JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
J. Int. Math. Virtual Inst., Vol. **11**(1)(2021), 101–117
DOI: 10.7251/JIMVI2101101S

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SOME FIXED POINT THEOREMS ON PARTIAL METRIC SPACES SATISFYING AN IMPLICIT CONTRACTIVE CONDITION WITH APPLICATIONS

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Abstract. In this paper, we establish a unique fixed point, a unique common fixed point and a coincidence point theorems satisfying an implicit contractive condition on partial metric spaces. The results presented in this paper extend, generalize and unify several results from the existing literature. We also present one of the possible applications of our result to well-posed and limit shadowing property of fixed point problems.

1. Introduction

The Banach contraction mappings principle is one of the most useful theorems in nonlinear analysis. Many authors generalized this famous result in different ways. Subsequently, several authors have concentrated on expanding and improving this theory (see, e.g., [**12, 18, 27, 37**] and many others).

The notion of partial metric space was originally developed by Matthews ([**24, 25**]) to provide a mechanism generalizing metric space theories. A partial metric space is a extension of metric by replacing the condition $d(x, x) = 0$ of the (usual) metric with the inequality $d(x, x) \leqslant d(x, y)$ for all x, y. Also, this concept provide the basis to study denotational semantics of dataflow networks [**24, 25, 40, 43**]. In partial metric spaces the distance of a point in the self may not be zero. Introducing partial metric space, Matthews extended the Banach contraction principle [**9**] and proved the fixed point theorem in this space.

2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. Fixed point, common fixed point, coincidence point, implicit contractive condition, partial metric space.
Communicated by Daniel A. Romano.

Matthews gave some basic definitions and properties on partial metric space such as Cauchy sequence, convergent sequence etc. Due to importance of the fixed point theory it is very interesting to study fixed point theorems on different concepts.

Many authors studied the fixed points for mappings satisfying contractive conditions in complete partial metric spaces. More recently, in [**1**], [**5**], [**6**], [**8**], [**13**], [**14**], [**17**], [**20**], [**41**] some fixed point theorems under various contractive conditions in complete partial metric spaces are proved.

On the other hand, Popa [**30**] and [**31**] considered an implicit contraction type condition instead of the usual explicit condition. This direction of research produced a consistent literature on fixed point, common fixed point and coincidence point theorems in various ambient spaces. For more details see [**4, 10, 11, 16, 32, 35**].

In 2013, Vetro and Vetro [**42**] initiated the study of fixed points of self mappings in partial metric spaces satisfying an implicit relation. In [**7**], Altun and Turkoglu launched a new type of implicit relation satisfying ϕ-map.

Very recently, Popa and Patriciu [**36**] have studied a new type of ϕ-implicit relation and established a unique point of coincidence and unique common fixed point results and also as application of results they obtained fixed point theorem for a sequence of mappings in partial metric spaces.

The purpose of this paper is to study Altun and Turkoglu [**7**] type implicit relation and establish a unique fixed point, a unique common fixed point and a coincidence point theorems in partial metric spaces. Our results extend, generalize and unify several results from the existing literature.

2. Preliminaries

Now, we give some basic properties and results on the concept of partial metric space (PMS).

Definition 2.1. ([**25**]) Let X be a nonempty set and $p\colon X \times X \to \mathbb{R}^+$ be such that for all $x, y, z \in X$ the followings are satisfied:

$(P1)$ $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

$(P2)$ $p(x,x) \leqslant p(x,y)$,

$(P3)$ $p(x,y) = p(y,x)$,

$(P4)$ $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

Then p is called partial metric on X and the pair (X,p) is called partial metric space.

Remark 2.1. It is clear that if $p(x,x) = 0$, then $x = y$. But, on the contrary $p(x,x)$ need not be zero.

Example 2.1. ([**8**]) Let $X = \mathbb{R}^+$ and $p\colon X \times X \to \mathbb{R}^+$ given by $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.

Example 2.2. ([**8**]) Let $X = \{[a,b] : a,b \in \mathbb{R}, a \leqslant b\}$. Then $p\Big([a,b],[c,d]\Big) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

Various applications of this space has been extensively investigated by many authors (see [**21**], [**41**] for details).

REMARK 2.2. ([**19**]) Let (X, p) be a partial metric space.

$(a1)$ The function $d^M : X \times X \to \mathbb{R}^+$ defined as $d^M(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a (usual) metric on X and (X, d^M) is a (usual) metric space.

$(a2)$ The function $d^S : X \times X \to \mathbb{R}^+$ defined as $d^S(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$ is a (usual) metric on X and (X, d^S) is a (usual) metric space.

Note also that each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \leqslant p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [**24**].

DEFINITION 2.2. ([**24**]) Let (X, p) be a partial metric space. Then:

$(b1)$ a sequence $\{x_n\}$ in (X, p) is said to be convergent to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$,

$(b2)$ a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite,

$(b3)$ (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to τ_p. Furthermore,

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

$(b4)$ A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f\Big(B_p(x_0, \delta)\Big) \subset B_p\Big(f(x_0), \varepsilon\Big)$.

DEFINITION 2.3. ([**26**]) Let (X, p) be a partial metric space. Then:

$(c1)$ a sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{m,n\to\infty} p(x_m, x_n) = 0$,

$(c2)$ (X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, such that $p(x, x) = 0$.

DEFINITION 2.4. A point x in X is called a coincidence point of f and T if $f(x) = T(x)$ for each $x \in X$.

