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SHEFFER STROKE BCH-ALGEBRAS

Tahsin Oner, Tuğce Kalkan, and Melisa Çakar

Abstract. In this paper, Sheffer stroke BCH-algebra is defined and its features are studied. It is stated the connection between a Sheffer stroke BCH-algebra and a BCH-algebra by defining a unary operation on a Sheffer stroke BCH-algebra. After describing a subalgebra and a BCA-part of a Sheffer stroke BCH-algebra, the relationship of this structures is shown. After determining a minimal element and a medial element of Sheffer stroke BCH-algebra, it is shown that this structures are equivalent. It is proved that the collection of all minimal elements are equal to all medial elements of a Sheffer stroke BCH-algebra. Morever, a centre and a branch of a Sheffer stroke BCH-algebra are defined and it is demonstrated that the centre is a subalgebra. It is indicated that the collection of all minimal elements and medial elements are subalgebras of a Sheffer stroke BCH-algebra. Finally, an ideal and a closed ideal are described and it is proved that an ideal is a closed ideal if and only if it is a subalgebra.

1. Introduction

In 1966, two classes of abstract algebras, BCK-algebras and BCI-algebras, were introduced by Y. Imai and K. Iséki ([**8**], [**9**]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Hu and Li [[**6**], [**7**]] introduced a new class of algebras so-called BCH-algebras. It is known that BCK-algebras and BCI-algebras are contained in the class of BCH-algebras. They have studied a few properties of these algebras. Some other properties of these algebras have been studied by Chaudhry ([**1**], [**2**]), Dudek and Thomys [**5**] and many other researchers.

The Sheffer stroke operation, which was first introduced by H. M. Sheffer [**16**], engages many scientists' attention, because any Boolean function or axiom can

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be expressed by means of this operation [**10**]. It reducts axiom systems of many algebraic structures. So, many researchers want to use this operation on their studies. Also, some applications of this operation has been appeared in algebraic structures such as Sheffer stroke non-associative MV-algebras [**3**] and filters [**12**], Sheffer stroke Hilbert algebras [**11**], Sheffer stroke UP-algebras [**13**], Sheffer stroke BG-algebras [**14**], Class of Sheffer stroke BCK-algebras [**15**] and Sheffer operation in ortholattices [**4**].

After giving basic definitions and notions about a Sheffer stroke and a BCH-algebra, it is defined a Sheffer stroke BCH-algebra. By presenting fundamental notions about this algebraic structure, it is stated the connection between a Sheffer stroke BCH-algebra and a BCH-algebra. It is demonstrated that every Sheffer stroke BCK-algebra is a Sheffer stroke BCH-algebra and one condition is necessary for the converse to be true. A subalgebra and a BCA-part of a Sheffer stroke BCH-algebra are defined and it is shown that a BCA-part is a subalgebra. By describing a minimal element and a medial element of a Sheffer stroke BCH-algebra, it is shown that the minimal element and medial element are equivalent and the collection of all minimal elements are equal to all medial elements in a Sheffer stroke BCH-algebra. The centre of a Sheffer stroke BCH-algebra is defined and it is proved that a centre is a subalgebra. After defining branch of a Sheffer stroke BCH-algebra, related concepts are given. It is presented that the collection of all minimal elements and medial elements are subalgebras of a Sheffer stroke BCH-algebra. Finally, an ideal and a closed ideal are defined and it is shown that an ideal is a closed ideal if and only if it is a subalgebra.

2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke and a BCH-algebra.

Definition 2.1. ([**3**]) Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be *Sheffer stroke* if it satisfies the following conditions:

$(S1)$ $a_1|a_2 = a_2|a_1$,
$(S2)$ $(a_1|a_1)|(a_1|a_2) = a_1$,
$(S3)$ $a_1|((a_2|a_3)|(a_2|a_3)) = ((a_1|a_2)|(a_1|a_2))|a_3$,
$(S4)$ $(a_1|((a_1|a_1)|(a_2|a_2)))|(a_1|((a_1|a_1)|(a_2|a_2))) = a_1$.

Definition 2.2. ([**6**]) A BCH-algebra is an algebra $(A, *, 0)$ of type (2, 0) satisfying the following conditions:
$(BCH.1)$ $(a_1 * a_2) * a_3 = (a_1 * a_3) * a_2$,
$(BCH.2)$ $a_1 * a_1 = 0$,
$(BCH.3)$ $a_1 * a_2 = 0$ and $a_2 * a_1 = 0$ imply $a_2 = a_1$.

A BCH-algebra is called bounded if it has the greatest element.

Definition 2.3. ([**1**]) A nonempty subset S of a BCH-algebra A is called a BCH-subalgebra if $a_1 * a_2 \in S$, for all $a_1, a_2 \in S$.

DEFINITION 2.4. ([**7**]) A subset I of A is called an ideal of A if it satisfies
$(I1)$ $0 \in I$,
$(I2)$ if $a_1 * a_2 \in I$ and $a_2 \in I$, then $a_1 \in I$,
for all $a_1, a_2 \in A$.

DEFINITION 2.5. ([**15**] A Sheffer stroke BCK-algebra is a structure $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A, $|$ is a Sheffer operation on A and the following axioms are satisfied:

(sBCK-1) $((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_3|a_3)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_3|a_3))))|(a_3|(a_2|a_2)) = 0|0$,

(sBCK-2) $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$ and $(a_2|(a_1|a_1))|(a_2|(a_1|a_1)) = 0$ imply $a_1 = a_2$,

for all $a_1, a_2, a_3 \in A$.

LEMMA 2.1 ([**15**]). *Let A be a Sheffer stroke BCK-algebra. Then the following features hold for all $a_1, a_2, a_3 \in A$:*
(1) $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$,
(2) $(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0$,
(3) $a_1|(((a_1|(a_2|a_2))|(a_2|a_2))|((a_1|(a_2|a_2))|(a_2|a_2))) = 0|0$,
(4) $(0|0)|(a_1|a_1) = a_1$,
(5) $a_1|0 = 0|0$,
(6) $(a_1|(0|0))|(a_1|(0|0)) = a_1$,
(7) $(0|(a_1|a_1))|(0|(a_1|a_1)) = 0$,
(8) $a_1|((a_2|(a_3|a_3))|(a_2|(a_3|a_3))) = a_2|((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))$,
(9) $((a_1|(((a_2|(a_3|a_3))|(a_2|(a_3|a_3))))))|((a_2|(a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|(a_1|(a_3|a_3))|(a_1|(a_3|a_3))))) = 0|0$,
(10) $((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|a_2) = 0|0$.

