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ON SOME PROPERTIES OF PARTICULAR TETRANACCI SEQUENCES

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Abstract. In this paper, a particular Tetranacci sequence, namely Pell-Padovan Tetranacci sequence is introduced. The Binet-like-formula, partial sum, generating function related to this sequence are represented. Some identities and examples about this sequence are stated by using the matrix form. Also norms, determinants and eigenvalues of the circulant matrices for the Pell-Padovan Tetranacci sequence are obtained.

1. Introduction

Many authors investigated recurrence relations like as Fibonacci numbers, Pell sequence, Padovan sequence, Tribonacci sequence, Tetrabonacci sequence and generalizations of these sequences. They established numerous results and identities about these sequences. Also they illustrated various applications of these sequences. To learn more about Pell sequence, Padovan sequence, Tetranacci sequences and generalizations of these sequences and applications of these sequences we refer to [**1, 2, 3, 5, 8, 9, 10, 15, 17, 18**].

Fibonacci numbers are a sequence which are defined by the following recurrence relation:

$$F_0 = 0, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for all } n \geqslant 2.$$

The first Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots.$$

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The characteristic equation of F_n is $x^2 - x - 1 = 0$ and hence the roots of it are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Also Binet's formula for the sequence is $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $n \geqslant 0$. Lucas numbers L_n are defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$. The first Lucas numbers are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \cdots \ ([\mathbf{11}]).$$

Pell sequence P_n is defined by the recursive relation

$$P_n = 2P_{n-1} + P_{n-2}$$

for all $n \geqslant 2$ with the initial values $P_0 = 0, P_1 = 1$. In [**12**], the Padovan sequence R_n is defined by the recursion relation

$$R_{n+2} = R_n + R_{n-1}$$

for all $n \geqslant 3$ with initial values $R_0 = 1, R_1 = 1, R_2 = 1$.

In [**12, 13**], the Pell–Padovan sequence $P(n)$ is defined by a third-order recurrence equation

$$P(n+3) = 2P(n+1) + P(n)$$

for $n \geqslant 0$, where $P(0) = P(1) = P(2) = 1$.

In [**7**] author defined a new integer sequence related to Fibonacci and Pell sequences with four parameters

$$W_n = 3W_{n-1} - 3W_{n-3} - W_{n-4} \text{ for } n \geqslant 4$$

with initial values $W_0 = W_1 = 0, W_2 = 1, W_3 = 3$ and then derive some algebraic identities on it. In [**16**], the authors considered the integer sequence with four parameters and introduced the concept of a fourth-order recurrence relations and established some identities for these sequence.

Let $A = (a_{ij})$ be an $n \times n$ matrix. The circulant matrix formed a square matrix

$$C(A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

The eigenvalues of A are

$$\lambda_j(A) = \sum_{k=0}^{n-1} a_k w^{-jk}, \tag{1.1}$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$ and $j = 0, 1, \cdots, n-1$. The spectral norm for a matrix $A = [a_{ij}]_{n \times m}$ is defined to be $||A||_{spec} = \max\{\sqrt{\lambda_i}\}$, where λ_i are the eigenvalues of $A^H A$ for $0 \leqslant j \leqslant n-1$ and A^H denotes the conjugate transpose of A.

2. Particular Tetranacci Sequences

Definition 2.1. We define the Pell-Padovan Tetranacci sequence (PT_n) by the recursive relation

$$PT_{n+4} = PT_{n+2} + 2PT_{n+1} + PT_n \tag{2.1}$$

with initial values $PT_0 = 0,\ \ PT_1 = 1,\ PT_2 = 1,\ \ PT_3 = 1.$

Hence the first few values of Pell-Padovan Tetranacci sequence are:

$$0, 1, 1, 1, 3, 4, 6, 11, 17, 27, 45, 72, 116.$$

Remark 2.1. Pell-Padovan Tetranacci sequence (PT_n) has the characteristic equation $x^4 - x^2 - 2x - 1 = 0$. By factorization of this polynomial we see that

$$x^4 - x^2 - 2x - 1 = (x^2 - x - 1)(x^2 + x + 1).$$

Therefore by solving the characteristic equation of this sequence we get that this equation has two real roots α, β and two complex roots γ, λ which are

$$\alpha = (1 + \sqrt{5})/2,\ \beta = (1 - \sqrt{5})/2,\ \gamma = -(1 - i\sqrt{3})/2,\ \lambda = -(1 + i\sqrt{3})/2$$

where $i = \sqrt{-1}$.

