-1},x_{n_k-1},y_{n_k})+S(y_{n_k},y_{n_k},y_{m_k-1}).$$

Thus we have

$$\max\{S(x_{m_k-1},x_{m_k-1},y_{n_k-1}),S(y_{m_k-1},y_{m_k-1},x_{n_k-1})\}$$

$$\leqslant \max\{2S(y_{n_k-1}, y_{n_k-1}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k-1}),$$
$$2S(x_{n_k-1}, x_{n_k-1}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k-1})\}.$$

On taking limit supremum as $k \to \infty$, and using Proposition 1.1,

$$\limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, y_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1})\}$$
$$\leqslant \limsup_{k\to\infty} \max\{S(x_{n_k}, x_{n_k}, x_{m_k-1}), S(y_{n_k}, y_{n_k}, y_{m_k-1})\} = \epsilon$$

(by (iii) of Lemma 1.5).

We consider

$$\begin{aligned} S(x_{m_k}, x_{m_k}, y_{n_k}) &\leqslant 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, y_{n_k}) \\ &= 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(y_{n_k}, y_{n_k}, x_{m_k-1}) \\ &\leqslant 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + 2S(y_{n_k}, y_{n_k}, y_{n_k-1}) \\ &\quad + S(y_{n_k-1}, y_{n_k-1}, x_{m_k-1}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} S(y_{m_k}, y_{m_k}, x_{n_k}) &\leqslant 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + S(y_{m_k-1}, y_{m_k-1}, x_{n_k}) \\ &= 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + S(x_{n_k}, x_{n_k}, y_{m_k-1}) \\ &\leqslant 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) \\ &\quad + S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1}). \end{aligned}$$

We have

$$\max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}$$
$$\leqslant \max\{2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + 2S(y_{n_k}, y_{n_k}, y_{n_k-1}) + S(y_{n_k-1}, y_{n_k-1}, x_{m_k-1}),$$
$$2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1})\}.$$

On taking limit infimum as $k \to \infty$, by using Proposition 1.2 (ii) and (2.9), we get

$$\epsilon \leqslant \liminf_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, y_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1})\}$$
$$\leqslant \limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, y_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1})\} \leqslant \epsilon.$$

Therefore

$$\epsilon \leqslant \liminf_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, y_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1})\}$$
$$\leqslant \limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, y_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1})\} \leqslant \epsilon.$$

Thus $\lim_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, y_{n_k-1}), S(y_{m_k-1}, y_{m_k-1}, x_{n_k-1})\} = \epsilon$.

We consider

$$\begin{aligned} \psi(S(x_{m_k}, x_{m_k}, y_{n_k})) &= \psi(S(y_{n_k}, y_{n_k}, x_{m_k})) \\ &= \psi(S(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k-1}, y_{n_k-1}), F(y_{m_k-1}, x_{m_k-1}))) \\ &\leqslant \psi(\tfrac{1}{2}[S(x_{n_k-1}, x_{n_k-1}, y_{m_k-1}) + S(y_{n_k-1}, y_{n_k-1}, x_{m_k-1})]) \\ &\qquad - \varphi(S(x_{n_k-1}, x_{n_k-1}, y_{m_k-1}), S(y_{n_k-1}, y_{n_k-1}, x_{m_k-1})). \end{aligned}$$

Similarly, we have

$$\psi(S(y_{m_k}, y_{m_k}, x_{n_k})) = \psi(S(F(x_{m_k-1}, y_{m_k-1}), F(x_{m_k-1}, y_{m_k-1}), F(y_{n_k-1}, x_{n_k-1})))$$

$$\leqslant \psi(\tfrac{1}{2}[S(x_{m_k-1},x_{m_k-1},y_{n_k-1})+S(y_{m_k-1},y_{m_k-1},x_{n_k-1})]) - \varphi(S(x_{m_k-1},x_{m_k-1},y_{n_k-1}),S(y_{m_k-1},y_{m_k-1},x_{n_k-1})).$$

We now consider

$$\begin{aligned}\psi(\max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}) &= \max\{\psi(S(x_{m_k},x_{m_k},y_{n_k})),\psi(S(y_{m_k},y_{m_k},x_{n_k}))\}\\ &\leqslant \max\{\psi(\tfrac{1}{2}[S(x_{n_k-1},x_{n_k-1},y_{m_k-1})+S(y_{n_k-1},y_{n_k-1},x_{m_k-1})]),\\ &\qquad \psi(\tfrac{1}{2}[S(x_{m_k-1},x_{m_k-1},y_{n_k-1})+S(y_{m_k-1},y_{m_k-1},x_{n_k-1})])\}\\ &\quad - \min\{\varphi(S(x_{n_k-1},x_{n_k-1},y_{m_k-1}),S(y_{n_k-1},y_{n_k-1},x_{m_k-1})),\\ &\qquad \varphi(S(x_{m_k-1},x_{m_k-1},y_{n_k-1}),S(y_{m_k-1},y_{m_k-1},x_{n_k-1}))\}.\end{aligned}$$

On letting $k\to\infty$,

$$\begin{aligned}\psi(\epsilon)\leqslant \psi(\epsilon)-\min\{&\lim_{k\to\infty}\varphi(S(x_{n_k-1},x_{n_k-1},y_{m_k-1}),S(y_{n_k-1},y_{n_k-1},x_{m_k-1})),\\ &\lim_{k\to\infty}\varphi(S(x_{m_k-1},x_{m_k-1},y_{n_k-1}),S(y_{m_k-1},y_{m_k-1},x_{n_k-1}))\}\end{aligned}$$

which implies that

$$\lim_{k\to\infty} S(x_{n_k-1},x_{n_k-1},y_{m_k-1})=0 \text{ and } \lim_{k\to\infty} S(y_{m_k-1},y_{m_k-1},x_{n_k-1})=0.$$

Therefore

$$\lim_{k\to\infty}\max\{S(x_{n_k-1},x_{n_k-1},y_{m_k-1}),S(y_{m_k-1},y_{m_k-1},x_{n_k-1})\}=0,$$

i.e., $\epsilon=0$, a contradiction. Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy. Since $A$ and $B$ are closed subsets of $X$ and $\{x_n\}\subseteq A$, $\{y_n\}\subseteq B$, there exist $x\in A$ and $y\in B$ such that

$$x_n\to x,\ y_n\to y \text{ as } n\to\infty. \tag{2.10}$$

From (2.7), we have $S(x,x,y)=0$. Therefore $x=y$.