PROPOSITION 2.1 ([**15**]). *Let $(A, |, 0)$ be a Sheffer stroke BCK-algebra. Then the following features are hold for all $a_1, a_2, a_3 \in A$:*
(i) $a_1 \leqslant a_3$ *implies* $(a_2|(a_3|a_3))|(a_2|(a_3|a_3)) \leqslant (a_2|(a_1|a_1))|(a_2|(a_1|a_1))$,
(ii) $((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) = ((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)$,
(iii) $((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \leqslant a_3 \Leftrightarrow ((a_1|(a_3|a_3))|(a_1|(a_3|a_3))) \leqslant a_2$,
(iv) $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \leqslant a_1$,
(v) $a_1 \leqslant a_2|(a_1|a_1)$,
(vi) $a_1 \leqslant (a_1|(a_2|a_2))|(a_2|a_2)$,
(vii) *If* $a_1 \leqslant a_2$, *then* $a_3|(a_1|a_1) \leqslant a_3|(a_2|a_2)$.

3. Sheffer stroke BCH-Algebras

In this part, we define a Sheffer stroke BCH-algebra and give some properties.

DEFINITION 3.1. A Sheffer stroke BCH-algebra is an algebra $(A,|,0)$ of type $(2,0)$ such that 0 is the constant in A and the following axioms are satisfied:

$(sBCH.1)$ $(a|(a|a))|(a|(a|a)) = 0$,

$(sBCH.2)$ $(a|(b|b))|(a|(b|b)) = (b|(a|a))|(b|(a|a)) = 0$ imply $a = b$,

$(sBCH.3)$ $(((a|(b|b))|(a|(b|b)))|(c|c)) = (((a|c|c))|(a|(c|c)))|(b|b))$,

for all $a, b, c \in A$.

A partial order $\leqslant$ on A can be defined by

$$a \leqslant b \Leftrightarrow (a|(b|b))|(a|(b|b)) = 0.$$

Let A be a Sheffer stroke BCH-algebra, unless otherwise is indicated.

EXAMPLE 3.1. Consider $(A,|,0)$, where $A = \{0, x, y, z, 1\}$. The binary operation $|$ has Cayley table as follow:

$\mid$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Then $(A,|,0)$ is a Sheffer stroke BCH-algebra.

LEMMA 3.1. *Let A be a Sheffer stroke BCH-algebra. Then the following features hold for all $a_1, a_2, a_3 \in A$:*

(1) $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$,

(2) $a_1|(((a_1|(a_2|a_2))|(a_2|a_2))|((a_1|(a_2|a_2))|(a_2|a_2))) = 0|0$,

(3) $(0|0)|(a_1|a_1) = a_1$,

(4) $(a_1|(0|0))|(a_1|(0|0)) = a_1$,

(5) $a_1|((a_2|(a_3|a_3))|(a_2|(a_3|a_3))) = a_2|((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))$,

(6) $((a_1|(a_1|(a_2|a_2)))|(a_1|(a_1|(a_2|a_2))))|(a_2|a_2) = 0|0$,

(7) $(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|a_1)) = 0|(a_2|a_2)$,

(8) $(0|(a_1|(a_2|a_2))) = ((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2))$,

(9) $a_1 \leqslant a_2$ *implies* $(0|(a_1|a_1)) = (0|(a_2|a_2))$.

PROOF. (1) Substituting $[a_2 := (a_1|a_1)]$ in (S2), we obtain $(a_1|a_1)|(a_1|(a_1|a_1)) = a_1$. Then $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$ from (S1).

(2) In (S3), by substituting $[a_2 := a_1|(a_2|a_2)]$ and $[a_3 := a_2|a_2]$ and applying (S1), (S2), (S3) and (sBCH.1), we have

$$\begin{aligned} a_1|((a_1|(a_2|a_2))|(a_2|a_2))|((a_1|(a_2|a_2))|(a_2|a_2)) \\ &= a_1|(((a_2|a_2)|(a_1|(a_2|a_2)))|((a_2|a_2)|(a_1|(a_2|a_2)))) \\ &= ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_2|a_2)) \\ &= (a_1|(a_2|a_2))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \\ &= 0|0. \end{aligned}$$

(3) $(0|0)|(a_1|a_1) = (a_1|(a_1|a_1))|(a_1|a_1) = a_1$ from (1), (S2) and (sBCH.1).

(4) By using (S1), (S2) and (3), we get

$$\begin{aligned}(a_1|(0|0))|(a_1|(0|0)) &= ((0|0)|((a_1|a_1)|(a_1|a_1)))|((0|0)|((a_1|a_1)|(a_1|a_1)))\\ &= (a_1|a_1)|(a_1|a_1)\\ &= a_1.\end{aligned}$$

(5) By using (S1) and (S3), we have

$$\begin{aligned}a_1|((a_2|(a_3|a_3))|(a_2|(a_3|a_3))) &= (((a_1|a_2)|(a_1|a_2))|(a_3|a_3))\\ &= (((a_2|a_1)|(a_2|a_1))|(a_3|a_3))\\ &= a_2|((a_1|(a_3|a_3))|(a_1|(a_3|a_3))).\end{aligned}$$

(6) It is obtained from (2) and (S3).

(7) It is obtained from (sBCH.1) and (sBCH.3).