Theorem 2.1. *The generating function for the Pell-Padovan Tetranacci sequence* (PT_n) *is*

$$\sum_{n=0}^{\infty} PT_n x^n = (x + x^2)/(1 - x^2 - 2x^3 - x^4).$$

Proof. Suppose that the generating function for the Pell-Padovan Tetranacci sequence (PT_n) has the form

$$g(x) = \sum_{n=0}^{\infty} PT_n x^n = PT_0 + PT_1 x + PT_2 x^2 + PT_3 x^3 + \cdots + PT_n x^n + \cdots .$$

Then we have

$$\begin{aligned} x^2 g(x) &= PT_0 x^2 + PT_1 x^3 + PT_2 x^4 + PT_3 x^5 + \cdots + PT_n x^{n+2} + \cdots \\ 2x^3 g(x) &= 2PT_0 x^3 + 2PT_1 x^4 + 2PT_2 x^5 + 2PT_3 x^6 + \cdots + 2PT_n x^{n+3} + \cdots \end{aligned}$$

and

$$x^4 g(x) = PT_0 x^4 + PT_1 x^5 + PT_2 x^6 + PT_3 x^7 + \cdots + PT_n x^{n+4} + \cdots .$$

Thus we obtain

$$
\begin{aligned}
& g(x)-x^2g(x)-2x^3g(x)-x^4g(x) \\
= \ & (PT_0+PT_1x+PT_2x^2+PT_3x^3+\cdots+PT_nx^n+\cdots) \\
& -(PT_0x^2+PT_1x^3+PT_2x^4+PT_3x^5+\cdots+PT_nx^{n+2}+\cdots) \\
& -(2PT_0x^3+2PT_1x^4+2PT_2x^5+2PT_3x^6+\cdots+2PT_nx^{n+4}+\cdots) \\
& -(PT_0x^4+PT_1x^5+PT_2x^6+PT_3x^7+\cdots+PT_nx^{n+4}+\cdots) \\
= \ & PT_0+PT_1x+(PT_2-PT_0)x^2+(PT_3-PT_1-2PT_0)x^3 \\
& +(PT_4-PT_2-2PT_1-PT_0)x^4+\cdots \\
& +(PT_n-PT_{n-2}-2PT_{n-3}-PT_{n-4})x^n+\cdots.
\end{aligned}
$$

Therefore we get

$$g(x)(1-x^2-2x^3-x^4)=0+x+(1-0)x^2+(1-1-0)x^3+0+\cdots+0=x+x^2.$$

Consequently

$$\sum_{n=0}^{\infty} PT_nx^n=(x+x^2)/(1-x^2-2x^3-x^4).$$

□

THEOREM 2.2. *Let $n \geqslant 0$ be an integer. Then the Binet-like formula for the Pell-Padovan Tetranacci sequence (PT_n) is*

$$
\begin{aligned}
PT_n \ = \ & \left(\frac{\alpha+1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}\right)\alpha^{n+1}+\left(\frac{\beta+1}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right)\beta^{n+1} \\
& +\left(\frac{\gamma+1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}\right)\gamma^{n+1}+\left(\frac{\lambda+1}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)}\right)\lambda^{n+1}
\end{aligned}
$$

where $\alpha,\ \beta,\ \gamma,\ \lambda$ are the roots of the equation $x^4-x^2-2x-1=0$.

PROOF. From Remark 2.1 we see that the equation

$$f(x)=x^4-x^2-2x-1=0$$

has four distinct roots $\alpha,\ \beta,\ \gamma,\ \lambda$. Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{\lambda}$ are the roots of

$$h(x)=f(1/x)=1-x^2-2x^3-x^4.$$

In exact, we have

$$h(x)=1-x^2-2x^3-x^4=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\lambda x).$$

According to the generating function of Pell-Padovan Tetranacci sequence, we have

$$
\begin{aligned}
(2.2)\quad g(x) \ = \ & \frac{x+x^2}{1-x^2-2x^3-x^4}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}+\frac{C}{1-\gamma x}+\frac{D}{1-\lambda x} \\
= \ & A\sum_{n=0}^{\infty}(\alpha x)^n+B\sum_{n=0}^{\infty}(\beta x)^n+C\sum_{n=0}^{\infty}(\gamma x)^n+D\sum_{n=0}^{\infty}(\lambda x)^n.
\end{aligned}
$$

Thus, we have

$$\begin{aligned} g(x) &= \frac{x+x^2}{1-x^2-2x^3-x^4} \\ &= \frac{\left\{\begin{array}{c} A(1-\beta x)(1-\gamma x)(1-\lambda x)+B(1-\alpha x)(1-\gamma x)(1-\lambda x) \\ +C(1-\alpha x)(1-\beta x)(1-\lambda x)+D(1-\alpha x)(1-\beta x)(1-\lambda x) \end{array}\right\}}{(1-\alpha x)(1-\beta x)(1-\gamma x)}. \end{aligned}$$

Therefore, by comparison of the left and right sides of this equality we get that

$$x+x^2=\left\{\begin{array}{c} A(1-\beta x)(1-\gamma x)(1-\lambda x)+B(1-\alpha x)(1-\gamma x)(1-\lambda x) \\ +C(1-\alpha x)(1-\beta x)(1-\lambda x)+D(1-\alpha x)(1-\beta x)(1-\gamma x) \end{array}\right\}.$$

If we substitute x by $1/\alpha$ we find that

$$\frac{1}{\alpha}+\frac{1}{\alpha^2}=A\left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right)\left(1-\frac{\lambda}{\alpha}\right).$$