We consider

$$\begin{aligned}\psi(S(x_{n+1},x_{n+1},F(x,x))) &= \psi(S(F(x,x),F(x,x),F(y_n,x_n)))\\ &\leqslant \psi(\tfrac{1}{2}[S(x,x,y_n)+S(x,x,x_n)])\\ &\quad - \varphi(S(x,x,y_n),S(x,x,x_n)).\end{aligned}$$

On taking limits as $n\to\infty$, we have

$$\begin{aligned}\psi(S(x,x,F(x,x))) &\leqslant \psi(\tfrac{1}{2}[S(x,x,y)+S(x,x,x)])-\varphi(S(x,x,y),S(x,x,x))\\ &= \psi(0)-\varphi(0,0).\end{aligned}$$

That is $\psi(S(x,x,F(x,x)))=0$ so that $S(x,x,F(x,x))=0$. Thus $x=F(x,x)$. Therefore $(x,x)$ is a strong coupled fixed point of $F$.

We now prove the uniqueness of strong coupled fixed point of $F$. Suppose $(x,x)$ and $(y,y)$ are two strong coupled fixed points of $F$. We consider

$$\begin{aligned}\psi(S(x,x,y)) &\leqslant \psi(S(F(x,x),F(x,x),F(y,y)))\\ &\leqslant \psi(\tfrac{1}{2}[S(x,x,y)+S(x,x,y)])-\varphi(S(x,x,y),S(x,x,y))\\ &= \psi(S(x,x,y))-\varphi(S(x,x,y),S(x,x,y)).\end{aligned}$$

This implies that $\varphi(S(x,x,y),S(x,x,y))=0$ which shows that $x=y$. □

By choosing $\psi(t) = t$ in Theorem 2.1, we have the following.

COROLLARY 2.1. *Let $(X, S)$ be a complete $S$-metric space. Let $A$ and $B$ be two nonempty closed subsets of $X$. Let $F : X \times X \to X$ be cyclic with respect to $A$ and $B$ satisfying the following the inequality:*

$$\begin{aligned} S(F(x,y), F(u,v), F(w,z)) \leqslant \frac{1}{2}[&\max\{S(x,x,w), S(u,u,w)\} \\ &+ \max\{S(y,y,z), S(v,v,z)\}] \\ &-\varphi(\max\{S(x,x,w), S(u,u,w)\}, \\ &\max\{S(y,y,z), S(v,v,z)\}) \end{aligned} \tag{2.11}$$

*where $x, u, z \in A$ and $y, v, w \in B$ and $\varphi \in \Phi$. Then $A \cap B \neq \emptyset$ and $F$ has a unique strong coupled fixed point in $A \cap B$.*

By choosing $\varphi(t_1, t_2) = (\frac{1}{2} - k)(t_1 + t_2)$ in Corollary 2.1, where $k \in (0, \frac{1}{2})$, then we have the following.

COROLLARY 2.2. *Let $(X, S)$ be a complete $S$-metric space. Let $A$ and $B$ be two nonempty closed subsets of $X$. Let $F : X \times X \to X$ be cyclic with respect to $A$ and $B$ satisfying the following the inequality: there exists $k \in (0, \frac{1}{2})$ such that*

$$\begin{aligned} S(F(x,y), F(u,v), F(w,z)) \leqslant k[&\max\{S(x,x,w), S(u,u,w)\} \\ &+ \max\{S(y,y,z), S(v,v,z)\}] \end{aligned} \tag{2.12}$$

*where $x, u, z \in A$ and $y, v, w \in B$. Then $A \cap B \neq \emptyset$ and $F$ has a unique strong coupled fixed point in $A \cap B$.*

The following example is in support of Theorem 2.1.

EXAMPLE 2.1. Let $X = [0, 2]$. We define $S : X^3 \to [0, \infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

Then clearly $(X, S)$ is an $S$-metric space. To show $(X, S)$ is complete, let $\{x_n\}$ be a Cauchy sequence in $X = [0, 2]$. Let $\epsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \frac{\epsilon}{4}$ for all $n \geqslant N_1$, $m \geqslant N_1$. Hence it follows that $\{x_n\}$ is a bounded sequence in [0,2] with respect to the $S$-metric. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$, $x \in [0, 2]$ as $k \to \infty$. Hence there exists $N_2 \in \mathbb{N}$ such that $S(x_{n_k}, x_{n_k}, x) < \frac{\epsilon}{2}$ for all $k \geqslant N_2$. Let $N = \max\{N_1, N_2\}$. Then for $n \geqslant N$, we have

$$\begin{aligned} S(x_n, x_n, x) &\leqslant 2S(x_n, x_n, x_{n_k}) + S(x_{n_k}, x_{n_k}, x),\ n_k \geqslant N \\ &< 2(\tfrac{\epsilon}{4}) + \tfrac{\epsilon}{2} = \epsilon, \end{aligned}$$

so that $\{x_n\}$ converges to $x$ in $X$. This shows that $(X, S)$ is a complete $S$-metric space.

Let $A = [1, 2]$ and $B = [0, 1]$. We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x-y}{4} & \text{if } x \in A \text{ and } y \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then $F(A,B)=[0,\frac{1}{2}]\subset B$ and $F(B,A)=\{0\}\subset A$ so that $F$ is cyclic with respect to $A$ and $B$. We define $\psi:[0,\infty)^2\to[0,\infty)$ by $\psi(t)=\frac{t}{2}$ and $\varphi:[0,\infty)^2\to[0,\infty)$ by $\varphi(t_1,t_2)=\frac{1}{16}(t_1+t_2)$.