LEMMA 2.1 ([**24, 25**]). *Let (X, p) be a partial metric space. Then:*

(d1) a sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d^M),

(d2) (X, p) is complete if and only if the metric space (X, d^M) is complete,

(d3) a subset E of a partial metric space (X, p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

LEMMA 2.2 ([**1**]). *Assume that $x_n \to u$ as $n \to \infty$ in a partial metric space (X, p) such that $p(u, u) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(u, y)$ for every $y \in X$.*

3. Implicit relation

Now, an implicit relation has been introduced to investigate a unique fixed point, a unique common fixed point and a coincidence point theorems in partial metric spaces.

Definition 3.1. A function $\psi\colon [0,\infty) \to [0,\infty)$ is a ϕ-function, $\psi \in \phi$, if ψ is nondecreasing function such that $\Sigma_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$ and $\psi(0) = 0$.

Remark 3.1. Since $\Sigma_{n=1}^{\infty} \psi^n(t) < +\infty$, then $\lim_{n\to\infty} \psi^n(t) = 0$. Then as in [**23**], $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$.

Definition 3.2. Let $\mathcal{F}_\phi$ be the set of all continuous functions

$$F(t_1,\dots,t_5)\colon \mathbb{R}_+^5 \longrightarrow \mathbb{R}$$

such that:

(F_1) : F is nonincreasing in variables $t_2,\dots,t_5$,

(F_2) : There exists a function $\psi \in \phi$ such that

(F_{2a}) : $F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) \leqslant 0$,

(F_{2b}) : $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) \leqslant 0$,

implies $u \leqslant \psi(v)$.

The proof of property (F_1) is easy, in the following examples. We shall only verify the property (F_2).

Example 3.1. Let

$$F(t_1,\dots,t_5) = t_1 - h\,\max\{t_2,\dots,t_5\},$$

where $h \in [0, \frac{1}{2})$.

(F_2) : Let $u, v \geqslant 0$ and

$$F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) = u - h\,(u+v) \leqslant 0,$$

which implies $u \leqslant \Big(\frac{h}{1-h}\Big)v$ and (F_{2a}) is satisfied for $\psi(t) = \Big(\frac{h}{1-h}\Big)t$.

Similarly, $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) = u - h\,(u+v) \leqslant 0$ which implies $u \leqslant \Big(\frac{h}{1-h}\Big)v$ and (F_{2b}) is satisfied for $\psi(t) = \Big(\frac{h}{1-h}\Big)t$.

Example 3.2. Let

$$F(t_1,\dots,t_5) = t_1 - k\,\max\{t_2, t_3, t_4, \frac{t_3+2t_4}{3}, \frac{t_4+2t_5}{3}\},$$

where $k \in [0, \frac{1}{2})$.

(F_2) : Let $u, v \geqslant 0$ and

$$F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) = u - k\,\max\{v, u+v, v, \frac{u+3v}{3}, \frac{u+2v}{3}\} \leqslant 0,$$

which implies $u \leqslant \left(\frac{k}{1-k}\right)v$ and (F_{2a}) is satisfied for $\psi(t) = \left(\frac{k}{1-k}\right)t$.

Similarly, $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) \leqslant 0$ which implies $u \leqslant \psi(v)$.

Example 3.3. Let

$$F(t_1, \dots, t_5) = t_1 - \max\{at_2, b(t_3 + 2t_4), b(t_4 + 2t_5)\},$$

where $a \in (0,1)$ and $b \in (0, \frac{1}{4})$.

(F_2) : Let $u, v \geqslant 0$ and

$$F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) = u - \max\{av, b(u+3v), b(u+2v)\}.$$

If $u > v$, then $u\Big(1 - \max\{a, 2b\}\Big) \leqslant 0$, a contradiction. Hence $u \leqslant v$, which implies $u \leqslant \max\{a, 2b\}v$ and (F_{2a}) is satisfied for $\psi(t) = \max\{a, 2b\}t$.

Similarly, $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) \leqslant 0$ which implies $u \leqslant \psi(v)$.

Example 3.4. Let

$$F(t_1, \dots, t_5) = t_1 - k\,\max\{t_2, t_3 + t_4, 2t_5\},$$

where $k \in (0, \frac{1}{3})$.

(F_2) : Let $u, v \geqslant 0$ and

$$F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) = u - k\,\max\{v, u+2v, u+v\} \leqslant 0,$$

which implies $u \leqslant \left(\frac{2k}{1-k}\right)v$ and (F_{2a}) is satisfied for $\psi(t) = \left(\frac{2k}{1-k}\right)t$.

Similarly, $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) \leqslant 0$ which implies $u \leqslant \psi(v)$.

Example 3.5. Let

$$F(t_1, \dots, t_5) = t_1^2 - a\,\max\{t_2^2, t_3^2, t_4^2\} - 2bt_4t_5,$$

where $a, b \geqslant 0$ with $4a + 2b < 1$.

(F_2) : Let $u, v \geqslant 0$ and $F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) = u^2 - a\,\max\{v^2, (u+v)^2, v^2\} - bv(u+v)$.If $u > v$, then $u^2\Big(1-(4a+2b)\Big) \leqslant 0$, a contradiction. Hence $u \leqslant v$, which implies $u \leqslant \sqrt{(4a+2b)}v$ and (F_{2a}) is satisfied for $\psi(t) = \sqrt{(4a+2b)}t$.

Similarly, $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) \leqslant 0$ which implies $u \leqslant \psi(v)$.

Example 3.6. Let

$$F(t_1, \dots, t_5) = t_1^3 - a\,t_1^2t_2 - b\,t_1t_2^2 - c\,t_2t_3t_4 - 2d\,t_1t_4t_5,$$

where $a, b, c, d \geqslant 0$ with $a + b + 2c + 2d < 1$.