Consequently, we get

$$A=\frac{\alpha(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}$$

and similarly we get

$$B=\tfrac{\beta(\beta+1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}, \qquad C=\tfrac{\gamma(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)},$$

and

$$D=\frac{\lambda(\lambda+1)}{(\lambda-\alpha)(\gamma-\beta)(\lambda-\gamma)}.$$

By (2.2) we obtain that

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty}\frac{\alpha(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}(\alpha x)^n+\sum_{n=0}^{\infty}\frac{\beta(\beta+1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}(\beta x)^n \\ &\quad +\sum_{n=0}^{\infty}\frac{\gamma(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}(\gamma x)^n+\sum_{n=0}^{\infty}\frac{\lambda(\lambda+1)}{(\lambda-\alpha)(\gamma-\beta)(\lambda-\gamma)}(\lambda x)^n \\ &= \sum_{n=0}^{\infty}\left[\frac{\alpha(\alpha+1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}+\frac{\beta(\beta+1)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right. \\ &\quad \left.+\frac{\gamma(\gamma+1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}+\frac{\lambda(\lambda+1)\lambda^n}{(\lambda-\alpha)(\gamma-\beta)(\lambda-\gamma)}\right]x^n. \end{aligned}$$

Consequently we obtain

$$\begin{aligned} PT_n &= \left(\frac{\alpha+1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}\right)\alpha^{n+1}+\left(\frac{\beta+1}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right)\beta^{n+1} \\ &+ \left(\frac{\gamma+1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}\right)\gamma^{n+1}+\left(\frac{\lambda+1}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)}\right)\lambda^{n+1}. \end{aligned}$$

□

THEOREM 2.3. *Let $n \geqslant 0$ be an integer. Then*

$$PT_{n+1} + PT_n =$$
$$\left(\frac{(\alpha+1)^2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}\right)\alpha^{n+1} + \left(\frac{(\beta+1)^2}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right)\beta^{n+1}$$
$$+\left(\frac{(\gamma+1)^2}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}\right)\gamma^{n+1} + \left(\frac{(\lambda+1)^2}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)}\right)\lambda^{n+1},$$

$$PT_{n+1} - PT_n =$$
$$\left(\frac{\alpha^2-1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}\right)\alpha^{n+1} + \left(\frac{\beta^2-1}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right)\beta^{n+1}$$
$$+\left(\frac{\gamma^2-1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}\right)\gamma^{n+1} + \left(\frac{\lambda^2-1}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)}\right)\lambda^{n+1}.$$

PROOF. They can be proved by direct calculations from Theorem 2.2. □

THEOREM 2.4. *Let $n \geqslant 0$ be an integer and k be an arbitrary integer. Then*

$$PT_{n+k} + PT_{n-k} =$$
$$\left(\frac{(\alpha+1)(\alpha^{2k}+1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}\right)\alpha^{n-k+1} + \left(\frac{(\beta+1)(\beta^{2k}+1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right)\beta^{n-k+1}$$
$$+\left(\frac{(\gamma+1)(\gamma^{2k}+1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}\right)\gamma^{n-k+1} + \left(\frac{(\lambda+1)(\lambda^{2k}+1)}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)}\right)\lambda^{n-k+1},$$

$$PT_{n+k} - PT_{n-k} =$$
$$\left(\frac{(\alpha+1)(\alpha^{2k}-1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}\right)\alpha^{n-k+1} + \left(\frac{(\beta+1)(\beta^{2k}-1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right)\beta^{n-k+1}$$
$$+\left(\frac{(\gamma+1)(\gamma^{2}-1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}\right)\gamma^{n-k+1} + \left(\frac{(\lambda+1)(\lambda^{2}-1)}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)}\right)\lambda^{n-k+1}.$$

PROOF. They can be proved by direct calculations from Theorem 2.2. □

COROLLARY 2.1. *From Theorem 2.4 for $k = 1$, we have*

$$PT_{n+1} + PT_{n-1} =$$
$$\left(\frac{(\alpha+1)(\alpha^{2}+1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}\right)\alpha^{n} + \left(\frac{(\beta+1)(\beta^{2}+1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right)\beta^{n}$$
$$+\left(\frac{(\gamma+1)(\gamma^{2}+1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}\right)\gamma^{n} + \left(\frac{(\lambda+1)(\lambda^{2}+1)}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)}\right)\lambda^{n},$$

$$PT_{n+1} - PT_{n-1} =$$
$$\left(\frac{(\alpha+1)^2(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}\right)\alpha^n + \left(\frac{(\beta+1)^2(\beta-1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)}\right)\beta^n$$
$$+\left(\frac{(\gamma+1)^2(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}\right)\gamma^n + \left(\frac{(\lambda+1)^2(\lambda-1)}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)}\right)\lambda^n.$$

LEMMA 2.1. *Let $k \geqslant 0$ be an integer. Then*

$$\sum_{k=0}^{n} PT_k = \frac{1}{3}\left(PT_{k+4} + PT_{k+3} - 2PT_{k+1} - 2\right).$$