We now verify the inequality (2.1). Let $x,u,z\in A$ and $y,v,w\in B$. Then $S(x,x,w)=x$; $S(u,u,w)=u$; $S(y,y,z)=z$ and $S(v,v,z)=z$. We consider

$$\begin{aligned}\psi(S(F(x,y),F(u,v),F(w,z))) &= \psi(S(\tfrac{x-y}{4},\tfrac{u-v}{4},0))\\ &= \tfrac{1}{2}\max\{\tfrac{x-y}{4},\tfrac{u-v}{4}\}\\ &= \tfrac{1}{8}\max\{x-y,u-v\}\\ &\leqslant \tfrac{1}{8}\max\{x,u\}\\ &\leqslant \tfrac{1}{8}[\max\{x,w\},\max\{u,w\}]\\ &= \tfrac{1}{8}\max\{S(x,x,w),S(u,u,w)\}\\ &\leqslant \tfrac{3}{16}[\max\{S(x,x,w),S(u,u,w)\}\\ &\qquad+\max\{S(y,y,z),S(v,v,z)\}]\\ &= \tfrac{1}{4}(t_1+t_2)-\tfrac{1}{16}(t_1+t_2)\\ &= \psi(\tfrac{1}{2}(t_1+t_2))-\varphi(t_1,t_2)\end{aligned}$$

where $t_1=\max\{S(x,x,w),S(u,u,w)\}$ and $t_2=\max\{S(y,y,z),S(v,v,z)\}$. Therefore $F$ is a $(\psi,\varphi)$-weakly cyclic coupled contraction mapping with respect to $A$ and $B$ and $F$ satisfies all the hypotheses of Theorem 2.1 and $(0,0)$ is a unique strong coupled fixed point of $F$. □

## 3. Kannan Type $(\psi,\varphi)$-weakly cyclic Coupled Contraction

In the following, we define Kannan type $(\psi,\varphi)$-weakly cyclic coupled contraction mapping.

Definition 3.1. Let $(X,S)$ be an $S$-metric space. Let $A$ and $B$ be two nonempty subsets of $X$. Let $F:X\times X\to X$ be a mapping. If (i) $F$ is cyclic with respect to $A$ and $B$ and (ii) there exist $\psi\in\Psi$ and $\varphi\in\Phi$ such that

$$\begin{aligned}\psi(S(F(x,y),F(u,v),F(w,z))) \leqslant{}& \psi(\frac{1}{3}[S(x,x,F(x,y))+S(u,u,F(u,v))\\ &+S(w,w,F(w,z))])\\ &-\varphi(\max\{S(x,x,F(x,y)),S(u,u,F(u,v))\},\\ &S(w,w,F(w,z)))\end{aligned}\tag{3.1}$$

where $x,u,z\in A$ and $y,v,w\in B$, then we say that $F$ is a Kannan type $(\psi,\varphi)$-weakly cyclic coupled contraction with respect to $A$ and $B$.

Theorem 3.1. *Let $(X,S)$ be a complete $S$-metric space. Let $A$ and $B$ be two nonempty closed subsets of $X$. Let $F:X\times X\to X$ be Kannan type $(\psi,\varphi)$-weakly cyclic coupled contraction mapping with respect to $A$ and $B$. Then $A\cap B\neq\emptyset$ and $F$ has a unique strong coupled fixed point in $A\cap B$.*

Proof. Let $x_0 \in A$ and $y_0 \in B$ be arbitrary. We define sequences $\{x_n\}$ and $\{y_n\}$ by $x_{n+1} = F(y_n, x_n),\ y_{n+1} = F(x_n, y_n),\ n = 0, 1, 2, \ldots$ . We now show that the following inequalities hold by using induction on $n$.

$$\psi(S(x_{2n+1}, x_{2n+1}, y_{2n+2})) < \psi(S(y_0, y_0, x_1)) \tag{3.2}$$

and

$$\psi(S(y_{2n+1}, y_{2n+1}, x_{2n+2})) < \psi(S(x_0, x_0, y_1)) \tag{3.3}$$

$$\psi(S(x_{2n}, x_{2n}, y_{2n+1})) < \psi(S(x_0, x_0, y_1)) \tag{3.4}$$

and

$$\psi(S(y_{2n}, y_{2n}, x_{2n+1})) < \psi(S(y_0, y_0, x_1)). \tag{3.5}$$

If $y_n = x_{n+1}$ and $x_n = y_{n+1}$ for some $n$, then

$$\begin{aligned}
\psi(S(x_n, x_n, y_n)) &= \psi(S(y_{n+1}, y_{n+1}, x_{n+1})) \\
&= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n))) \\
&\leqslant \psi(\tfrac{1}{3}[2S(x_n, x_n, F(x_n, y_n)) + S(y_n, y_n, F(y_n, x_n))]) \\
&\qquad - \varphi(S(x_n, x_n, F(x_n, y_n)), S(y_n, y_n, F(y_n, x_n))) \\
&= \psi(\tfrac{1}{3}[2S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1})]) \\
&\qquad - \varphi(S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})) \\
&= \psi(\tfrac{1}{3}[2S(x_n, x_n, x_n) + S(y_n, y_n, y_n)]) \\
&\qquad - \varphi(S(x_n, x_n, x_n), S(y_n, y_n, y_n)).
\end{aligned}$$

That is $\psi(S(x_n, x_n, y_n) = 0$ so that $S(x_n, x_n, y_n) = 0$. Thus $x_n = y_n$. Therefore $(x_n, x_n)$ is a strong coupled fixed point of $F$ and we are through.

Here we note that if either $y_n \neq x_{n+1}$ or $x_n \neq y_{n+1}$ for all $n$, then some of the inequalities (3.2)-(3.5) may fail to hold.

For example, for the case $y_n \neq x_{n+1}$ and $x_n = y_{n+1}$ for all $n$, the inequality (3.3) is not possible to hold. For, by choosing $n = 2m$, $m$ is a positive integer, we have

$$\begin{aligned}
\psi(S(y_{2m+1}, y_{2m+1}, x_{2m+2}) &= \psi(S(F(x_{2m}, y_{2m}), F(x_{2m}, y_{2m}), F(y_{2m+1}, x_{2m+1}))) \\
&\leqslant \psi(\tfrac{1}{3}[2S(x_{2m}, x_{2m}, y_{2m+1}) + S(y_{2m+1}, y_{2m+1}, x_{2m+2})]) \\
&\qquad - \varphi(S(x_{2m}, x_{2m}, y_{2m+1}), S(y_{2m+1}, y_{2m+1}, x_{2m+2})) \\
&\leqslant \psi(S(y_{2m+1}, y_{2m+1}, x_{2m+2})) \\
&\qquad - \varphi(0, S(y_{2m+1}, y_{2m+1}, x_{2m+2}))
\end{aligned}$$

which implies that

$$\varphi(0, S(y_{2m+1}, y_{2m+1}, x_{2m+2})) = 0.$$

Therefore $S(y_{2m+1}, y_{2m+1}, x_{2m+2}) = 0$. Thus $y_{2m+1} = x_{2m+2}$ which is a contradiction. Hence (3.3) fails to hold so that this case does not arise.