(F_2) : Let $u, v \geqslant 0$ and

$$F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) = u^3 - a\,u^2v - b\,uv^2 - c\,v^2(u+v) - d\,uv(u+v) \leqslant 0.$$

If $u > v$, then $u^3\Big(1-(a+b+2c+2d)\Big) \leqslant 0$, a contradiction. Hence $u \leqslant v$, which implies $u \leqslant \sqrt[3]{(a+b+2c+2d)}v$ and (F_{2a}) is satisfied for $\psi(t) = \sqrt[3]{(a+b+2c+2d)}t$.

Similarly, $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) \leqslant 0$ which implies $u \leqslant \psi(v)$.

EXAMPLE 3.7. Let $F(t_1, \dots, t_5) = t_1 - \psi\big(at_2 + bt_3 + ct_4 + 2dt_5\big)$, where $a, b, c, d \geqslant 0$ with $a + 2b + c + 2d < 1$.

(F_2) : Let $u, v \geqslant 0$ and

$$F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) = u - \psi\big(av + b(u+v) + cv + d(u+v)\big) \leqslant 0.$$

If $u > v$, then $u - \psi\big((a+2b+c+2d)u\big) \leqslant 0$, which implies

$$u \leqslant \psi\Big((a+2b+c+2d)u\Big) \leqslant \psi(u) < u,$$

a contradiction. Hence $u \leqslant v$, which implies $u \leqslant \psi(v)$.

Similarly, $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) \leqslant 0$ which implies $u \leqslant \psi(v)$.

EXAMPLE 3.8. Let

$$F(t_1, \dots, t_5) = t_1 - \psi\big(at_2 + bt_3 + c\max\{t_4, 2t_5\}\big),$$

where $a, b, c \geqslant 0$ with $a + 2b + 2c < 1$.

(F_2) : Let $u, v \geqslant 0$ and

$$F\Big(u, v, u+v, v, \frac{1}{2}(u+v)\Big) = u - \psi\big(av + b(u+v) + c\max\{v, (u+v)\}\big) \leqslant 0.$$

If $u > v$, then $u - \psi\big((a+2b+2c)u\big) \leqslant 0$, which implies

$$u \leqslant \psi\big((a+2b+2c)u\big) \leqslant \psi(u) < u,$$

a contradiction. Hence $u \leqslant v$, which implies $u \leqslant \psi(v)$.

Similarly, $F\Big(u, v, v, u+v, \frac{1}{2}(u+v)\Big) \leqslant 0$ which implies $u \leqslant \psi(v)$.

The purpose of this paper is to study ψ-implicit contractive condition on partial metric space and establish a unique fixed point, a unique common fixed point and a coincidence point theorems in the said space. The results of findings extend and generalize several results from the existing literature.

4. Main Results

In this section, we shall prove a unique fixed point, a unique common fixed point and a coincidence point theorems for implicit contractive condition defined in definition 3.2 in the framework of partial metric spaces.

THEOREM 4.1. *Let (X, p) be a complete partial metric space and $\mathcal{T}\colon X \to X$ be a mapping satisfying the condition:*

$$\begin{aligned} &F\Big(p(\mathcal{T}x, \mathcal{T}y), p(x, y), p(x, \mathcal{T}y), p(y, \mathcal{T}x), \\ &\qquad \frac{1}{2}\big[p(x, \mathcal{T}x) + p(y, \mathcal{T}y)\big]\Big) \leqslant 0, \end{aligned} \tag{4.1}$$

for all $x, y \in X$, where $F \in \mathcal{F}_\phi$. Then $\mathcal{T}$ has a unique fixed point.

Proof. Let $x_0 \in X$. We construct the iterative sequence $\{x_n\}$ which is defined as $x_n = \mathcal{T}x_{n-1}$ for $n = 1, 2, 3, \dots$, then $x_n = \mathcal{T}^n x_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of $\mathcal{T}$. So, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From (4.1) for $x = x_{n-1}$ and $y = x_n$ we have successively

$$F\Big(p(\mathcal{T}x_{n-1}, \mathcal{T}x_n), p(x_{n-1}, x_n), p(x_{n-1}, \mathcal{T}x_n), p(x_n, \mathcal{T}x_{n-1}), \frac{1}{2}\big[p(x_{n-1}, \mathcal{T}x_{n-1}) + p(x_n, \mathcal{T}x_n)\big]\Big) \leqslant 0.$$

$$F\Big(p(x_n, x_{n+1}), p(x_{n-1}, x_n), p(x_{n-1}, x_{n+1}), p(x_n, x_n), \frac{1}{2}\big[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\big]\Big) \leqslant 0. \tag{4.2}$$

Since by $(P4)$,

$$\begin{aligned} p(x_{n-1}, x_{n+1}) &\leqslant p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n) \\ &\leqslant p(x_{n-1}, x_n) + p(x_n, x_{n+1}), \end{aligned}$$

and by $(P2)$,

$$p(x_n, x_n) \leqslant p(x_{n-1}, x_n).$$

By (4.2) and (F_1), we obtain

$$F\Big(p(x_n, x_{n+1}), p(x_{n-1}, x_n), p(x_{n-1}, x_n) + p(x_n, x_{n+1}), p(x_{n-1}, x_n), \frac{1}{2}\big[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\big]\Big) \leqslant 0. \tag{4.3}$$