PROOF. From the definition of Pell-Padovan Tetranacci sequence we now that $PT_k = PT_{k+4} - PT_{k+2} - 2PT_{k+1}$. Thus we have

$$\begin{aligned} PT_0 &= PT_4 - PT_2 - 2PT_1 \\ PT_1 &= PT_5 - PT_3 - 2PT_2 \\ PT_2 &= PT_6 - PT_4 - 2PT_3 \\ &\vdots \\ PT_{k-2} &= PT_{k+2} - PT_k - 2PT_{k-1} \\ PT_{k-1} &= PT_{k+3} - PT_{k+1} - 2PT_k \\ PT_k &= PT_{k+4} - PT_{k+2} - 2PT_{k+1}. \end{aligned}$$

Therefore, we get

$$\sum_{k=0}^{n} PT_k = -PT_2 - PT_3 + PT_{k+4} + PT_{k+3} - 2PT_{k+1} - 2\sum_{k=0}^{n} PT_k.$$

Thus, we have

$$3\sum_{k=0}^{n} PT_k = -1 - 1 + PT_{k+4} + PT_{k+3} - 2PT_{k+1}.$$

Consequently, we get

$$\sum_{k=0}^{n} PT_k = \frac{1}{3}\left(PT_{k+4} + PT_{k+3} - 2PT_{k+1} - 2\right).$$

□

3. Special Matrices on Pell-Padovan Tetranacci Sequence

There are two subsections in this section, one of which consists of algebraic results obtained with the help of matrices. In the other subsection, circulant matrices are created for Pell-Padovan Tetranacci sequence and the eigenvalues and norms of this matrix are examined.

3.1. More identities about Pell-Padovan Tetranacci sequence.

THEOREM 3.1. *Let $n \geqslant 0$ be an integer and* $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$. *Then*

$$\begin{bmatrix} PT_n \\ PT_{n+1} \\ PT_{n+2} \\ PT_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = M^n \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

PROOF. We prove this theorem by mathematical induction on n. For $n = 1$ we have

$$M^1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}^1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = M \begin{bmatrix} PT_1 \\ PT_2 \\ PT_3 \\ PT_4 \end{bmatrix}.$$

Thus the result is true for $n = 1$. Now suppose that the result is true for $n = k$. Hence we have

$$\begin{bmatrix} PT_k \\ PT_{k+1} \\ PT_{k+2} \\ PT_{k+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = M^k \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then for $n = k + 1$ we have

$$\begin{aligned} M^{k+1} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} &= MM^k \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = M \begin{bmatrix} PT_k \\ PT_{k+1} \\ PT_{k+2} \\ PT_{k+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} PT_k \\ PT_{k+1} \\ PT_{k+2} \\ PT_{k+3} \end{bmatrix} \\ &= \begin{bmatrix} PT_{k+1} \\ PT_{k+2} \\ PT_{k+3} \\ PT_k + 2PT_{k+1} + PT_{k+2} \end{bmatrix} = \begin{bmatrix} PT_{k+1} \\ PT_{k+2} \\ PT_{k+3} \\ PT_{k+4} \end{bmatrix}. \end{aligned}$$

Therefore, the result is true for $n = k + 1$. Consequently, by induction the result is true for every n. This proves the theorem. □

REMARK 3.1. As we know the characteristic polynomial of the recursive relation

$$PT_{n+4} = PT_{n+2} + 2PT_{n+1} + PT_n \quad \text{is} \quad p(x) = x^4 - x^2 - 2x - 1 = 0.$$

This polynomial can be written as

$$p(x) = det\ (xI - M) = 0$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}.$$

From the Cayley Hamilton Theorem in matrix algebra (Theorem 3.9 in [**19**]), we have $p(M) = 0$. Thus we obtain

$$M^4 - M^2 - 2M - I = 0. \tag{3.1}$$

THEOREM 3.2. *Let* $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$. *Then*

$$I = M^4 - M^2 - 2M = M^4 + 2M^3 + 3M^2 - 2M^5$$

and

$$M^n = M^{n+4} + 2M^{n+3} + 3M^{n+2} - 2M^{n+5}.$$

PROOF. According to the Remark 3.1, we have

$$\begin{aligned} I &= M^4 - M^2 - 2M = M(M^3 - M - 2I) = M(M^3 - M - 2(M^4 - M^2 - 2M)) \\ &= M(M^3 + 2M^2 + 3M - 2M^4) = M^4 + 2M^3 + 3M^2 - 2M^5. \end{aligned}$$

Thus

$$I = M^4 + 2M^3 + 3M^2 - 2M^5.$$

This proves the first equality. Multiplying both sides of the above equality by M^n we obtain

$$M^n = M^{n+4} + 2M^{n+3} + 3M^{n+2} - 2M^{n+5}. \tag{3.2}$$

Thus the proof is completed. □

COROLLARY 3.1. *Let* $n \geqslant 0$ *be an integer. Then*

$$M^{n+5} = \frac{1}{2}\left(M^{n+4} + 2M^{n+3} + 3M^{n+2} - M^n\right). \tag{3.3}$$

According to this corollary, we have the following interesting example and theorem about the Pell-Padovan Tetranacci sequence.