Now for the case $y_n = x_{n+1}$ and $x_n \neq y_{n+1}$ for all $n$, the inequality (3.2) is not possible to hold. For, by choosing $n = 2m$, $m$ is a positive integer, we have

$$\psi(S(x_{2m+1}, x_{2m+1}, y_{2m+2})) = \psi(S(y_{2m+2}, y_{2m+2}, x_{2m+1}))$$

$$
\begin{aligned}
&= \psi(S(F(x_{2m+1}, y_{2m+1}), F(x_{2m+1}, y_{2m+1}), F(y_{2m}, x_{2m})))\\
&\leqslant \psi(\tfrac{1}{3}(2S(x_{2m+1}, x_{2m+1}, F(x_{2m+1}, y_{2m+1}))\\
&\qquad + S(y_{2m}, y_{2m}, F(y_{2m}, x_{2m})))\\
&\quad - \varphi(S(x_{2m+1}, x_{2m+1}, F(x_{2m+1}, y_{2m+1})),\\
&\qquad S(y_{2m}, y_{2m}, F(y_{2m}, x_{2m})))\\
&= \psi(\tfrac{1}{3}(2S(x_{2m+1}, x_{2m+1}, y_{2m+2}) + S(y_{2m}, y_{2m}, x_{2m+1})))\\
&\quad - \varphi(S(x_{2m+1}, x_{2m+1}, y_{2m+2}), S(y_{2m}, y_{2m}, x_{2m+1}))\\
&\leqslant \psi(S(x_{2m+1}, x_{2m+1}, y_{2m+2})) - \varphi(S(x_{2m+1}, x_{2m+1}, y_{2m+2}), 0)
\end{aligned}
$$

which implies that

$$\varphi(S(x_{2m+1}, x_{2m+1}, y_{2m+2}), 0) = 0.$$

Therefore $S(x_{2m+1}, x_{2m+1}, y_{2m+2}) = 0$. Thus $x_{2m+1} = y_{2m+2}$ which is a contradiction. Hence (3.2) fails to hold so that this case also does not arise.

Hence, we assume that $y_n \neq x_{n+1}$ and $x_n \neq y_{n+1}$ for all $n$. Now by using (3.1), we have

$$
\begin{aligned}
\psi(S(x_1, x_1, y_2)) &= \psi(S(y_2, y_2, x_1)) = \psi(S(F(x_1, y_1), F(x_1, y_1), F(y_0, x_0)))\\
&\leqslant \psi(\tfrac{1}{3}[2S(x_1, x_1, F(x_1, y_1)) + S(y_0, y_0, F(y_0, x_0))])\\
&\quad - \varphi(S(x_1, x_1, F(x_1, y_1)), S(y_0, y_0, F(y_0, x_0)))\\
&= \psi(\tfrac{1}{3}[2S(x_1, x_1, y_2) + S(y_0, y_0, x_1)) - \varphi(S(x_1, x_1, y_2), S(y_0, y_0, x_1))\\
&\leqslant \psi(\max\{S(x_1, x_1, y_2), S(y_0, y_0, x_1)\}) - \varphi(S(x_1, x_1, y_2), S(y_0, y_0, x_1))\\
&< \psi(\max\{S(x_1, x_1, y_2), S(y_0, y_0, x_1)\}).
\end{aligned}
$$

That is

$$\psi(S(x_1, x_1, y_2)) < \psi(S(y_0, y_0, x_1)). \tag{3.6}$$

Similarly, by (3.1), we have

$$
\begin{aligned}
\psi(S(y_1, y_1, x_2)) &= \psi(S(F(x_0, y_0), F(x_0, y_0), F(y_1, x_1)))\\
&\leqslant \psi(\tfrac{1}{3}[2S(x_0, x_0, F(x_0, y_0)) + S(y_1, y_1, F(y_1, x_1))])\\
&\quad - \varphi(S(x_0, x_0, F(x_0, y_0)), S(y_1, y_1, F(y_1, x_1)))\\
&= \psi(\tfrac{1}{3}[2S(x_0, x_0, y_1) + S(y_1, y_1, x_2)]) - \varphi(S(x_0, x_0, y_1), S(y_1, y_1, x_2))\\
&\leqslant \psi(\max\{S(x_0, x_0, y_1), S(y_1, y_1, x_2)\}) - \varphi(S(x_0, x_0, y_1), S(y_1, y_1, x_2))\\
&< \psi(\max\{S(x_0, x_0, y_1), S(y_1, y_1, x_2)\}).
\end{aligned}
$$

That is

$$\psi(S(y_1, y_1, x_2)) < \psi(S(x_0, x_0, y_1)). \tag{3.7}$$

Again by (3.1), we have

$$
\begin{aligned}
\psi(S(x_2, x_2, y_3)) &= \psi(S(y_3, y_3, x_2)) = \psi(S(F(x_2, y_2), F(x_2, y_2), F(y_1, x_1)))\\
&\leqslant \psi(\tfrac{1}{3}[2S(x_2, x_2, F(x_2, y_2)) + S(y_1, y_1, F(y_1, x_1))])\\
&\quad - \varphi(S(x_2, x_2, F(x_2, y_2)), S(y_1, y_1, F(y_1, x_1)))\\
&= \psi(\tfrac{1}{3}[2S(x_2, x_2, y_3) + S(y_1, y_1, x_2)]) - \varphi(S(x_2, x_2, y_3), S(y_1, y_1, x_2))
\end{aligned}
$$

$$\leqslant \psi(\max\{S(x_2,x_2,y_3),S(y_1,y_1,x_2)\}) - \varphi(S(x_2,x_2,y_3),S(y_1,y_1,x_2))$$
$$< \psi(\max\{S(x_2,x_2,y_3),S(y_1,y_1,x_2)\}).$$

That is

$$\psi(S(x_2,x_2,y_3)) < \psi(S(y_1,y_1,x_2)) < \psi(S(x_0,x_0,y_1)) \text{ (by (3.7))}. \tag{3.8}$$