By (F_{2a}), we obtain

$$p(x_n, x_{n+1}) \leqslant \psi\big(p(x_{n-1}, x_n)\big). \tag{4.4}$$

By (4.1) for $x = x_n$ and $y = x_{n+1}$, we obtain

$$F\Big(p(\mathcal{T}x_n, \mathcal{T}x_{n+1}), p(x_n, x_{n+1}), p(x_n, \mathcal{T}x_{n+1}), p(x_{n+1}, \mathcal{T}x_n), \frac{1}{2}\big[p(x_n, \mathcal{T}x_n) + p(x_{n+1}, \mathcal{T}x_{n+1})\big]\Big) \leqslant 0.$$

$$F\Big(p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), p(x_n, x_{n+2}), p(x_{n+1}, x_{n+1}), \frac{1}{2}\big[p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})\big]\Big) \leqslant 0. \tag{4.5}$$

Since by $(P4)$,

$$\begin{aligned} p(x_n, x_{n+2}) &\leqslant p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) \\ &\leqslant p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}), \end{aligned}$$

and by $(P2)$,

$$p(x_{n+1}, x_{n+1}) \leqslant p(x_n, x_{n+1}).$$

By (4.5) and (F_1), we obtain

$$F\Big(p(x_{n+1},x_{n+2}),p(x_n,x_{n+1}),p(x_n,x_{n+1})+p(x_{n+1},x_{n+2}),$$
$$p(x_n,x_{n+1}),\frac{1}{2}\big[p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})\big]\Big)\leqslant 0. \tag{4.6}$$

By (F_{2a}), we obtain

$$p(x_{n+1},x_{n+2})\leqslant \psi\big(p(x_n,x_{n+1})\big),$$

which implies

$$p(x_n,x_{n+1})\leqslant \psi\big(p(x_{n-1},x_n)\big)\leqslant \psi^2\big(p(x_{n-2},x_{n-1})\big)\leqslant \dots\leqslant \psi^n\big(p(x_0,x_1)\big).$$

For $n,m\in\mathbb{N}$ with $m>n$, by repeated use of $(P4)$, we have that

$$\begin{aligned}
p(x_n,x_m) &\leqslant p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})+\cdots+p(x_{m-1},x_m)\\
&\quad -p(x_{n+1},x_{n+1})-p(x_{n+2},x_{n+2})-\cdots-p(x_{m-1},x_{m-1})\\
&\leqslant p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})+\cdots+p(x_{m-1},x_m)\\
&= \sum_{k=n}^{m-1}\psi^k\big(p(x_0,x_1)\big).
\end{aligned}$$

Since $\sum_{k=0}^{\infty}\psi^k\big(p(x_0,x_1)\big)<\infty$, then

$$\lim_{n\to\infty}\sum_{k=n}^{m-1}\psi^k\big(p(x_0,x_1)\big)=0 \text{ and } \lim_{n,m\to\infty}p(x_n,x_m)=0$$

and so

$$d^M(x_n,x_m)=2p(x_n,x_m)\to 0 \text{ as } n,m\to\infty. \tag{4.7}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Thus by Lemma 2.1 this sequence will also Cauchy in (X,d^M). In addition, since (X,p) is complete, (X,d^M) is also complete. Thus there exists $u\in X$ such that $x_n\to u$ as $n\to\infty$. Moreover by Lemma 2.2,

$$p(u,u)=\lim_{n\to\infty}p(u,x_n)=\lim_{n,m\to\infty}p(x_n,x_m)=0, \tag{4.8}$$

implies

$$\lim_{n\to\infty}d^M(u,x_n)=0. \tag{4.9}$$

Now, we show that x is a fixed point of $\mathcal{T}$. Notice that due to (4.8), we have $p(u,u)=0$. By (4.1) with $x=u$ and $y=x_n$, we have

$$F\Big(p(\mathcal{T}u,\mathcal{T}x_n),p(u,x_n),p(u,\mathcal{T}x_n),p(x_n,\mathcal{T}u),$$
$$\frac{1}{2}\big[p(u,\mathcal{T}u)+p(x_n,\mathcal{T}x_n)\big]\Big)\leqslant 0.$$

$$F\Big(p(\mathcal{T}u,x_{n+1}),p(u,x_n),p(u,x_{n+1}),p(x_n,\mathcal{T}u),$$
$$\frac{1}{2}\big[p(u,\mathcal{T}u)+p(x_n,x_{n+1})\big]\Big)\leqslant 0. \tag{4.10}$$

Letting $n \to \infty$ in (4.10), we obtain by Lemma 2.2 and using $(P3)$ that

$$F\Big(p(\mathcal{T}u,u),0,0,p(\mathcal{T}u,u),\frac{1}{2}p(\mathcal{T}u,u)\Big) \leqslant 0,$$

which implies by (F_{2b}) that $p(\mathcal{T}u,u) \leqslant \psi(0) = 0$, that is, $\mathcal{T}u = u$. This shows that u is a fixed point of $\mathcal{T}$.

Now we show that the fixed point of $\mathcal{T}$ is unique. Assume that v is another fixed point of $\mathcal{T}$ such that $v = \mathcal{T}v$ with $v \neq u$. Then form (4.1), (4.8) and using $(P3)$, we have

$$F\Big(p(\mathcal{T}u,\mathcal{T}v),p(u,v),p(u,\mathcal{T}v),p(v,\mathcal{T}u),\frac{1}{2}\big[p(u,\mathcal{T}u)+p(v,\mathcal{T}v)\big]\Big) \leqslant 0.$$

$$F\Big(p(\mathcal{T}u,\mathcal{T}v),p(u,v),p(u,\mathcal{T}v),p(v,\mathcal{T}u),\frac{1}{2}\big[p(u,\mathcal{T}u)+p(v,\mathcal{T}v)\big]\Big) \leqslant 0.$$

$$F\Big(p(u,v),p(u,v),p(u,v),p(v,u),\frac{1}{2}\big[p(u,u)+p(v,v)\big]\Big) \leqslant 0.$$

$$F\Big(p(u,v),p(u,v),p(u,v),p(u,v),0\Big) \leqslant 0.$$

By (F_1) and (F_{2a}), we obtain

$$p(u,v) \leqslant \psi\big(p(u,v)\big) < p(u,v),$$

if $p(u,v) \neq 0$, a contradiction. Hence $p(u,v) = 0$, which implies $u = v$. This shows that the fixed point of $\mathcal{T}$ is unique. This completes the proof. □