EXAMPLE 3.1. From the first values of Pell-Padovan Tetranacci sequence we have

$$6 = \frac{1}{2}(4 + 2(3) + 3(1) - 1).$$

In exact, we have

$$PT_6 = \frac{1}{2}(PT_5 + 2PT_4 + 3PT_3 - PT_1)$$

or

$$PT_{1+5} = \frac{1}{2}(PT_{1+4} + 2PT_{1+3} + 3PT_{1+2} - PT_1).$$

THEOREM 3.3. *Let $n \geqslant 0$ be an integer. Then*

$$PT_{n+5} = \frac{1}{2}\left(PT_{n+4} + 2PT_{n+3} + 3PT_{n+2} - PT_n\right).$$

PROOF. We prove this theorem by mathematical induction on n. According to the last example we see that

$$PT_{1+5} = \frac{1}{2}\left(PT_{1+4} + 2PT_{1+3} + 3PT_{1+2} - PT_1\right).$$

Then if we assume that $PT_{t+5} = \frac{1}{2}\left(PT_{t+4} + 2PT_{t+3} + 3PT_{t+2} - PT_t\right)$ for all $t < n$. Then we have

$$\begin{aligned} PT_{n+5} &= PT_{n+3} + 2PT_{n+2} + PT_{n+1} \\ &= \frac{1}{2}\left(PT_{n+2} + 2PT_{n+1} + 3PT_n - PT_{n-2}\right) \\ &\quad +2\left[\frac{1}{2}\left(PT_{n+1} + 2PT_n + 3PT_{n-1} - PT_{n-3}\right)\right] \\ &\quad +\frac{1}{2}\left(PT_n + 2PT_{n-1} + 3PT_{n-2} - PT_{n-4}\right) \\ &= \frac{1}{2}\left[(PT_{n+2} + 2PT_{n+1} + PT_n) + 2(PT_{n+1} + 2PT_n + PT_{n-1})\right. \\ &\quad \left.+3(PT_n + 2PT_{n-1} + PT_{n-2}) - (PT_{n-2} + 2PT_{n-3} + PT_{n-4})\right] \\ &= \frac{1}{2}\left(PT_{n+4} + 2PT_{n+3} + 3PT_{n+2} - PT_n\right). \end{aligned}$$

Thus, the result is true for negative n. □

THEOREM 3.4. *Let $r, n \geqslant 0$ be integer. Then*

$$M^{n+r} = 4M^{n+r+10} - 4M^{n+r+9} - 7M^{n+r+8} - 8M^{n+r+7} + 10M^{n+r+6} + 12M^{n+r+5} + 9M^{n+r+4}$$

PROOF. By Theorem 3.2 we have

$$M^n = M^{n+4} + 2M^{n+3} + 3M^{n+2} - 2M^{n+5}.$$

Hence

$$\begin{aligned} & M^{n+r} = M^n M^r \\ = \;& (M^{n+4} + 2M^{n+3} + 3M^{n+2} - 2M^{n+5})(M^{r+4} + 2M^{r+3} + 3M^{r+2} - 2M^{r+5}) \\ = \;& M^{n+r+8} + 2M^{n+r+7} + 3M^{n+r+6} - 2M^{n+r+9} + 2M^{n+r+7} + 4M^{n+r+6} \\ & +6M^{n+r+5} - 4M^{n+r+8} + 3M^{n+r+6} + 6M^{n+r+5} + 9M^{n+r+4} - 6M^{n+r+7} \\ & -2M^{n+r+9} - 4M^{n+r+8} - 6M^{n+r+7} + 4M^{n+r+10} \\ = \;& 4M^{n+r+10} - 4M^{n+r+9} - 7M^{n+r+8} - 8M^{n+r+7} + 10M^{n+r+6} + 12M^{n+r+5} \\ & +9M^{n+r+4}. \end{aligned}$$

Thus, the proof is completed. □

COROLLARY 3.2. *Let $n \geqslant 0$ be an integer. Then*

$M^{2n} =$

$4M^{2n+10} - 4M^{2n+9} - 7M^{2n+8} - 8M^{2n+7} + 10M^{2n+6} + 12M^{2n+5} + 9M^{2n+4}$,

$M^{2n+9} =$

$\frac{1}{4}\left[4M^{2n+10} - 7M^{2n+8} - 8M^{2n+7} + 10M^{2n+6} + 12M^{2n+5} + 9M^{2n+4} - M^{2n}\right]$.