Similarly, we have

$$\begin{aligned}\psi(S(y_2,y_2,x_3)) &= \psi(S(F(x_1,y_1),F(x_1,y_1),F(y_2,x_2)))\\ &\leqslant \psi(\tfrac{1}{3}[2S(x_1,x_1,F(x_1,y_1)) + S(y_2,y_2,F(y_2,x_2))])\\ &\quad - \varphi(S(x_1,x_1,F(x_1,y_1)),S(y_2,y_2,F(y_2,x_2)))\\ &= \psi(\tfrac{1}{3}[2S(x_1,x_1,y_2) + S(y_2,y_2,x_3)]) - \varphi(S(x_1,x_1,y_2),S(y_2,y_2,x_3))\\ &= \psi(\max\{S(x_1,x_1,y_2),S(y_2,y_2,x_3)\}) - \varphi(S(x_1,x_1,y_2),S(y_2,y_2,x_3))\\ &< \psi(\max\{S(x_1,x_1,y_2),S(y_2,y_2,x_3)\}).\end{aligned}$$

That is

$$\psi(S(y_2,y_2,x_3)) < \psi(S(x_1,x_1,y_2)) < \psi(S(y_0,y_0,x_1)) \text{ (by (3.6))}. \tag{3.9}$$

We assume that (3.2), (3.3) and (3.4), (3.5) are true for $n = m$. We first show that (3.2) and (3.3) are true for $n = m+1$ when $m$ is even. Now, let $m$ be even. In this case, we consider $\psi(S(x_{m+1},x_{m+1},y_{m+2})) = \psi(S(y_{m+2},y_{m+2},x_{m+1}))$

$$\begin{aligned}&= \psi(S(F(x_{m+1},y_{m+1}),F(x_{m+1},y_{m+1}),F(y_m,x_m)))\\ &\leqslant \psi(\tfrac{1}{3}[2S(x_{m+1},x_{m+1},F(x_{m+1},y_{m+1})) + S(y_m,y_m,F(y_m,x_m))])\\ &\quad - \varphi(S(x_{m+1},x_{m+1},F(x_{m+1},y_{m+1})),S(y_m,y_m,F(y_m,x_m)))\\ &= \psi(\tfrac{1}{3}[2S(x_{m+1},x_{m+1},y_{m+2}) + S(y_m,y_m,x_{m+1})])\\ &\quad - \varphi(S(x_{m+1},x_{m+1},y_{m+2}),S(y_m,y_m,x_{m+1}))\end{aligned}$$

which implies that

$$\begin{aligned}\psi(S(x_{m+1},x_{m+1},y_{m+2})) &\leqslant \psi(\max\{S(x_{m+1},x_{m+1},y_{m+2}),S(y_m,y_m,x_{m+1})\})\\ &\quad - \varphi(S(x_{m+1},x_{m+1},y_{m+2}),S(y_m,y_m,x_{m+1}))\\ &< \psi(\max\{S(y_m,y_m,x_{m+1}),S(x_{m+1},x_{m+1},y_{m+2})\}).\end{aligned} \tag{3.10}$$

By using (3.5), we have

$$\psi(S(x_{m+1},x_{m+1},y_{m+2})) < \psi(S(y_m,y_m,x_{m+1})) < \psi(S(y_0,y_0,x_1)). \tag{3.11}$$

Similarly, we have

$$\begin{aligned}\psi(S(y_{m+1},y_{m+1},x_{m+2})) &= \psi(S(F(x_m,y_m),F(x_m,y_m),F(y_{m+1},x_{m+1})))\\ &\leqslant \psi(\tfrac{1}{3}[2S(x_m,x_m,F(x_m,y_m)) + S(y_{m+1},y_{m+1},F(y_{m+1},x_{m+1}))])\\ &\quad - \varphi(S(x_m,x_m,F(x_m,y_m)),S(y_{m+1},y_{m+1},F(y_{m+1},x_{m+1}))\end{aligned}$$

which implies that

$$\begin{aligned}\psi(S(y_{m+1},y_{m+1},x_{m+2})) &\leqslant \psi(\max\{S(x_m,x_m,y_{m+1}),S(y_{m+1},y_{m+1},x_{m+2})\})\\ &\quad - \varphi(S(x_m,x_m,y_{m+1}),S(y_{m+1},y_{m+1},x_{m+2})).\end{aligned} \tag{3.12}$$

$$< \psi(\max\{S(x_m, x_m, y_{m+1}), S(y_{m+1}, y_{m+1}, x_{m+2})\}).$$

By using (3.4), we have

$$\psi(S(y_{m+1}, y_{m+1}, x_{m+2})) < \psi(S(x_m, x_m, y_{m+1})) < \psi(S(x_0, x_0, y_1)). \tag{3.13}$$

Hence, when $m$ is even, we have (3.2) and (3.3) hold for $n = m + 1$.

We now show that (3.4) and (3.5) are true for $n = m+1$ when $m$ is odd. Now, let $m$ be odd. In this case,

$$\begin{aligned}
\psi(S(x_{m+1}, x_{m+1}, y_{m+2})) &= \psi(S(y_{m+2}, y_{m+2}, x_{m+1})) \\
&= \psi(S(F(x_{m+1}, y_{m+1}), F(x_{m+1}, y_{m+1}), F(y_m, x_m))) \\
&\leqslant \psi(\tfrac{1}{3}[2S(x_{m+1}, x_{m+1}, F(x_{m+1}, y_{m+1})) + S(y_m, y_m, F(y_m, x_m))]) \\
&\quad - \varphi(S(x_{m+1}, x_{m+1}, F(x_{m+1}, y_{m+1})), S(y_m, y_m, F(y_m, x_m))) \\
&= \psi(\tfrac{1}{3}[2S(x_{m+1}, x_{m+1}, y_{m+2}) + S(y_m, y_m, x_{m+1})]) \\
&\quad - \varphi(S(x_{m+1}, x_{m+1}, y_{m+2}), S(y_m, y_m, x_{m+1}))
\end{aligned}$$

which implies that

$$\begin{aligned}
\psi(S(x_{m+1}, x_{m+1}, y_{m+2})) &\leqslant \psi(\max\{S(x_{m+1}, x_{m+1}, y_{m+2}), S(y_m, y_m, x_{m+1})\}) \\
&\quad - \varphi(S(x_{m+1}, x_{m+1}, y_{m+2}), S(y_m, y_m, x_{m+1})) \\
&< \psi(\max\{S(y_m, y_m, x_{m+1}), S(x_{m+1}, x_{m+1}, y_{m+2})\}).
\end{aligned} \tag{3.14}$$

That is

$$\psi(S(x_{m+1}, x_{m+1}, y_{m+2})) < \psi(S(y_m, y_m, x_{m+1})) < \psi(S(x_0, x_0, y_1)),$$
(by our assumption on (3.3)).