THEOREM 4.2. *Let $\mathcal{T}$ and f be two self-maps on a complete partial metric space (X,p) satisfying the condition:*

$$F\Big(p(\mathcal{T}x,\mathcal{T}y),p(fx,fy),p(fx,\mathcal{T}y),p(fy,\mathcal{T}x),\frac{1}{2}\big[p(fx,\mathcal{T}x)+p(fy,\mathcal{T}y)\big]\Big) \leqslant 0, \tag{4.11}$$

for all $x,y \in X$, where $F \in \mathcal{F}_\phi$. If the range of f contains the range of $\mathcal{T}$ and $f(X)$ is a complete subspace of X, then $\mathcal{T}$ and f have a coincidence fixed point.

PROOF. Let $x_0 \in X$ and choose a point x_1 in X such that

$$\mathcal{T}x_0 = fx_1, \ldots, \mathcal{T}x_n = fx_{n+1} = y_{n+1}.$$

Then from (4.11) for $x = x_{n-1}$ and $y = x_n$ we have successively

$$F\Big(p(\mathcal{T}x_{n-1},\mathcal{T}x_n),p(fx_{n-1},fx_n),p(fx_{n-1},\mathcal{T}x_n),p(fx_n,\mathcal{T}x_{n-1}),\frac{1}{2}\big[p(fx_{n-1},\mathcal{T}x_{n-1})+p(fx_n,\mathcal{T}x_n)\big]\Big) \leqslant 0.$$

$$F\Big(p(y_n, y_{n+1}), p(y_{n-1}, y_n), p(y_{n-1}, y_{n+1}), p(y_n, y_n),$$
$$\frac{1}{2}\big[p(y_{n-1}, y_n) + p(y_n, y_{n+1})\big]\Big) \leqslant 0. \tag{4.12}$$

Since by $(P4)$,

$$\begin{aligned} p(y_{n-1}, y_{n+1}) &\leqslant p(y_{n-1}, y_n) + p(y_n, y_{n+1}) - p(y_n, y_n) \\ &\leqslant p(y_{n-1}, y_n) + p(y_n, y_{n+1}), \end{aligned}$$

and by $(P2)$,

$$p(y_n, y_n) \leqslant p(y_{n-1}, y_n).$$

By (4.12) and (F_1), we obtain

$$F\Big(p(y_n, y_{n+1}), p(y_{n-1}, y_n), p(y_{n-1}, y_n) + p(y_n, y_{n+1}),$$
$$p(y_{n-1}, y_n), \frac{1}{2}\big[p(y_{n-1}, y_n) + p(y_n, y_{n+1})\big]\Big) \leqslant 0. \tag{4.13}$$

By (F_{2a}), we obtain

$$p(y_n, y_{n+1}) \leqslant \psi\big(p(y_{n-1}, y_n)\big),$$

which implies

$$p(y_n, y_{n+1}) \leqslant \psi\big(p(y_{n-1}, y_n)\big) \leqslant \psi^2\big(p(y_{n-2}, y_{n-1})\big) \leqslant \ldots \leqslant \psi^n\big(p(y_0, y_1)\big).$$

For $n, m \in \mathbb{N}$ with $m > n$, by repeated use of $(P4)$, we have that

$$\begin{aligned} p(y_n, y_m) &\leqslant p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) \\ &\quad - p(y_{n+1}, y_{n+1}) - p(y_{n+2}, y_{n+2}) - \cdots - p(y_{m-1}, y_{m-1}) \\ &\leqslant p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) \\ &= \sum_{j=n}^{m-1} \psi^j\big(p(y_0, y_1)\big). \end{aligned}$$

Since $\sum_{j=0}^{\infty} \psi^j\big(p(y_0, y_1)\big) < \infty$, then

$$\lim_{n\to\infty} \sum_{j=n}^{m-1} \psi^j\big(p(y_0, y_1)\big) = 0 \text{ and } \lim_{n,m\to\infty} p(y_n, y_m) = 0$$

and so

$$d^M(y_n, y_m) = 2p(y_n, y_m) \to 0 \text{ as } n, m \to \infty. \tag{4.14}$$

This implies that $\{y_n\} = \{fx_n\}$ is a Cauchy sequence in X. Thus by Lemma 2.1 this sequence will also Cauchy in (X, d^M). In addition, since (X, p) is complete, (X, d^M) is also complete. Thus there exists $v \in X$ such that $x_n \to v \Rightarrow fx_n \to fv$ as $n \to \infty$, since $f(X)$ is a complete subspace of X. Moreover by Lemma 2.2,

$$p(fv, fv) = \lim_{n\to\infty} p(fv, fx_n) = \lim_{n,m\to\infty} p(fx_n, fx_m) = 0, \tag{4.15}$$

implies

$$\lim_{n\to\infty} d^M(fv, fx_n) = 0. \tag{4.16}$$

Now, we show that v is a coincidence point of $\mathcal{T}$ and f. Notice that due to (4.15), we have $p(fv, fv) = 0$. By (4.11) with $x = u$ and $y = x_n$, we have

$$F\Big(p(\mathcal{T}v, \mathcal{T}x_n), p(fv, fx_n), p(fv, \mathcal{T}x_n), p(fx_n, \mathcal{T}v), \frac{1}{2}\big[p(fv, \mathcal{T}v) + p(fx_n, \mathcal{T}x_n)\big]\Big) \leqslant 0.$$

$$F\Big(p(\mathcal{T}v, fx_{n+1}), p(fv, fx_n), p(fv, fx_{n+1}), p(fx_n, \mathcal{T}v), \frac{1}{2}\big[p(fv, \mathcal{T}v) + p(fx_n, fx_{n+1})\big]\Big) \leqslant 0. \tag{4.17}$$