(8) By using (S1), (S3), (sBCH.1), (sBCH.3) and (7), we get

$$\begin{aligned}0|(a_1|(a_2|a_2)) &= (((0|(a_2|a_2))|((0|(a_2|a_2))|(0|(a_2|a_2))))|((0|(a_2|a_2))|((0|(a_2|a_2))|\\ &\quad (0|(a_2|a_2)))))|(a_1|(a_2|a_2))\\ &= (((0|(a_2|a_2))|(a_1|(a_2|a_2)))|((0|(a_2|a_2))|(a_1|(a_2|a_2))))|\\ &\quad ((0|(a_2|a_2))|(0|(a_2|a_2)))\\ &= (0|(a_2|a_2))|((((a_1|(a_2|a_2)))|((0|(a_2|a_2))|(0|(a_2|a_2))))|\\ &\quad (((a_1|(a_2|a_2)))|((0|(a_2|a_2))|(0|(a_2|a_2)))))\\ &= (0|(a_2|a_2))|(((a_1|(a_2|a_2))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|a_1)))|\\ &\quad (((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|a_1)))|\\ &\quad (((a_1|(a_2|a_2))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|a_1)))|\\ &\quad (((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|a_1)))\\ &= (0|(a_2|a_2))|((((a_1|(a_2|a_2))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))))|\\ &\quad ((a_1|(a_2|a_2))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))))|(a_1|a_1))|\\ &\quad (((((a_1|(a_2|a_2))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))))|((a_1|(a_2|a_2))|\\ &\quad ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))))|(a_1|a_1))\\ &= (0|(a_2|a_2))|((0|(a_1|a_1))|(0|(a_1|a_1)))\\ &= ((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2)).\end{aligned}$$

(9) By using (7), we get

$$0|(a_2|a_2) = ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|a_1) = 0|(a_1|a_1).$$

□

THEOREM 3.1. *Let* $(A, |, 0)$ *be a Sheffer stroke BCH-algebra. If we define*

$$a_1 * a_2 := (a_1|(a_2|a_2))|(a_1|a_2|a_2),$$

then $(A,*,0)$ *is a BCH-algebra.*

PROOF. (BCH.1) By using (sBCH.3), we have

$$\begin{aligned}(a_1*a_2)*a_3 &= (((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3))|(((a_1|(a_2|a_2))|\\ &\quad (a_1|(a_2|a_2)))|(a_3|a_3)\\ &= (((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2))|(((a_1|(a_3|a_3))|\\ &\quad (a_1|(a_3|a_3)))|(a_2|a_2)\\ &= (a_1*a_3)*a_2.\end{aligned}$$

(BCH.2) $a_1*a_1=(a_1|(a_1|a_1))|(a_1|(a_1|a_1))=0$ from (sBCH.1).

(BCH.3) By using (sBCH.2) we get $a_1*a_2=((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))=0$ and $a_2*a_1=((a_2|(a_1|a_1))|(a_2|(a_1|a_1)))=0$ imply $a_1=a_2$. □

EXAMPLE 3.2. Consider Sheffer stroke BCH-algebra $(A,|,0)$ in Example 3.1. Then the structure $(A,*,0)$ has the following Cayley table: It is clear that it is a

TABLE 1

$*$	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

BCH-algebra.

THEOREM 3.2. *Let* $(A,*,0,1)$ *be a bounded BCH-algebra. If we define* $a_1|a_2:=(a_1*a_2^0)^0$ *and* $a_1^0=1*a_1$*, where* $a_1*(1*a_1)=a_1$ *and* $1*(1*a_1)=a_1$*, then* $(A,|,0)$ *is a Sheffer stroke BCH-algebra.*

PROOF.

$(sBCH.1)$ By using (BCH.2), we have

$$\begin{aligned}(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) &= (a_1|a_1^0)|(a_1|a_1^0)\\ &= (a_1*a_1)^0|(a_1*a_1)^0\\ &= ((a_1*a_1)^0)^0\\ &= a_1*a_1\\ &= 0.\end{aligned}$$

$(sBCH.2)$:

$$\begin{aligned}(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) &= (a_1|a_2^0)|(a_1|a_2^0)\\ &= (a_1*a_2)^0|(a_1*a_2)^0\\ &= ((a_1*a_2)^0)^0\\ &= a_1*a_2\\ &= 0\end{aligned}$$

and

$$\begin{aligned}(a_2|(a_1|a_1))|(a_2|(a_1|a_1)) &= (a_2|a_1^0)|(a_2|a_1^0)\\ &= (a_2 * a_1)^0|(a_2 * a_1)^0\\ &= ((a_2 * a_1)^0)^0\\ &= a_2 * a_1\\ &= 0\end{aligned}$$

imply $a_1 = a_2$ by $(BCH.3)$.

$(sBCH.3)$ By using $(BCH.1)$, we have

$$\begin{aligned}(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3)) &= ((a_1|a_2^0)|(a_1|a_2^0))|a_3^0\\ &= ((a_1 * a_2)^0|(a_1 * a_2)^0)|a_3^0\\ &= (((a_1 * a_2)^0)^0)|a_3^0\\ &= ((a_1 * a_2) * a_3)^0\\ &= ((a_1 * a_3) * a_2)^0\\ &= (((a_1 * a_3)^0)^0)|a_2^0\\ &= ((a_1 * a_3)^0|(a_1 * a_3)^0)|a_2^0\\ &= ((a_1|a_3^0)|(a_1|a_3^0))|a_2^0\\ &= (((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)).\end{aligned}$$

□

Example 3.3. Consider a bounded BCH-algebra $(A, *, 0, 1)$ with $A = \{0, x, y, z, t, u, v, 1\}$ and the binary operation $*$ on A defined in Table 2. Then the structure $(A, |, 0)$ defined by the bounded BCH-algebra $(A, *, 0, 1)$ has Cayley table given in Table 3. Therefore,it is a Sheffer stroke BCH-algebra.

Table 2

$*$	0	x	y	z	t	u	v	1
0	0	0	0	0	0	0	0	0
x	x	0	x	x	0	0	x	0
y	y	y	0	y	0	y	0	0
z	z	z	z	0	z	0	0	0
t	t	y	x	t	0	y	x	0
u	u	z	u	x	z	0	x	0
v	v	v	z	y	z	y	0	0
1	1	v	u	t	z	y	x	0

Theorem 3.3. *Every Sheffer stroke BCK-algebra is a Sheffer stroke BCH-algebra.*

Proof. It is obtained from (sBCK-2), Lemma 2.1 (2), Proposition 2.1 (ii). □

TABLE 3

$\mid$	0	x	y	z	t	u	v	1
0	1	1	1	1	1	1	1	1
x	1	v	1	1	v	v	1	v
y	1	1	u	1	u	1	u	u
z	1	1	1	t	1	t	t	t
t	1	v	u	1	z	v	u	z
u	1	v	1	t	v	y	t	y
v	1	1	u	t	u	t	x	x
1	1	v	u	t	z	y	x	0

THEOREM 3.4. *A Sheffer stroke BCH-algebra satisfying*

$$((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) =$$
$$((((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_3|a_3))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_3|a_3)))|$$
$$(a_2|(a_3|a_3))$$

is a Sheffer stroke BCK-algebra.