Similarly, we have

$$\begin{aligned}
\psi(S(y_{m+1}, y_{m+1}, x_{m+2})) &= \psi(S(F(x_m, y_m), F(x_m, y_m), F(y_{m+1}, x_{m+1}))) \\
&\leqslant \psi(\tfrac{1}{3}[2S(x_m, x_m, F(x_m, y_m)) + S(y_{m+1}, y_{m+1}, F(y_{m+1}, x_{m+1}))]) \\
&\quad - \varphi(S(x_m, x_m, F(x_m, y_m)), S(y_{m+1}, y_{m+1}, F(y_{m+1}, x_{m+1}))) \\
&= \psi(\tfrac{1}{3}[2S(x_m, x_m, , y_{m+1}) + S(y_{m+1}, y_{m+1}, x_{m+2})]) \\
&\quad - \varphi(S(x_m, x_m, y_{m+1})), S(y_{m+1}, y_{m+1}, x_{m+2})
\end{aligned}$$

which implies that

$$\begin{aligned}
\psi(S(y_{m+1}, y_{m+1}, x_{m+2})) &\leqslant \psi(\max\{S(x_m, x_m, y_{m+1}), S(y_{m+1}, y_{m+1}, x_{m+2})\}) \\
&\quad - \varphi(S(x_m, x_m, y_{m+1}), S(y_{m+1}, y_{m+1}, x_{m+2})) \\
&< \psi(\max\{S(x_m, x_m, y_{m+1}), S(y_{m+1}, y_{m+1}, x_{m+2})\}).
\end{aligned} \tag{3.15}$$

That is

$$\psi(S(y_{m+1}, y_{m+1}, x_{m+2})) < \psi(S(x_m, x_m, y_{m+1})) < \psi(S(y_0, y_0, x_1)),$$
(by our induction assumption on (3.2)).

Hence, when $m$ is odd, we have (3.4) and (3.5) hold for $n = m+1$. Thus (3.2)-(3.5) hold for $n = m + 1$. Hence by mathematical induction, the inequalities (3.2)-(3.5) hold for all $n$.

Now, we observe that

$$\{\psi(S(x_{2n}, x_{2n}, y_{2n+1}))\},\ \{\psi(S(y_{2n-1}, y_{2n-1}, x_{2n}))\}$$

are decreasing and converges to $r \geqslant 0$. Also,

$$\{\psi(S(y_{2n}, y_{2n}, x_{2n+1}))\},\ \{\psi(S(x_{2n-1}, x_{2n-1}, y_{2n}))\}$$

are decreasing and converges to $s \geqslant 0$. On letting $n \to \infty$ in (3.14), we get

$$r \leqslant r - \varphi(\lim_{n\to\infty} S(y_n, y_n, x_{n+1}), \lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+2}))$$

which implies that

$$\varphi(\lim_{n\to\infty} S(y_n, y_n, x_{n+1}), \lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+2})) = 0.$$

Thus $\lim_{n\to\infty} S(y_n, y_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+2}) = 0$. Therefore $r = 0$. On letting $n \to \infty$ in (3.15), we get

$$s \leqslant s - \varphi(\lim_{n\to\infty} S(y_n, y_n, x_{n+1}), \lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+2}))$$

which implies that

$$\varphi(\lim_{n\to\infty} S(y_n, y_n, x_{n+1}), \lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+2})) = 0.$$

Thus $\lim_{n\to\infty} S(y_n, y_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+2}) = 0$. Therefore $s = 0$.

Hence we have the following:

$$\lim_{n\to\infty} \psi(S(x_{2n+1}, x_{2n+1}, y_{2n+2})) = 0 \implies \lim_{n\to\infty} S(x_{2n+1}, x_{2n+1}, y_{2n+2}) = 0;$$
$$\lim_{n\to\infty} \psi(S(y_{2n+1}, y_{2n+1}, x_{2n+2})) = 0 \implies \lim_{n\to\infty} S(y_{2n+1}, y_{2n+1}, x_{2n+2}) = 0;$$

and

$$\lim_{n\to\infty} \psi(S(x_{2n}, x_{2n}, y_{2n+1})) = 0 \implies \lim_{n\to\infty} S(x_{2n}, x_{2n}, y_{2n+1}) = 0;$$
$$\lim_{n\to\infty} \psi(S(y_{2n}, y_{2n}, x_{2n+1})) = 0 \implies \lim_{n\to\infty} S(x_{2n}, x_{2n}, y_{2n+1}) = 0.$$

Let $n$ be a positive integer. We consider

$$\begin{aligned}
\psi(S(x_{n+1}, x_{n+1}, y_{n+1})) &= \psi(S(y_{n+1}, y_{n+1}, x_{n+1})) \\
&= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n))) \\
&\leqslant \psi(\tfrac{1}{3}[2S(x_n, x_n, F(x_n, y_n)) + S(y_n, y_n, F(y_n, x_n))]) \\
&\quad - \varphi(S(x_n, x_n, F(x_n, y_n)), S(y_n, y_n, F(y_n, x_n))) \\
&\leqslant \psi(\max\{S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})\}) \\
&\quad - \varphi(S(x_n, x_n, y_{n+1}), S(y_n, y_n, x_{n+1})) \\
&< \psi(\max\{S(y_n, y_n, x_{n+1}), S(x_n, x_n, y_{n+1})\}).
\end{aligned}$$

On taking limits as $n \to \infty$, we get $\lim_{n\to\infty} \psi(S(x_{n+1}, x_{n+1}, y_{n+1})) \leqslant 0$ which implies that $\psi(\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+1})) = 0$. Thus $\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+1}) = 0$.

We now consider

$$\begin{aligned}S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1}) &\leqslant 2S(x_n,x_n,y_n)+S(y_n,y_n,x_{n+1})\\ &\quad+2S(y_n,y_n,x_n)+S(x_n,x_n,y_{n+1})\\ &=4S(x_n,x_n,y_n)+S(y_n,y_n,x_{n+1})+S(x_n,x_n,y_{n+1}).\end{aligned}$$

On letting $n\to\infty$, $\lim\limits_{n\to\infty}[S(x_n,x_n,x_{n+1})+S(y_n,y_n,y_{n+1})]=0$.