PROOF. They can be derived from Theorem 3.4 by substituting $r = n$. □

EXAMPLE 3.2. From the first values of Pell-Padovan Tetranacci sequence PT_n we have

$$72 = 1\frac{1}{4}[9(16) + 12(11) + 10(17) - 8(27) - 7(45) + 4(116) - 1].$$

In exact, we have

$$PT_{11} = \frac{1}{4}[9PT_6 + 12PT_7 + 10PT_8 - 8PT_9 - 7PT_{10} + 4PT_{12} - PT_1]$$

or equivalently we have

$$\begin{aligned} PT_{2\times1+9} \quad = \quad & \frac{1}{4}[9PT_{2\times1+4} + 12PT_{2\times1+5} + 10PT_{2\times1+6} - 8PT_{2\times1+7} \\ & -7PT_{2\times1+8} + 4PT_{2\times1+10} - PT_{2\times1}]. \end{aligned}$$

THEOREM 3.5. *Let $n \geqslant 0$ be an integer. Then*

$PT_{2n+9} =$

$\frac{1}{4}[9PT_{2n+4} + 12PT_{2n+5} + 10PT_{2n+6} - 8PT_{2n+7} - 7PT_{2n+8} + 4PT_{2n+10} - PT_{2n}]$.

PROOF. It can be proved similar to Theorem 3.4. □

3.2. Circulant matrices via Pell-Padovan Tetranacci sequence. We define circulant matrix

$$PT = C(PT_n) = \begin{bmatrix} PT_0 & PT_1 & PT_2 & \cdots & PT_{n-1} \\ PT_{n-1} & PT_0 & PT_1 & \cdots & PT_{n-2} \\ PT_{n-2} & PT_{n-1} & PT_0 & \cdots & PT_{n-3} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ PT_1 & PT_2 & PT_3 & \cdots & PT_0 \end{bmatrix}$$

for PT_n.

The equations we will use in the following theorems are given.

$$P = \frac{\alpha+1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\lambda)}, \quad Q = \frac{\beta+1}{(\beta-\alpha)(\beta-\gamma)(\beta-\lambda)},$$

$$R = \frac{\gamma+1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\lambda)}, \quad S = \frac{\lambda+1}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)},$$

$$\begin{aligned} a &= -PT_n + P\alpha + Q\beta + R\gamma + S\lambda \\ b &= -P(\alpha+1)(1-\alpha^n) - Q(1-\beta^n)(\beta+1) \\ &\quad -R(\gamma+1)(\gamma^n-1) - S(\lambda+1)(\lambda^n-1) \\ c &= -PT_n + P\alpha + Q\beta - R\gamma - S\lambda \\ d &= PT_{n-1} - P - Q - R - S. \end{aligned}$$

THEOREM 3.6. *Let PT denote the circulant matrices of (PT_n). The eigenvalues of PT are*

$$\lambda_j(PT) = \frac{dw^{-3j} + cw^{-2j} + bw^{-j} + a}{-w^{-4j} - 2w^{-3j} - w^{-2j} + 1}$$

for $j = 0, 1, 2, \cdots, n-1$.

PROOF. For the sequence (PT_n), we have

$$\begin{aligned} \lambda_j(C(PT_n)) &= \sum_{k=0}^{n-1} PT_k w^{-jk} \\ &= \sum_{k=0}^{n-1} \left[P\alpha^{k+1} + Q\beta^{k+1} + R\gamma^{k+1} + S\lambda^{k+1}\right] w^{-jk} \\ &= P\alpha\left(\frac{(\alpha w^{-j})^n - 1}{(\alpha w^{-j}) - 1}\right) + Q\beta\left(\frac{(\beta w^{-j})^n - 1}{(\beta w^{-j}) - 1}\right) \\ &\quad + R\gamma\left(\frac{(\gamma w^{-j})^n - 1}{(\gamma w^{-j}) - 1}\right) + S\lambda\left(\frac{(\lambda w^{-j})^n - 1}{(\lambda w^{-j}) - 1}\right). \end{aligned}$$

We know about

$$(\alpha w^{-j})^n = \alpha^n\ ,\ (\beta w^{-j})^n = \beta^n\ \ ,(\gamma w^{-j})^n = \gamma^n\ \ ,(\lambda w^{-j})^n = \lambda^n,$$

then the equation

$$\lambda_j(C(PT_n)) = \frac{\left\{\begin{array}{c} P\alpha(\alpha^n-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)(\lambda w^{-j}-1) \\ +Q\beta(\beta^n-1)(\alpha w^{-j}-1)(\lambda w^{-j}-1)(\gamma w^{-j}-1) \\ +R\gamma(\gamma^n-1)(\alpha w^{-j}-1)(\lambda w^{-j}-1)(\beta w^{-j}-1) \\ +S\lambda(\lambda^n-1)(\alpha w^{-j}-1)(\gamma w^{-j}-1)(\beta w^{-j}-1) \end{array}\right\}}{(\alpha w^{-j}-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)(\lambda w^{-j}-1)}.$$