PROOF. It is sufficient to prove that (sBCK-1) holds. By using (sBCH.1), (sBCH.3) and (S2), we get

$$((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|(a_3|a_3)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|$$
$$(a_1|(a_3|a_3))))|(a_3|(a_2|a_2))$$
$$= ((((a_1|(a_1|(a_3|a_3)))|(a_1|(a_1|(a_3|a_3))))|(a_2|a_2))|(((a_1|(a_1|(a_3|a_3)))|$$
$$(a_1|(a_1|(a_3|a_3))))|(a_2|a_2)))|(a_3|(a_2|a_2))$$
$$= (((((a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_2|a_2))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_2|a_2)))$$
$$(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)))|((((a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_2|a_2))|$$
$$(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_2|a_2)))(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2))))|$$
$$(a_3|(a_2|a_2))$$
$$= (((((a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_2|a_2))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_2|a_2)))$$
$$(a_3|(a_2|a_2)))|((((a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_2|a_2))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|$$
$$(a_2|a_2)))(a_3|(a_2|a_2))))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2))$$
$$= ((((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2))|(((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)))$$
$$((a_1|(a_3|a_3))|(a_1|(a_3|a_3)))|(a_2|a_2)$$
$$= 0|0.$$

□

DEFINITION 3.2. Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then a non-empty subset S of A is called a subalgebra of A, if

$$(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in S,$$

for all $a_1, a_2 \in S$.

EXAMPLE 3.4. In Example 3.1,

$$S_1 = \{0, x\}, S_2 = \{0, y\}, \text{ and } S_3 = \{0, x, y\}$$

are subalgebras of A.

DEFINITION 3.3. Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then the subset

$$M = \{a_1 : a_1 \in A \text{ and } 0|(a_1|a_1) = 0|0\}$$

is called a BCA-part of A.

EXAMPLE 3.5. Given Sheffer stroke BCH-algebra in Example 3.1. Then it is obvious that the $M = \{0, x, y, 1\}$ is a BCA-part of A.

THEOREM 3.5. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then M is a subalgebra of A.*

PROOF. Clearly, $0 \in M$ and M is non-empty. Let $a_1, a_2 \in M$. By using Lemma 3.1 (8), (S2) and (sBCH.1), we have

$(0|((a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2))))|\ (0|((a_1|(a_2|a_2))|$
$(a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2))))$
$= (0|(a_1|(a_2|a_2)))|(0|(a_1|(a_2|a_2)))$
$= (((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2)))|(((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2)))$
$= (((0|0)|(0|0))|(0|0))|(((0|0)|(0|0))|(0|0))$
$= ((0|(0|0))|(0|(0|0))$
$= 0.$

Then $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in M$ and M is a subalgebra of A. □

DEFINITION 3.4. An element $a_0 \in A$ is called a minimal element of A if $a \leqslant a_0$ implies $a = a_0$.

The collection of all minimal elements of a Sheffer stroke BCH-algebra A is denoted by $Min(A)$.

EXAMPLE 3.6. Given Sheffer stroke BCH-algebra in Example 3.1. Then 0 is the minimal element of A.

DEFINITION 3.5. Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. We define

$$Med(A) = \{a_1 : a_1 \in A \text{ and } 0|(0|(a_1|a_1)) = a_1|a_1\} \subseteq A.$$

The set $Med(A)$ is called the medial part of A.

An element $a \in Med(A)$ is called a medial element of A. For a medial element $a \in A$, we write $0|(0|(a|a)) = a|a$.

EXAMPLE 3.7. In Example 3.1, $\{0\}$ is a medial element of A.

PROPOSITION 3.1. *In a Sheffer stroke BCH-algebra $(A, |, 0)$ the following features hold:*

(1) 0 *is a minimal element of A,*

(2) $a_1|(0|0) = a_1|a_1$, *for all $a_1 \in A$.*

PROOF. (1) Let $a_1 \in A$ such that $a_1 \leqslant 0$. Thus $(a_1|(0|0))|(a_1|(0|0)) = 0$. By using (S2), (sBCH.1) and Lemma 3.1 (6), we get

$$\begin{aligned}
(0|(a_1|a_1))|(0|(a_1|a_1)) &= (((a_1|(0|0))|(a_1|(0|0)))|(a_1|a_1))|(((a_1|(0|0))|(a_1|(0|0)))| \\
&\quad (a_1|a_1)) \\
&= (((a_1|(a_1|(a_1|a_1)))|((a_1|(a_1|(a_1|a_1)))))|(a_1|a_1))| \\
&\quad (((a_1|(a_1|(a_1|a_1)))|((a_1|(a_1|(a_1|a_1)))))|(a_1|a_1)) \\
&= (0|0)|(0|0) \\
&= 0.
\end{aligned}$$

Then $(a_1|(0|0))|(a_1|(0|0)) = (0|(a_1|a_1))|(0|(a_1|a_1)) = 0$. Hence $a_1 = 0$ from (sBCH.2). Therefore, 0 is a minimal element of A.