We now prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose that either $\{x_n\}$ or $\{y_n\}$ is not Cauchy. Then there exist $\epsilon>0$ and subsequences $\{m_k\}$ and $\{n_k\}$ with $m_k>n_k>k$ such that

$$\max\{S(x_{m_k},x_{m_k},x_{n_k}),S(y_{m_k},y_{m_k},y_{n_k})\}\geqslant\epsilon. \tag{3.16}$$

We choose $m_k$ as the smallest integer with $m_k>n_k$ satisfying (1.5). That is

$$\max\{S(x_{m_k},x_{m_k},x_{n_k}),S(y_{m_k},y_{m_k},y_{n_k})\}\geqslant\epsilon \text{ with}$$

$$\max\{S(x_{m_k-1},x_{m_k-1},x_{n_k}),S(y_{m_k-1},y_{m_k-1},y_{n_k})\}<\epsilon.$$

On the similar lines as in the proof of Theorem 2.1, we have

$$\lim_{n\to\infty}\max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\}=\epsilon.$$

We consider

$$\begin{aligned}\psi(S(x_{m_k},x_{m_k},y_{n_k}))&=\psi(S(y_{n_k},y_{n_k},x_{m_k}))\\ &=\psi(S(F(x_{n_k-1},y_{n_k-1}),F(x_{n_k-1},y_{n_k-1}),F(y_{m_k-1},x_{m_k-1})))\\ &\leqslant\psi(\tfrac{1}{3}[2S(x_{n_k-1},x_{n_k-1},F(x_{n_k-1},y_{n_k-1}))\\ &\qquad+S(y_{m_k-1},y_{m_k-1},F(y_{m_k-1},x_{m_k-1}))])\\ &\quad-\varphi(S(x_{n_k-1},x_{n_k-1},F(x_{n_k-1},y_{n_k-1})),\\ &\qquad S(y_{m_k-1},y_{m_k-1},F(y_{m_k-1},x_{m_k-1})))\\ &=\psi(\tfrac{1}{3}[2S(x_{n_k-1},x_{n_k-1},y_{n_k})+S(y_{m_k-1},y_{m_k-1},x_{m_k})])\\ &\qquad-\varphi(S(x_{n_k-1},x_{n_k-1},y_{n_k}),S(y_{m_k-1},y_{m_k-1},x_{m_k})).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\psi(S(y_{m_k},y_{m_k},x_{n_k}))&=\psi(S(F(x_{m_k-1},y_{m_k-1}),F(x_{m_k-1},y_{m_k-1}),F(y_{n_k-1},x_{n_k-1})))\\ &\leqslant\psi(\tfrac{1}{2}[2S(x_{m_k-1},x_{m_k-1},F(x_{m_k-1},y_{m_k-1}))\\ &\qquad+S(y_{n_k-1},y_{n_k-1},F(y_{n_k-1},x_{n_k-1}))])\\ &\quad-\varphi(S(x_{m_k-1},x_{m_k-1},F(x_{m_k-1},y_{m_k-1})),\\ &\qquad S(y_{n_k-1},y_{n_k-1},F(y_{n_k-1},x_{n_k-1})))\\ &=\psi(\tfrac{1}{3}[2S(x_{m_k-1},x_{m_k-1},y_{m_k})+S(y_{n_k-1},y_{n_k-1},x_{n_k})])\\ &\qquad-\varphi(S(x_{m_k-1},x_{m_k-1},y_{m_k}),S(y_{n_k-1},y_{n_k-1},x_{n_k})).\end{aligned}$$

We now consider

$$\begin{aligned}&\psi(\max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\})\\ &\qquad=\max\{\psi(S(x_{m_k},x_{m_k},y_{n_k})),\psi(S(y_{m_k},y_{m_k},x_{n_k}))\}\\ &\qquad\leqslant\max\{\psi(\tfrac{1}{3}[2S(x_{n_k-1},x_{n_k-1},y_{n_k})+S(y_{m_k-1},y_{m_k-1},x_{m_k})]),\end{aligned}$$

$$\psi(\tfrac{1}{3}[2S(x_{m_k-1},x_{m_k-1},y_{m_k})+S(y_{n_k-1},y_{n_k-1},x_{n_k})])\}$$
$$-\min\{\varphi(S(x_{n_k-1},x_{n_k-1},y_{n_k}),S(y_{m_k-1},y_{m_k-1},x_{m_k})),$$
$$\varphi(S(x_{m_k-1},x_{m_k-1},y_{m_k}),S(y_{n_k-1},y_{n_k-1},x_{n_k}))\}.$$

On letting $k\to\infty$,

$$\begin{aligned}&\psi(\lim_{n\to\infty}\max\{S(x_{m_k},x_{m_k},y_{n_k}),S(y_{m_k},y_{m_k},x_{n_k})\})\\ &\leqslant \max\{\lim_{k\to\infty}\psi(\tfrac{1}{3}[2S(x_{n_k-1},x_{n_k-1},y_{n_k})+S(y_{m_k-1},y_{m_k-1},x_{m_k})]),\\ &\qquad \lim_{k\to\infty}\psi(\tfrac{1}{3}[2S(x_{m_k-1},x_{m_k-1},y_{m_k})+S(y_{n_k-1},y_{n_k-1},x_{n_k})])\}\\ &\quad -\min\{\lim_{k\to\infty}\varphi(S(x_{n_k-1},x_{n_k-1},y_{n_k}),S(y_{m_k-1},y_{m_k-1},x_{m_k})),\\ &\qquad \lim_{k\to\infty}\varphi(S(x_{m_k-1},x_{m_k-1},y_{m_k}),S(y_{n_k-1},y_{n_k-1},x_{n_k}))\}.\end{aligned}$$

That is $\psi(\epsilon)\leqslant 0$ so that $\psi(\epsilon)=0$ which implies that $\epsilon=0$, a contradiction. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy. On the similar lines as in Theorem 2.1, we can show that $F$ has a unique strong coupled fixed point in $A\cap B$. ☐

By choosing $\psi(t)=t$ in Theorem 3.1, we have the following.