We found the numarator

$$\begin{aligned} &\left\{\begin{array}{c} P\alpha(\alpha^n-1)(\beta\gamma\lambda) + Q\beta(\beta^n-1)(\alpha\gamma\lambda) \\ +R\gamma(\gamma^n-1)(\alpha\beta\lambda) - S\lambda(\lambda^n-1)(\alpha\beta\gamma) \end{array}\right\} w^{-3j} \\ &+\left\{\begin{array}{c} P(\alpha^n-1)(\gamma+\lambda-\alpha) + Q(\beta^n-1)(\gamma+\lambda-\beta) \\ +R(\gamma^n-1)(\gamma+\lambda-\beta) + S(\lambda^n-1)(\lambda+\gamma-\beta) \end{array}\right\} w^{-2j} \\ &+\left\{\begin{array}{c} P(\alpha^n-1)(-1+\alpha\gamma+\alpha\lambda) + Q(\beta^n-1)(-1+\beta\gamma+\beta\lambda) \\ +R(\gamma^n-1)(1+\gamma\alpha+\gamma\beta) + S(\lambda^n-1)(1+\lambda\alpha+\lambda\beta) \end{array}\right\} w^{-j} \\ &-P\alpha(\alpha^n-1) - Q\beta(\beta^n-1) - R\gamma(\gamma^n-1) - S\lambda(\lambda^n-1) \end{aligned}$$

and denominator

$$-w^{-4j} - 2w^{-3j} - w^{-2j} + 1$$

for $\lambda_j\left(C(PT_n)\right)$. For the given values, we obtain

$$\lambda_j\left(C(PT_n)\right)=\frac{\left\{\begin{array}{c}(-PT_{n-1}+P+Q+R+S)w^{-3j}\\+(-PT_n+P\alpha+Q\beta-R\gamma-S\lambda)w^{-2j}\\+\left\{\begin{array}{c}P(\alpha+1)(1-\alpha^n)+Q(1-\beta^n)(\beta+1)\\+R(\gamma+1)(\gamma^n-1)+S(\lambda+1)(\lambda^n-1)\end{array}\right\}w^{-j}\\-PT_n+P\alpha+Q\beta+R\gamma+S\lambda\end{array}\right\}}{-w^{-4j}-2w^{-3j}-w^{-2j}+1}$$

and then

$$\lambda_j\left(C(PT_n)\right)=\frac{dw^{-3j}+cw^{-2j}+bw^{-j}+a}{-w^{-4j}-2w^{-3j}-w^{-2j}+1}.$$

□

Now, we deduce the determinants of the PT_n matrices.

THEOREM 3.7. *Let PT denote the circulant matrices of (PT_n). Then the determinants are*

$$\det(C(PT_n))=\frac{a^n-(-d)^n+2^{1-n}\left(\frac{2ad-c}{b}\right)^n+2^n\left(\frac{ad}{b}\right)^n+(2^{-n}-2^{1-2n})(-b)^n}{(-1)^n(1+2^{1-n})+2^{2-n}+2^{2-4n}+1}.$$

PROOF. Recall that the eigenvalue of $C(PT_n)$ and Lemma 1.2 of [**8**], we have

$$\begin{aligned}&\det(C(PT_n))\\=&\prod_{j=0}^{n-1}\lambda_j(C(PT_n))\\=&\prod_{j=0}^{n-1}\frac{dw^{-3j}+cw^{-2j}+bw^{-j}+a}{-w^{-4j}-2w^{-3j}-w^{-2j}+1}\\=&\frac{a^n-(-d)^n+2^{1-n}\left(\frac{2ad-c}{b}\right)^n+2^n\left(\frac{ad}{b}\right)^n+(2^{-n}-2^{1-2n})(-b)^n}{(-1)^n(1+2^{1-n})+2^{2-n}+2^{2-4n}+1}.\end{aligned}$$

□

There are many articles dealing with the norms of matrices [**4, 6, 14**]. Let $A=(a_{ij})_{nxn}$ be an matrix, Euclidean norm and spectral norm of the matrix A are defined as

$$\|A\|_E=\left(\sum_{i,j=1}^{n}|b_{ij}|^2\right)^{1/2}$$

$$\|A\|_{spec}=\left(\max_{1\leqslant i\leqslant n}\lambda_i(A^*A)\right)^{1/2},$$

where A^*represent the conjugate transpose of A.

THEOREM 3.8. *Let PT denote the circulant matrices of (PT_n). Then the Euclidean norms are*

$$\|C(PT_n)\|_E = \sqrt{\left\{\begin{array}{c} nP^2\alpha^4\frac{1-\alpha^{2n-2}}{1-\alpha^2} + nQ^2\beta^4\frac{1-\beta^{2n-2}}{1-\beta^2} + nR^2\gamma^4\frac{1-\gamma^{2n-2}}{1-\gamma^2} + nS^2\delta^4\frac{1-\lambda^{2n-2}}{1-\lambda^2} \\ +2n\left\{\begin{array}{c} PQ\frac{1-(\alpha\beta)^{n-1}}{1-\alpha\beta} + RS\frac{1-(\gamma\lambda)^{n-1}}{1-\gamma\lambda} \\ +PR(\alpha\gamma)^2\frac{1-(\alpha\gamma)^{n-1}}{1-\alpha\gamma} + PS(\alpha\lambda)^2\frac{1-(\alpha\lambda)^{n-1}}{1-\alpha\lambda} \\ +QR(\beta\gamma)^2\frac{1-(\beta\gamma)^{n-1}}{1-\beta\gamma} + QS(\beta\lambda)^2\frac{1-(\beta\lambda)^{n-1}}{1-\beta\lambda} \end{array}\right\} \end{array}\right\}}.$$