(2) Obviously, $(a_1|(a_1|(0|0)))|(a_1|(a_1|(0|0))) \leqslant 0$ from Lemma 3.1 (6). Since 0 is a minimal element of A, $(a_1|(a_1|(0|0)))|(a_1|(a_1|(0|0))) = 0$. Thus $a_1 \leqslant (a_1|(0|0))|(a_1|(0|0))$ by (S2). Moreover,

$$\begin{aligned}
&((((a_1|(0|0)))|(a_1|(0|0)))|(a_1|a_1))|((((a_1|(0|0)))|(a_1|(0|0)))|(a_1|a_1)) \\
&\quad = (((a_1|(a_1|(a_1|a_1)))|((a_1|(a_1|(a_1|a_1)))))|(a_1|a_1))|(((a_1|(a_1|(a_1|a_1)))| \\
&\qquad ((a_1|(a_1|(a_1|a_1)))))|(a_1|a_1)) \\
&\quad = (0|0)|(0|0) \\
&\quad = 0.
\end{aligned}$$

Hence $(a_1|(0|0))|(a_1|(0|0)) \leqslant a_1$. Then $(a_1|(0|0))|(a_1|(0|0)) = a_1$. By (S2), we obtain $a_1|(0|0) = a_1|a_1$. □

Remark 3.1. In a Sheffer stroke BCH-algebra A, we have $(0|(0|(0|0))) = (0|(0|0)) = 0|0$ from (sBCH.1). So, $0 \in Med(A)$. From Proposition 3.1, we get that $0 \in Min(A)$. Hence $Med(A)$ and $Min(A)$ are non-empty.

Proposition 3.2. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then an element $a_0 \in A$ is a medial element if and only if it is a minimal element.*

Proof. Let a_0 be a medial element of A. Then $0|(0|(a_0|a_0)) = a_0|a_0$. Let $a_1 \leqslant a_0$. Then $(a_1|(a_0|a_0))|(a_1|(a_0|a_0)) = 0$. From (sBCH.1) and (sBCH.3),

$$\begin{aligned}
0|(a_1|a_1) &= ((a_1|(a_0|a_0))|(a_1|(a_0|a_0)))|(a_1|a_1) \\
&= ((a_1|(a_1|a_1))|(a_1|(a_1|a_1)))|(a_0|a_0) \\
&= 0|(a_0|a_0).
\end{aligned}$$

Then $0|(0|(a_0|a_0)) = 0|(0|(a_1|a_1))$. Therefore, $a_0 = (0|(0|(a_0|a_0)))|(0|(0|(a_0|a_0))) = (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))) \leqslant a_1$ from Lemma 3.1 (6). Hence $a_0 = a_1$. So a_0 is a minimal element of A.

Conversely, let a_0 be a minimal element of A. Since

$$(0|(0|(a_0|a_0)))|(0|(0|(a_0|a_0))) \leqslant a_0$$

from Lemma 3.1 (6) and a_0 is a minimal element,

$$a_0 = (0|(0|(a_0|a_0)))|(0|(0|(a_0|a_0))).$$

Thus $a_0|a_0 = (0|(0|(a_0|a_0)))$ and a_0 is a medial element of A. □

Remark 3.2. From Proposition 3.1, $Min(A) = Med(A)$.

Theorem 3.6. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then*

$$(0|(a_1|a_1))|(0|(a_1|a_1)) \in Min(A),$$

for all $a_1 \in A$.

Proof. Let $a_1, a_2 \in A$ and $a_2 \leqslant (0|(a_1|a_1))|(0|(a_1|a_1))$. Then

$$(a_2|(0|(a_1|a_1)))|(a_2|(0|(a_1|a_1))) = 0,$$

which implies

$$((a_2|(0|(a_1|a_1)))|(a_2|(0|(a_1|a_1))))|(a_2|a_2) = 0|(a_2|a_2).$$

That is,

$$((a_2|(a_2|a_2))|(a_2|(a_2|a_2)))|(0|(a_1|a_1)) = 0|(a_2|a_2)$$

and then we get $0|(0|(a_1|a_1)) = 0|(a_2|a_2)$. Now,

$$((0|(a_2|a_2))|(0|(a_2|a_2)))|(a_1|a_1) = ((0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))))|(a_1|a_1) = 0|0$$

and then we have

$((((0|(a_2|a_2))|(0|(a_2|a_2)))|(a_1|a_1))|(((0|(a_2|a_2))|(0|(a_2|a_2)))|(a_1|a_1)))|$ $(0|(a_2|a_2)) = 0|(0|(a_2|a_2))$.

$(((0|(a_2|a_2))|((0|(a_2|a_2))|(0|(a_2|a_2))))|((0|(a_2|a_2))|((0|(a_2|a_2))|(0|(a_2|a_2)))))$ $|(a_1|a_1) = 0|(0|(a_2|a_2))$.

That is, $0|(a_1|a_1) = 0|(0|(a_2|a_2))$. Then $(0|(a_1|a_1))(0|(a_1|a_1)) = (0|(0|(a_2|a_2)))$ $|(0|(0|(a_2|a_2))) \leqslant a_2$. Hence, $a_2 = (0|(a_1|a_1))|(0|(a_1|a_1))$. Therefore,

$$(0|(a_1|a_1))|(0|(a_1|a_1)) \in Min(A).$$

□

Remark 3.3. Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Since $Min(A) = Med(A)$, then $(0|(a_1|a_1))|(0|(a_1|a_1)) \in Med(A)$. Thus

$$(0|(0|(0|(a_1|a_1)))) = 0|(a_1|a_1),$$

for all $a_1 \in A$.

Theorem 3.7. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then*

$$Min(A) = \{((0|(a_1|a_1))|(0|(a_1|a_1)))|a_1 \in A\}.$$

Proof. Let $B = \{((0|(a_1|a_1))|(0|(a_1|a_1)))|a_1 \in A\}$ and $a_2 \in B$. Then

$$a_2 = ((0|(a_1|a_1))|(0|(a_1|a_1)))$$

for some $a_1 \in A$. By Theorem 3.6, $a_2 = ((0|(a_1|a_1))|(0|(a_1|a_1))) \in Min(A)$. So $B \subseteq Min(A)$.

Let $a_1 \in Min(A)$. Since $Min(A) = Med(A)$, so $a_1 \in Med(A)$. Thus

$$a_1 = (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))) = (0|(a_2|a_2))|(0|(a_2|a_2)),$$

where $a_2 = ((0|(a_1|a_1))|(0|(a_1|a_1)))$. Hence $a_1 \in B$. Then $Min(A) \subseteq B$. Therefore, $Min(A) = B = \{((0|(a_1|a_1))|(0|(a_1|a_1)))|a_1 \in A\}$. □

Remark 3.4. From above theorem and Remark 3.2, we have that if $(A, |, 0)$ is a Sheffer stroke BCH-algebra, then

$$Min(A) = Med(A) = \{((0|(a_1|a_1))|(0|(a_1|a_1)))|a_1 \in A\}.$$

Proposition 3.3. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra and $a \in A$. Then the following conditions are equivalent:*

(*i*) *a is minimal;*

(*ii*) $a_1|(a_1|(a|a)) = a|a$,

(*iii*) $0|(0|(a|a)) = a|a$,

(*iv*) $a|(a_1|a_1) = ((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a|a))$,

(*v*) $a|(a_1|a_1) = 0|(a_1|(a|a))$, *for every* $a, a_1 \in A$.