Corollary 3.1. *Let $(X,S)$ be a complete $S$-metric space. Let $A$ and $B$ be two nonempty closed subsets of $X$. Let $F:X\times X\to X$ be cyclic with respect to $A$ and $B$satisfying the following inequality:*

$$\begin{aligned}S(F(x,y),F(u,v),F(w,z))\leqslant \frac{1}{3}[&S(x,x,F(x,y))+S(u,u,F(u,v))\\ &+S(w,w,F(w,z))]\\ &-\varphi(\max\{S(x,x,F(x,y)),S(u,u,F(u,v))\},\\ &S(w,w,F(w,z)))\end{aligned}\tag{3.17}$$

*where $x,u,z\in A$ and $y,v,w\in B$, $\varphi\in\Phi$. Then $A\cap B\neq\emptyset$ and $F$ has a unique strong coupled fixed point in $A\cap B$.*

By choosing $\varphi(t_1,t_2)=(\frac{1}{3}-k)(t_1+t_2)$ in Corollary 3.1, where $k\in(0,\frac{1}{3})$, we have the following.

Corollary 3.2. *Let $(X,S)$ be a complete $S$-metric space. Let $A$ and $B$ be two nonempty closed subsets of $X$. Let $F:X\times X\to X$ be cyclic with respect to $A$ and $B$ satisfying the following inequality: there exists $k\in(0,\frac{1}{3})$ such that*

$$\begin{aligned}S(F(x,y),F(u,v),F(w,z))\leqslant k[&S(x,x,F(x,y))+S(u,u,F(u,v))\\ &+S(w,w,F(w,z))]\end{aligned}\tag{3.18}$$

*where $x,u,z\in A$ and $y,v,w\in B$. Then $A\cap B\neq\emptyset$ and $F$ has a unique strong coupled fixed point in $A\cap B$.*

The following example is in support of Theorem 3.1.

Example 3.1. Let $X=[0,4]$. We define $S:X^3\to[0,\infty)$ by

$$S(x,y,z)=\begin{cases}0 & \text{if } x=y=z\\ \max\{x,y,z\} & \text{otherwise.}\end{cases}$$

Then $(X,S)$ is a complete $S$-metric space, which follows as in Example 2.1. Let $A=[0,4]$ and $B=[0,1]$. We define $F: X\times X\to X$ by

$$F(x,y)=\begin{cases} \frac{xy}{8} & \text{if } x\in A \text{ and } y\in B\\ 0 & \text{otherwise.}\end{cases}$$

Then $F(A,B)=[0,\frac{1}{2}]\subset B$ and $F(B,A)=\{0\}\subset A$ so that $F$ is cyclic with respect to $A$ and $B$. We define $\psi:[0,\infty)^2\to[0,\infty)$ by $\psi(t)=\frac{t}{2}$ and $\varphi:[0,\infty)^2\to[0,\infty)$ by $\varphi(t_1,t_2)=\frac{5}{48}(t_1+t_2)$.

We now verify the inequality (3.1). Let $x,u,z\in A$ and $y,v,w\in B$.

*Case* (i): Let $x,u,z\in[0,1]$ and $y,v,w\in[0,1]$. We have

$$S(x,x,F(x,y))=S(x,x,\tfrac{xy}{8})=x;\ S(u,u,F(u,v))=S(u,u,\tfrac{uv}{8})=u;$$
$$S(w,w,F(w,z))=S(w,w,\tfrac{wz}{8})=w.$$

We consider

$$\begin{aligned}\psi(S(F(x,y),F(u,v),F(w,z)))&=\psi(S(\tfrac{xy}{8},\tfrac{uv}{8},\tfrac{wz}{8}))\\ &=\tfrac{1}{2}S(\tfrac{xy}{8},\tfrac{uv}{8},\tfrac{wz}{8})\\ &=\tfrac{1}{2}[\max\{\tfrac{xy}{8},\tfrac{uv}{8},\tfrac{wz}{8}\}]\\ &\leqslant\tfrac{1}{16}[x+u+w]\\ &=\tfrac{1}{16}[S(x,x,F(x,y))+S(u,u,F(u,v))\\ &\qquad+S(w,w,F(w,z))\\ &=\tfrac{1}{6}[t_1+t_2+t_3]-\tfrac{5}{48}[t_1+t_2+t_3]\\ &\leqslant\tfrac{1}{6}[t_1+t_2+t_3]-\tfrac{5}{48}[\max\{t_1,t_2\}+t_3]\\ &=\psi(\tfrac{1}{3}[t_1+t_2+t_3])-\varphi(\max\{t_1,t_2\},t_3)\end{aligned}$$

where $t_1=S(x,x,F(x,u))$, $t_2=S(u,u,F(u,v))$ and $t_3=S(w,w,F(w,z))$.

*Case* (ii): Let $x,u,z\in(1,4]$ and $y,v,w\in[0,1]$. We have

$$S(x,x,F(x,y))=x;\ S(u,u,F(u,v))=u;\ S(w,w,F(w,z))=S(w,w,0)=w.$$

We consider

$$\begin{aligned}\psi(S(F(x,y),F(u,v),F(w,z)))&=\psi(S(\tfrac{xy}{8},\tfrac{uv}{8},0))\\ &=\tfrac{1}{2}S(\tfrac{xy}{8},\tfrac{uv}{8},0)\\ &=\tfrac{1}{2}[\max\{\tfrac{xy}{8},\tfrac{uv}{8}\}]\\ &\leqslant\tfrac{1}{16}[x+u]\\ &\leqslant\tfrac{1}{16}[S(x,x,F(x,y))+S(u,u,F(u,v))\\ &\qquad+S(w,w,F(w,z))\\ &=\tfrac{1}{6}[t_1+t_2+t_3]-\tfrac{5}{48}[t_1+t_2+t_3]\\ &\leqslant\tfrac{1}{6}[t_1+t_2+t_3]-\tfrac{5}{48}[\max\{t_1,t_2\}+t_3]\\ &=\psi(\tfrac{1}{3}[t_1+t_2+t_3])-\varphi(\max\{t_1,t_2\},t_3)\end{aligned}$$

where $t_1=S(x,x,F(x,u))$, $t_2=S(u,u,F(u,v))$ and $t_3=S(w,w,F(w,z))$.

Therefore $F$ is a Kannan type $(\psi, \varphi)$-weakly cyclic coupled contraction mapping with respect to $A$ and $B$. Hence $F$ satisfies all the hypotheses of Theorem 3.1 and $(0, 0)$ is a unique strong coupled fixed point of $F$.

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Received by editors 13.02.2020; Revised version 06.09.2021; Available online 13.09.2021.

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