PROOF. From the defination of the Euclidean norm, we know

$$\|C(PT_n)\|_E^2 = n\sum_{i=0}^{n-1} PT_i^2.$$

From Binet formulas for the Pell-Padovan Tetranacci sequence, we obtain

$$\begin{aligned}
\sum_{i=1}^{n-1} PT_i^2 &= \sum_{i=1}^{n-1}\left(P\alpha^{i+1} + Q\beta^{i+1} + R\gamma^{i+1} + S\lambda^{i+1}\right)^2 \\
&= P^2\sum_{i=1}^{n-1}\alpha^{2(i+1)} + Q^2\sum_{i=1}^{n-1}\beta^{2(i+1)} + R^2\sum_{i=1}^{n-1}\gamma^{2(i+1)} + S^2\sum_{i=1}^{n-1}\lambda^{2(i+1)} \\
&\quad +2PQ\sum_{i=1}^{n-1}(\alpha\beta)^{(i+1)} + 2RS\sum_{i=1}^{n-1}(\gamma\lambda)^{(i+1)} + 2PR\sum_{i=1}^{n-1}(\alpha\gamma)^{(i+1)} \\
&\quad +2PS\sum_{i=1}^{n-1}(\alpha\lambda)^{(i+1)} + 2QR\sum_{i=1}^{n-1}(\beta\gamma)^{(i+1)} + 2QS\sum_{i=1}^{n-1}(\beta\lambda)^{(i+1)}.
\end{aligned}$$

We know $\sum_{s=1}^{n} i^s = \frac{i-i^{n+1}}{1-i}$, hence

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n-1} PT_i^2 = \\
&P^2\alpha^4\frac{1-\alpha^{2n-2}}{1-\alpha^2} + Q^2\beta^4\frac{1-\beta^{2n-2}}{1-\beta^2} + R^2\gamma^4\frac{1-\gamma^{2n-2}}{1-\gamma^2} + S^2\delta^4\frac{1-\lambda^{2n-2}}{1-\lambda^2} \\
&+2\left(\begin{array}{c} PQ\frac{1-(\alpha\beta)^{n-1}}{1-\alpha\beta} + RS\frac{1-(\gamma\lambda)^{n-1}}{1-\gamma\lambda} + PR(\alpha\gamma)^2\frac{1-(\alpha\gamma)^{n-1}}{1-\alpha\gamma} \\ +PS(\alpha\lambda)^2\frac{1-(\alpha\lambda)^{n-1}}{1-\alpha\lambda} + QR(\beta\gamma)^2\frac{1-(\beta\gamma)^{n-1}}{1-\beta\gamma} + QS(\beta\lambda)^2\frac{1-(\beta\lambda)^{n-1}}{1-\beta\lambda} \end{array}\right).
\end{aligned}$$

So we get the result

$$\|C(PT_n)\|_E^2 = n\left(\sum_{i=1}^{n-1} PT_i^2 + PT_0^2\right) = n\sum_{i=1}^{n-1} PT_i^2.$$

□

The spectral norm of PT_n is given by the following theorem which can be proved by induction on n.

THEOREM 3.9. *Let PT denote the circulant matrices of (PT_n). The spectral norm of PT is*

$$||PT||_{spec} = \frac{1}{3}\left(PT_{n+4} + PT_{n+3} - 2PT_{n+1} - 2\right)$$

for $n \geqslant 4$.

PROOF. For the matrix $C(PT_n)$, its spectral radius $\rho(C(PT_n))$ is obtained from the following inequality,

$$\min_{1\leqslant i\leqslant n} \sum_{j=1}^{n} a_{ij} \leqslant \rho(C(PT_n)) \leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} a_{ij}$$

then

$$\sum_{j=1}^{n} a_{ij} = \sum_{l=0}^{n-1} PT_l = \frac{1}{3}\left(PT_{n+4} + PT_{n+3} - 2PT_{n+1} - 2\right)$$

for any $i = 1, 2, \cdots, n$. Then

$$\|C(PT_n)\|_{spec} = \frac{1}{3}\left(PT_{n+4} + PT_{n+3} - 2PT_{n+1} - 2\right).$$

□

CONCLUSION 3.1. *In this paper we introduced the Pell-Padovan Tetranacci sequence. We obtained Binet-like formula of this sequence. We studied the generating function and partial sum of this sequence. We investigated some interesting identities and examples about this sequence. Also we deduce norms, determinants and eigenvalues of the circulant matrices for the Pell-Padovan Tetranacci sequence.*

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