Letting $n \to \infty$ in (4.17) and using $(P3)$, we obtain by Lemma 2.2 that

$$F\Big(p(\mathcal{T}v, fv), 0, 0, p(\mathcal{T}v, fv), \frac{1}{2}p(\mathcal{T}v, fv)\Big) \leqslant 0, \tag{4.18}$$

which implies by (F_{2b}) that $p(\mathcal{T}v, fv) \leqslant \psi(0) = 0$, that is, $\mathcal{T}v = fv$. This shows that v is a coincidence point of $\mathcal{T}$ and f. This completes the proof. □

THEOREM 4.3. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two self-maps on a complete partial metric space (X, p) satisfying the condition:*

$$F\Big(p(\mathcal{T}_1x, \mathcal{T}_2y), p(x, y), p(x, \mathcal{T}_2y), p(y, \mathcal{T}_1x), \frac{1}{2}\big[p(x, \mathcal{T}_1x) + p(y, \mathcal{T}_2y)\big]\Big) \leqslant 0, \tag{4.19}$$

for all $x, y \in X$, where $F \in \mathcal{F}_\phi$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ have a unique common fixed point in X.

PROOF. For each $x_0 \in X$. Put $x_{2n+1} = \mathcal{T}_1x_{2n} = y_{2n}$ and $x_{2n+2} = \mathcal{T}_2x_{2n+1} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$. We prove that $\{y_n\}$ is a Cauchy sequence in (X, p). It follows from (4.19) for $x = x_{2n}$ and $y = x_{2n+1}$ that

$$F\Big(p(\mathcal{T}_1x_{2n}, \mathcal{T}_2x_{2n+1}), p(x_{2n}, x_{2n+1}), p(x_{2n}, \mathcal{T}_2x_{2n+1}), p(x_{2n+1}, \mathcal{T}_1x_{2n}), \frac{1}{2}\big[p(x_{2n}, \mathcal{T}_1x_{2n}) + p(x_{2n+1}, \mathcal{T}_2x_{2n+1})\big]\Big) \leqslant 0.$$

$$F\Big(p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n}), p(y_{2n-1}, y_{2n+1}), p(y_{2n}, y_{2n}), \frac{1}{2}\big[p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1})\big]\Big) \leqslant 0. \tag{4.20}$$

Since by $(P4)$,

$$\begin{aligned} p(y_{2n-1}, y_{2n+1}) &\leqslant p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}) - p(y_{2n}, y_{2n}) \\ &\leqslant p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n+1}), \end{aligned}$$

and by $(P2)$,

$$p(y_{2n}, y_{2n}) \leqslant p(y_{2n-1}, y_{2n}).$$

By (4.20) and (F_1), we obtain

$$F\Big(p(y_{2n},y_{2n+1}),p(y_{2n-1},y_{2n}),p(y_{2n-1},y_{2n})+p(y_{2n},y_{2n+1}),\\ p(y_{2n-1},y_{2n}),\frac{1}{2}\big[p(y_{2n-1},y_{2n})+p(y_{2n},y_{2n+1})\big]\Big)\leqslant 0. \tag{4.21}$$

By (F_{2a}), we obtain

$$p(y_{2n},y_{2n+1})\leqslant \psi\big(p(y_{2n-1},y_{2n})\big),$$

By (4.19) for $x=x_{2n+2}$ and $y=x_{2n+1}$, we obtain

$$F\Big(p(\mathcal{T}_1x_{2n+2},\mathcal{T}_2x_{2n+1}),p(x_{2n+2},x_{2n+1}),p(x_{2n+2},\mathcal{T}_2x_{2n+1}),p(x_{2n+1},\mathcal{T}_1x_{2n+2}),\\ \frac{1}{2}\big[p(x_{2n+2},\mathcal{T}_1x_{2n+2})+p(x_{2n+1},\mathcal{T}_2x_{2n+1})\big]\Big)\leqslant 0.$$

$$F\Big(p(y_{2n+2},y_{2n+1}),p(y_{2n+1},y_{2n}),p(y_{2n+1},y_{2n+1}),p(y_{2n},y_{2n+2}),\\ \frac{1}{2}\big[p(y_{2n+1},y_{2n+2})+p(y_{2n},y_{2n+1})\big]\Big)\leqslant 0. \tag{4.22}$$

Since by $(P4)$,

$$\begin{aligned} p(y_{2n},y_{2n+2}) &\leqslant p(y_{2n},y_{2n+1})+p(y_{2n+1},y_{2n+2})-p(y_{2n+1},y_{2n+1})\\ &\leqslant p(y_{2n},y_{2n+1})+p(y_{2n+1},y_{2n+2}), \end{aligned}$$

and by $(P2)$,

$$p(y_{2n+1},y_{2n+1})\leqslant p(y_{2n},y_{2n+1}).$$

By (4.22), (F_1) and using $(P3)$, we obtain

$$F\Big(p(y_{2n+2},y_{2n+1}),p(y_{2n+1},y_{2n}),p(y_{2n+1},y_{2n}),p(y_{2n+1},y_{2n})\\ +p(y_{2n+2},y_{2n+1}),\frac{1}{2}\big[p(y_{2n+2},y_{2n+1})+p(y_{2n+1},y_{2n})\big]\Big)\leqslant 0. \tag{4.23}$$