Proof. (*i*) ⇒ (*ii*): By Lemma 3.1 (6),

$$((a_1|(a_1|(a|a)))|(a_1|(a_1|(a|a))))|(a|a) = 0|0.$$

Since a is minimal, we get $((a_1|(a_1|(a|a)))|(a_1|(a_1|(a|a)))) = a$. By (S2), we obtain (*ii*).

(*ii*) ⇒ (*iii*): Obvious.

(*iii*) ⇒ (*iv*): From (sBCH.3), we get

$$\begin{aligned} a|(a_1|a_1) &= ((0|(0|(a|a)))|(0|(0|(a|a))))|(a_1|a_1) \\ &= ((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a|a)). \end{aligned}$$

(*iv*) ⇒ (*v*): Applying Lemma 3.1 (8), we obtain

$$(0|(a_1|(a|a))) = ((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a|a)) = a|(a_1|a_1).$$

(*v*) ⇒ (*i*): Let $a_1 \leqslant a$. Then $(a_1|(a|a))|(a_1|(a|a)) = 0$. Then

$$a|(a_1|a_1) = 0|(a_1|(a|a)) = 0|(0|0) = 0|0.$$

By (S2), $(a|(a_1|a_1))|(a|(a_1|a_1)) = 0$. Then $a \leqslant a_1$. Consequently, $a = a_1$. □

Proposition 3.4. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra and $a \in A$. Then a is minimal element if and only if there is an element $a_1 \in A$ such that $a = (0|(a_1|a_1))|(0|(a_1|a_1))$.*

Proof. Let a be a minimal element of A. By Proposition 3.3 and (S2), $(0|(0|(a|a)))|(0|(0|(a|a))) = a$. If we choose $a_1 = (0|(a|a))|(0|(a|a))$, then $a = (0|(a_1|a_1))|(0|(a_1|a_1))$.

Conversely, suppose that $a = (0|(a_1|a_1))|(0|(a_1|a_1))$ for some $a_1 \in A$. By using Lemma 3.1 (6) and (9), we get $((0|(0|(a|a)))|(0|(0|(a|a))))|(a|a) = 0|0$. That is,

$$((0|(0|(a|a)))|(0|(0|(a|a)))) \leqslant a.$$

Then $(0|(0|(0|(a|a)))) = 0|(a|a)$. Thus,

$$\begin{aligned} ((0|(0|(a|a)))|(0|(0|(a|a)))) &= (0|(0|(0|(a_1|a_1))))|(0|(0|(0|(a_1|a_1)))) \\ &= (0|(a_1|a_1))|(0|(a_1|a_1)) \\ &= a. \end{aligned}$$

From Proposition 3.3, we have a is minimal. □

EXAMPLE 3.8. Let $(A,|,0)$ be the Sheffer stroke BCH-algebra in Example 3.1. Then 0 is the minimal element of A. There is an element $x \in A$ such that $0 = (0|(x|x))|(0|(x|x))$. Conversely, there is an element $x \in A$ such that $0 = (0|(x|x))|(0|(x|x))$ for $x \in A$. By using Lemma 3.1 (6) and (9), we have $(0|(0|(0|(0|0)))) = 0|(0|0)$.

$$\begin{aligned} ((0|(0|(0|0)))|(0|(0|(0|0)))) &= (0|(0|(0|(x|x))))|(0|(0|(0|(x|x)))) \\ &= (0|(x|x))|(0|(x|x)) \\ &= 0. \end{aligned}$$

Then 0 is the minimal element of A from Proposition 3.3.

From $a \in A$, $\overline{a_1} = (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1)))$.

PROPOSITION 3.5. *Let $(A,|,0)$ be a Sheffer stroke BCH-algebra. Then the following features hold for all $a_1, a_2 \in A$:*

(i) $\overline{(a_1|(a_2|a_2))|(a_1|(a_2|a_2))} = (\overline{a_1}|(\overline{a_2}|\overline{a_2}))|(\overline{a_1}|(\overline{a_2}|\overline{a_2}))$,

(ii) $\overline{\overline{a_1}} = \overline{a_1}$.

PROOF. *(i)* $\overline{(a_1|(a_2|a_2))|(a_1|(a_2|a_2))} = (0|(0|(a_1|(a_2|a_2))))|(0|(0|(a_1|(a_2|a_2))))$
$= (0|(((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_2|a_2))))|(0|(((0|(a_1|a_1))|(0|(a_1|a_1)))|$
$(0|(a_2|a_2))))$
$= (((0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))))|(0|(0|(a_2|a_2))))|(((0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))))|$
$(0|(0|(a_2|a_2))))$
$= (\overline{a_1}|(\overline{a_2}|\overline{a_2}))|(\overline{a_1}|(\overline{a_2}|\overline{a_2}))$.

(ii) By using Lemma 3.1 (6) and (9), we get

$$((0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))))|(a_1|a_1) = 0|0.$$

That is,

$$((0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1)))) \leqslant a_1.$$

Then $(0|(0|(0|(a_1|a_1)))) = 0|(a_1|a_1)$. That is, $(0|(\overline{a_1}|\overline{a_1})) = (0|(a_1|a_1))$. Hence

$$\begin{aligned} (\overline{\overline{a_1}}|\overline{\overline{a_1}}) &= (0|(0|(\overline{a_1}|\overline{a_1})) \\ &= (0|(0|(a_1|a_1))) \\ &= (\overline{a_1}|\overline{a_1}). \end{aligned}$$

Therefore, $\overline{\overline{a_1}} = \overline{a_1}$. □

The set $\{a \in A : a = \overline{a}\}$ is called the centre of A, denoted by $CentA$. By Proposition 3.3, $CentA$ is the set of all minimal elements of A. We get

$$CentA = \{\overline{a} = a \in A\}.$$

Define $\varphi : A \to CentA$ by $\varphi(a) = \overline{a}$ for all $a \in A$. By Proposition 3.5, φ is a homomorphism from A onto $CentA$.