By (F_{2b}), we obtain

$$p(y_{2n+2},y_{2n+1})\leqslant \psi\big(p(y_{2n+1},y_{2n})\big),$$

which implies

$$p(y_n,y_{n+1})\leqslant \psi\big(p(y_{n-1},y_n)\big)\leqslant \psi^2\big(p(y_{n-2},y_{n-1})\big)\leqslant \dots\leqslant \psi^n\big(p(y_0,y_1)\big).$$

For $n,m\in\mathbb{N}$ with $m>n$, by repeated use of $(P4)$, we have that

$$\begin{aligned} p(y_n,y_m) &\leqslant p(y_n,y_{n+1})+p(y_{n+1},y_{n+2})+\cdots+p(y_{m-1},y_m)\\ &\quad -p(y_{n+1},y_{n+1})-p(y_{n+2},y_{n+2})-\cdots-p(y_{m-1},y_{m-1})\\ &\leqslant p(y_n,y_{n+1})+p(y_{n+1},y_{n+2})+\cdots+p(y_{m-1},y_m)\\ &= \sum_{r=n}^{m-1}\psi^r\big(p(y_0,y_1)\big). \end{aligned}$$

Since $\sum_{r=0}^{\infty}\psi^r\big(p(y_0,y_1)\big)<\infty$, then

$$\lim_{n\to\infty}\sum_{r=n}^{m-1}\psi^r\big(p(y_0,y_1)\big)=0 \text{ and } \lim_{n,m\to\infty}p(y_n,y_m)=0$$

and so

$$d^M(y_n,y_m)=2p(y_n,y_m)\to 0 \text{ as } n,m\to\infty. \tag{4.24}$$

This implies that $\{y_n\}$ is a Cauchy sequence in X. Thus by Lemma 2.1 this sequence will also Cauchy in (X,d^M). In addition, since (X,p) is complete, (X,d^M) is also complete. Thus there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$. Moreover by Lemma 2.2,

$$p(z,z)=\lim_{n\to\infty}p(z,y_n)=\lim_{n,m\to\infty}p(y_n,y_m)=0, \tag{4.25}$$

implies

$$\lim_{n\to\infty}d^M(z,y_n)=0. \tag{4.26}$$

Now, we show that z is a common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$. Notice that due to (4.25), we have $p(z,z)=0$. By (4.19) with $x=z$ and $y=x_{2n+1}$ and using (4.25), we have

$$F\Big(p(\mathcal{T}_1z,\mathcal{T}_2x_{2n+1}),p(z,x_{2n+1}),p(z,\mathcal{T}_2x_{2n+1}),p(x_{2n+1},\mathcal{T}_1z),\frac{1}{2}\big[p(z,\mathcal{T}_1z)+p(x_{2n+1},\mathcal{T}_2x_{2n+1})\big]\Big)\leqslant 0.$$

$$F\Big(p(\mathcal{T}_1z,x_{2n+2}),p(z,x_{2n+1}),p(z,x_{2n+2}),p(x_{2n+1},\mathcal{T}_1z),\frac{1}{2}\big[p(z,\mathcal{T}_1z)+p(x_{2n+1},x_{2n+2})\big]\Big)\leqslant 0. \tag{4.27}$$

Letting $n\to\infty$ in (4.27) and using $(P3)$, we obtain by Lemma 2.2 that

$$F\Big(p(\mathcal{T}_1z,z),0,0,p(\mathcal{T}_1z,z),\frac{1}{2}p(\mathcal{T}_1z,z)\Big)\leqslant 0,$$

which implies by (F_{2b}) that $p(\mathcal{T}_1z,z)\leqslant\psi(0)=0$, that is, $\mathcal{T}_1z=z$. This shows that z is a fixed point of $\mathcal{T}_1$. Similarly, we can show that $\mathcal{T}_2z=z$. Thus z is a common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Now, we have to show that the common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$ is unique. Assume that z' is another common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T}_1z'=z'=\mathcal{T}_2z'$ with $z\neq z'$. Now using (4.19), (4.25) and $(P3)$ with $x=z$ and $y=z'$, we have

$$F\Big(p(\mathcal{T}_1z,\mathcal{T}_2z'),p(z,z'),p(z,\mathcal{T}_2z'),p(z',\mathcal{T}_1z),\frac{1}{2}\big[p(z,\mathcal{T}_1z)+p(z',\mathcal{T}_2z')\big]\Big)\leqslant 0.$$

$$F\Big(p(z,z'),p(z,z'),p(z,z'),p(z',z),\frac{1}{2}\big[p(z,z)+p(z',z')\big]\Big)\leqslant 0.$$

$$F\Big(p(z,z'),p(z,z'),p(z,z'),p(z,z'),0\Big)\leqslant 0.$$

By (F_1) and (F_{2a}), we obtain

$$p(z, z') \leqslant \psi\big(p(z, z')\big) < p(z, z'),$$

if $p(z, z') \neq 0$, a contradiction. Hence $p(z, z') = 0$, which implies $z = z'$. This shows that the common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$ is unique. This completes the proof. □

REMARK 4.1. If we take $f = I$, the identity map and $\mathcal{T}$ is the single valued mapping in Theorem 4.2, then we obtain Theorem 4.1 of this paper.

REMARK 4.2. If we take $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$ in Theorem 4.3, then we obtain Theorem 4.1 of this paper.