PROPOSITION 3.6. *Let A be a Sheffer stroke BCH-algebra. Then $CentA$ is a subalgebra of A.*

PROPOSITION 3.7. *Let A be a Sheffer stroke BCH-algebra and $a_1, a_2 \in CentA$. Then $a_1|(a_3|(a_2|a_2)) = a_2|(a_3|(a_1|a_1))$, for every $a_3 \in A$.*

PROOF. Let $a_3 \in A$. By using Proposition 3.3 and (sBCH.3), we obtain

$$\begin{aligned} a_1|(a_3|(a_2|a_2)) &= (((a_3|(a_3|(a_1|a_1)))|(a_3|(a_3|(a_1|a_1))))|(a_3|(a_2|a_2)) \\ &= (((a_3|(a_3|(a_2|a_2)))|(a_3|(a_3|(a_2|a_2))))|(a_3|(a_1|a_1)) \\ &= a_2|(a_3|(a_1|a_1)). \end{aligned}$$

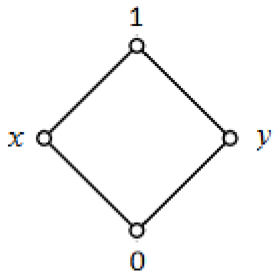
□

DEFINITION 3.6. Let a_0 be a minimal element of a Sheffer stroke BCH-algebra. Then the set

$$B(a_0) = \{a_1 : a_1 \in A \text{ and } a_0 \leqslant a_1\}$$

is called the branch of A, determined by a_0.

EXAMPLE 3.9. Let $(A, |, 0)$ be the Sheffer stroke BCH-algebra in Example 3.1.



Then $B(0) = \{x, y, 1\}$.

THEOREM 3.8. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then for each $a_1 \in A$, there is unique $a_0 \in Min(A)$ such that $a_0 \leqslant a_1$. That is, a_1 belongs to the unique branch $B(a_0)$, determined by a_0.*

PROOF. Let $a_1 \in A$. Then $(0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))) \leqslant a_1$. We take $a_0 = (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1)))$. So $a_0 \leqslant a_1$. By Theorem 3.7, $a_0 = (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))) \in Min(A)$. Then there is a $a_0 \in Min(A)$ such that $a_1 \in B(a_0)$.

Let $b_0 \in Min(A)$ such that $a_1 \in B(b_0)$. Thus $b_0 \leqslant a_1$, so $((b_0|(a_1|a_1))|(b_0|(a_1|a_1))) = 0$, which gives

$$((b_0|(a_1|a_1))|(b_0|(a_1|a_1)))|(b_0|b_0) = 0|(b_0|b_0).$$

That is, $0|(a_1|a_1) = 0|(b_0|b_0)$. Thus $(0|(0|(a_1|a_1))) = (0|(0|(b_0|b_0)))$. So

$$a_0 = (0|(0|(b_0|b_0)))|(0|(0|(b_0|b_0))) \leqslant b_0.$$

Since $b_0 \in Min(A)$, so $a_0 = b_0$. Hence for $a_1 \in A$, there is a unique $a_0 = (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1)))$ such that $a_1 \in B(a_0)$. □

REMARK 3.5. From above theorem, we conclude that in a Sheffer stroke BCH-algebra A, for any $a_1 \in A$, there exists a unique $a_0 = (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))) \in Min(A) = Med(A)$ such that $a_0 \leqslant a_1$. That is, every $a_1 \in A$ belongs to a unique branch $B(a_0)$, where

$$a_0 = (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))) \in Min(A) = Med(A).$$

THEOREM 3.9. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra and $a_0, b_0 \in Min(A)$. Then $a_0 = b_0$ if and only if $B(a_0) = B(b_0)$.*

PROOF. Let $a_0 = b_0$ and $B(a_0) \neq B(b_0)$. Suppose that there is an $a_1^{'} \in B(a_0)$ such that $a_1^{'} \notin B(b_0)$. Since $a_1^{'} \in B(a_0)$, $a_0 \leqslant a_1^{'}$. Thus $(a_0|(a_1^{'}|a_1^{'})) = 0|0$. Then, $((a_0|(a_1^{'}|a_1^{'}))|(a_0|(a_1^{'}|a_1^{'})))|(a_0|a_0) = 0|(a_0|a_0)$. Thus $0|(a_1^{'}|a_1^{'}) = 0|(a_0|a_0)$ that is, $(0|(0|(a_1^{'}|a_1^{'}))) = (0|(0|(a_0|a_0)))$. Since $a_0 \in Min(A) = Med(A)$, so $a_0 = (0|(0|(a_0|a_0)))|(0|(0|(a_0|a_0)))$. Then $a_0 = (0|(0|(a_1^{'}|a_1^{'})))|(0|(0|(a_1^{'}|a_1^{'})))$. Since $a_0 = b_0$, so $b_0 = (0|(0|(a_1^{'}|a_1^{'})))|(0|(0|(a_1^{'}|a_1^{'}))) \leqslant a_1^{'}$. Hence, $a_1^{'} \in B(b_0)$, a contradiction. Therefore, $B(a_0) = B(b_0)$.

Conversely, let $B(a_0) = B(b_0)$. Since $a_0 \in B(a_0)$, so $a_0 \in B(b_0)$. Then $b_0 \leqslant a_0$. Similarly, $b_0 \in B(b_0) = B(a_0)$ gives $a_0 \leqslant b_0$. Therefore, $a_0 = b_0$. □

THEOREM 3.10. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Let $a_0 \neq b_0$ and $a_0, b_0 \in Min(A)$. Then $B(a_0) \cap B(b_0) = \varnothing$.*

PROOF. Let $a_0 \neq b_0$ and $B(a_0) \cap B(b_0) \neq \varnothing$. Let $a_1 \in B(a_0) \cap B(b_0)$. Then $a_1 \in B(a_0)$ and $a_1 \in B(b_0)$. By Theorem 3.8, we obtain $B(a_0) = B(b_0)$. From Theorem 3.9, we get $a_0 = b_0$, a contradiction. Therefore, $B(a_0) \cap B(b_0) = \varnothing$. □

THEOREM 3.11. *Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then $Med(A)$ is a subalgebra of A.*

PROOF. Let $(A, |, 0)$ be a Sheffer stroke BCH-algebra. Then $(0|(0|(0|0))) = 0|0$, so $0 \in Med(A)$.
Let $a_1, a_2 \in Med(A)$. Then $(0|(0|(a_1|a_1))) = a_1|a_1$ and $(0|(0|(a_2|a_2))) = a_2|a_2$. From Lemma 3.1 (8) and (S2), we get

$$\begin{aligned}(0|(0|(a_1|(a_2|a_2)))) &= (0|((0|(a_1|a_1))|(0|(a_1|a_2)))|(0|(a_2|a_2))) \\ &= ((0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))))|(0|(0|(a_2|a_2))) \\ &= (a_1|(a_2|a_2)).\end{aligned}$$

So, $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in Med(A)$ by (S2). Therefore, $Med(A)$ is a subalgebra of A. □

REMARK 3.6. If A is a Sheffer stroke BCH-algebra, then $Med(A) = Min(A)$. Hence we obtain $Min(A)$ is a subalgebra of A.

4. Ideals in Sheffer stroke BCH-Algebras

DEFINITION 4.1. A non-empty subset I of a Sheffer stroke BCH-algebra A is called an ideal of A if it satisfies

$(I1)$ $0 \in I$,

$(I2)$ $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in I$ and $a_2 \in I$ imply $a_1 \in I$.

We will denote by $Id(A)$ the set of all ideals of A.

EXAMPLE 4.1. In Example 3.1, $\{0\}, \{0, x\}, \{0, y\}$ are ideals of A.

PROPOSITION 4.1. *Let A be a Sheffer strkoke BCH-algebra and $I \in Id(A)$. For any $a_1, a_2 \in A$, if $a_2 \in I$ and $a_1 \leqslant a_2$, then $a_1 \in I$.*

Proposition 4.2. *Let $\varphi : A \to B$ be a surjective homomorphism. If I is an ideal of a containing $\varphi^{-1}(0)$, then $\varphi(I)$ is an ideal of B.*

Proof. Since $0 \in I$, $0 = \varphi(0) \in \varphi(I)$. Let $a_1, a_2 \in B$ and

$$(a_1|(a_2|a_2))|(a_1|(a_2|a_2)), a_2 \in \varphi(I).$$

Then there are $x \in A$ and $y, z \in I$ such that

$$a_1 = \varphi(x),\ a_2 = \varphi(y) \text{ and } (a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = \varphi(z).$$

We have $\varphi((x|(y|y))|(x|(y|y))) = \varphi(z)$. Then

$$(((x|(y|y))|(x|(y|y)))|(z|z))|(((x|(y|y))|(x|(y|y)))|(z|z)) \in \varphi^{-1}(0) \subseteq I.$$

By the definition of an ideal, $x \in I$. Consequently, $a_1 = \varphi(x) \in \varphi(I)$. Therefore, $\varphi(I)$ is an ideal of A. □

Definition 4.2. An ideal I of a Sheffer stroke BCH-algebra A is said to be closed if $(0|(a|a))|(0|(a|a)) \in I$, for every $a \in I$.

Theorem 4.1. *An ideal I of a Sheffer stroke BCH-algebra A is closed if and only if I is a subalgebra of A.*

Proof. Let I be a closed ideal of A and $a_1, a_2 \in I$. By using (sBCH.3), we obtain

$((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(a_2|a_2)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|$
$(0|(a_2|a_2))))|(a_1|a_1)$
$= ((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|a_1))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_1|a_1)))|$
$(0|(a_2|a_2))$
$= ((((a_1|(a_1|a_1))|(a_1|(a_1|a_1)))|(a_2|a_2))|(((a_1|(a_1|a_1))|(a_1|(a_1|a_1)))|(a_2|a_2)))|$
$(0|(a_2|a_2))$
$= ((0|(a_2|a_2))|(0|(a_2|a_2)))|(0|(a_2|a_2))$
$= 0|0.$

Thus
$(((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(a_2|a_2)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(a_2|a_2))))$
$|(a_1|a_1))|(((((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(a_2|a_2)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|$
$(0|(a_2|a_2))))|(a_1|a_1)) \in I$. Since $a_1, (0|(a_2|a_2))|(0|(a_2|a_2)) \in I$, we get
$((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \in I$.

Conversely, if I is a subalgebra of A, then $a_1 \in I$ and $0 \in I$ imply $(0|(a_1|a_1))|(0|(a_1|a_1)) \in I$. Therefore, I is a closed ideal of A. □

5. Conclusion

In this study, we have given a Sheffer stroke BCH-algebra, and study a minimal element, a medial element, a subalgebra, an ideal and many features in Sheffer stroke BCH-algebras. After giving basic definitions and notions about Sheffer stroke and a BCH-algebra, we describe a Sheffer stroke BCH-algebra and present basic notions about this algebraic structure. We show that a Sheffer stroke BCH-algebra is a BCH-algebra and a BCH-algebra is a Sheffer stroke BCH-algebra. After defining a subalgebra and a BCA-part, we present the relationship between this structures on Sheffer stroke BCH-algebra. Besides, by defining a minimal element and a medial element on Sheffer stroke BCH-algebra, it is proved that the minimal element and medial element are equivalent. The centre and branch of a Sheffer stroke BCH-algebra are described and it is shown that a centre of a Sheffer stroke BCH-algebra is a subalgebra. It is indicated that $Med(A)$ is a subalgebra of a Sheffer stroke BCH-algebra. Finally, an ideal and a closed ideal are defined and it is shown that an ideal is a closed ideal if and only if it is a subalgebra.

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Department of Mathematics, Ege University, Izmir, Turkey
Email address: tahsin.oner@ege.edu.tr

Department of Mathematics, Ege University, Izmir, Turkey
Email address: tugcekalkan92@gmail.com

Department of Mathematics, Ege University, Izmir, Turkey
Email address: melisacinar1996@gmail.com