5. Application to well-posedness and limit shadowing of fixed point problem

The notion of well posedness of a fixed point problem has generated much interest to several mathematicians, for example, Akkouchi [**2**], Akkouchi and Popa [**3**], De Blasi and Myjak [**15**], Lahiri and Das [**22**], Popa [**33, 34**], Reich and Zaslawski [**38**] and many others. Here, we study well posedness and limit shadowing of a fixed point problem of mappings in Theorem 4.1.

DEFINITION 5.1. ([**15**]) Let (X, d) be a metric space and $\mathcal{T}\colon X \to X$ be a mapping. The fixed point problem of $\mathcal{T}$ is said to be well-posed if

(i) $\mathcal{T}$ has a unique fixed point u in X;

(ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(\mathcal{T}x_n, x_n) = 0$, we have $\lim_{n\to\infty} d(x_n, u) = 0$.

The limit shadowing property of fixed point problems has been discussed in the articles [**28, 29, 39**] and others.

DEFINITION 5.2. ([**31**]) Let (X, d) be a metric space and $\mathcal{T}\colon X \to X$ be a mapping. The fixed point problem of $\mathcal{T}$ is said to have limit shadowing property in X if assuming that sequence $\{x_n\}$ in X satisfies $d(\mathcal{T}x_n, x_n) = 0$ as $n \to \infty$ it follows that there exists $x \in X$ such that $d(\mathcal{T}^n x, x_n) = 0$ as $n \to \infty$.

We can give similar definitions in partial metric spaces.

Concerning the well-posedness and limit shadowing of the fixed point problem for a mapping in a partial metric space satisfying the conditions of Theorem 4.1, we have the following results.

THEOREM 5.1. *Let $\mathcal{T}\colon X \to X$ be a self mapping as in Theorem 4.1. Then the fixed point problem for $\mathcal{T}$ is well posed.*

PROOF. Owing to Theorem 4.1, we know that $\mathcal{T}$ has a unique fixed point $u = \mathcal{T}u \in X$, such that $p(u, \mathcal{T}u) = 0$. Let $\{x_n\} \subset X$ be such that $\lim_{n\to\infty} p(x_n, \mathcal{T}x_n) = 0$. Then taking $x = x_{n-1}$ and $y = u$ in inequality (4.1), we have

$$F\Big(p(\mathcal{T}x_{n-1}, \mathcal{T}u), p(x_{n-1}, u), p(x_{n-1}, \mathcal{T}u), p(u, \mathcal{T}x_{n-1}), \frac{1}{2}\big[p(x_{n-1}, \mathcal{T}x_{n-1} + p(u, \mathcal{T}u)\big]\Big) \leqslant 0.$$

or

$$F\Big(p(x_n, u), p(x_{n-1}, u), p(x_{n-1}, u), p(u, x_n), \frac{1}{2}\big[p(x_{n-1}, x_n + p(u, \mathcal{T}u)\big]\Big) \leqslant 0,$$

or

$$F\Big(p(x_n, u), p(x_{n-1}, u), p(x_{n-1}, u), p(u, x_n), \frac{1}{2}\big[p(x_{n-1}, u) + p(u, x_n)\big]\Big) \leqslant 0,$$

by (P3), we have

$$F\Big(p(x_n, u), p(x_{n-1}, u), p(x_{n-1}, u), p(x_n, u), \frac{1}{2}\big[p(x_{n-1}, u) + p(x_n, u)\big]\Big) \leqslant 0,$$

which implies by (F_{2b}) that

$$p(x_n, u) \leqslant \psi\big(p(x_{n-1}, u)\big).$$

Hence, we have

$$p(x_n, u) \leqslant \psi\big(p(x_{n-1}, u)\big) \leqslant \psi^2\big(p(x_{n-2}, u)\big) \leqslant \ldots \leqslant \psi^n\big(p(x_0, u)\big).$$

Taking the limit as $n \to \infty$ in the above inequality and by Remark 3.1, we get that $p(x_n, u) \to 0$ as $n \to \infty$ which is equivalent to saying that $x_n \to u$ as $n \to \infty$. This completes the proof. □

THEOREM 5.2. *Let $\mathcal{T}\colon X \to X$ be a self mapping as in Theorem 4.1. Then $\mathcal{T}$ has the limit shadowing property.*

PROOF. Owing to Theorem 4.1, we know that $\mathcal{T}$ has a unique fixed point $u = \mathcal{T}u \in X$, such that $p(u, \mathcal{T}u) = 0$. Let $\{x_n\} \subset X$ be such that $\lim_{n\to\infty} p(x_n, \mathcal{T}x_n) = 0$. Then, as in the previous proof,

$$p(x_n, u) \leqslant \psi\big(p(x_{n-1}, u)\big) \leqslant \psi^2\big(p(x_{n-2}, u)\big) \leqslant \ldots \leqslant \psi^n\big(p(x_0, u)\big).$$

Passing to the limit as $n \to \infty$ in the above inequality and by Remark 3.1, it follows that $p(x_n, \mathcal{T}^n u) = p(x_n, u) \to 0$ as $n \to \infty$. This completes the proof. □

6. Conclusion

In this paper, we study Popa and Patriciu [**36**] type implicit relation and establish a unique fixed point, a coincidence point and a unique common fixed point theorems in the framework of partial metric spaces. The results presented in this paper extend, unify and generalize several results from the existing literature regarding various ambient spaces and contraction condition. We also present one of the possible applications of our result to well-posed and limit shadowing property of fixed point problems.

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Received by editors 27.01.2021; Revised version 24.06.2021; Available online 05.07.2021.

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