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NEW OPERATIONS DEFINED OVER THE Q-RUNG ORTHOPAIR FUZZY SETS

I. Silambarasan

Abstract. In this paper, we define some new operations ($A@B$), ($A\$B$), ($A\#B$), ($A*B$), ($A\to B$) of q-rung orthopair fuzzy sets. Then we discuss several properties of these operations. Further, we prove necessity and possibility operations of q-rung orthopair fuzzy sets. Finally, we have identified and proved several of these properties, particularly those involving the operator $A\to B$ defined as q-rung orthopair fuzzy implication with other operations.

1. Introduction and Preliminaries

Zadeh [**11**] fuzzy set (FS) has acquired greater attention by researchers in a wide range of scientific areas, including management sciences, robotics, decision theory. and many other disciplines. FSs were further generalized to intuitionistic fuzzy sets (IFSs) by Atanassov [**1**] in 1983. An IFS is distinguished by a membership and nonmembership satisfying the condition that the sum of both membership degrees should not exceed one. IF values play an important role in both theoretical and practical progress of IFSs. Applications of IFSs appear in various fields, including medical diagnosis, optimization problems, and decision-making. But if the sum of the membership degree and the nonmembership degree is greater than 1, the IFS is no longer applicable. Yager [**8**] proposed the Pythagorean fuzzy set (PFS) $A=\{x,\mu_A(x),\nu_A(x)|x\in X\}$, where the squared sum of its membership degree $\mu_A(x)\in[0,1]$ and nonmembership degree $\nu_A(x)\in[0,1]$ is less than or equal to 1. Since the PFS was brought up, it has been widely applied in different fields, such

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as investment decision making, service quality of domestic airline, collaborative-based recommender systems, and so on. Although the PFS generalizes the IFS, it cannot describe the following decision information. A panel of experts were invited to give their opinions about the feasibility of an investment plan, and they were divided into two independent groups to make a decision. One group considered the degree of the feasibility of the investment plan as 0.9, while the other group considered the nonmembership degree as 0.6. It was clearly seen that $0.9 + 0.6 > 1, (0.9)^2 + (0.6)^2 > 1$, and thus it could not be described by IFS and PFS. After the IFS and PFS theory, many researchers [**2, 4, 5, 6, 7, 9**] attempted the important role in this theory. With continuous complication of society and the development of theory, a new concept was presented again by Yager [**10**], the q-rung orthopair fuzzy sets (q-ROFS),in which the sum of the q^{th} power of the membership degree and the q^{th} power of the degrees of non-membership is restricted to one. We can find that the q-ROFS are general because IFSs and PFSs are all their special cases. It is worth noting that as the rung "q" increases, the space of acceptable orthopairs increases, and more orthopairs satisfy the bounding constraint. So we can express a wider range of fuzzy information by using q-ROFS. In other words, we can continue to adjust the value of the parameter "q" to determine the information expression range, thus q-ROFS are more flexible and more suitable for the uncertain environment. In recent years, the topic of information aggregation has attracted a lot of attention and is one of the key research issues in the problems of MAGDM. As far as q-ROFS is concerned, different aggregation operations have been introduced and applied, such as q-ROFWA and q-ROFWG operator [**3**]. In this paper, some new operations for q-rung orthopair fuzzy sets are defined and several properties are discussed.

Definition 1.1. ([**10**]) A q-rung orthopair fuzzy set A on a universe X is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) \,|x \in X\}$, where $\mu_A(x) \in [0,1]$ is called the degree of membership of x in A, $\nu_A(x) \in [0,1]$ is called the degree of non-membership of x in A, and where $\mu_A(x)$ and $\nu_A(x)$ satisfy the following condition: $0 \leqslant \mu_A^q(x) + \nu_A^q(x) \leqslant 1$ for all $x \in X$

IFS and PFS operations on q-ROFS

Definition 1.2. ([**10**]) Let $q - ROFS(X)$ denote the family of all $q - ROFSs$ on the universe X, and let $A, B \in q - ROFS(X)$ be given as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\} \text{ and } B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \,|x \in X\}.$$

Then following q-ROFS operations are defined:

(i) $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}\rangle \,|x \in X\}$

(ii) $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle \,|x \in X\}$

(iii) $A^C = \{\langle x, (\nu_A(x)), (\mu_A(x))\rangle \,|x \in X\}$

(iv) $A \boxplus_q B = \left\{\left(x, (\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}, \nu_A(x)\nu_B(x)\right) |x \in H\right\}$

(v) $A \boxtimes_q B = \left\{\left(x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\right) |x \in H\right\}.$

LEMMA 1.1 ([**2**]). *For any two numbers* $a, b \in [0,1]$, *the following hold*

$$a \cdot b \leqslant \min\{a,b\} \leqslant \frac{2(a \cdot b)}{a+b} \leqslant \sqrt{a \cdot b} \leqslant \max\{a,b\} \leqslant a+b-a \cdot b,$$

and

$$a \cdot b \leqslant \frac{a+b}{2(a+b+1)} \leqslant \frac{a+b}{2}.$$

2. New operations for q-rung orthopair fuzzy sets

In this section, we define the new q-rung orthopair fuzzy operations and investigates the algebraic properties.

IFS and PFS operations on q-ROFS

DEFINITION 2.1. Let $q - ROFS(X)$ denote the family of all $q - ROFSs$ on the universe X, and let $A, B \in q - ROFS(X)$ be given as

$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \,|x \in X\}$.

Then following q-ROFS operations are defined:

$$(i)\ A@B = \left\{\left\langle x, \left(\frac{\mu_A^q(x)+\mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x)+\nu_B^q(x)}{2}\right)^{1/q}\right\rangle |x \in X\right\}$$

$$(ii)\ A\$B = \left\{\left\langle x, (\mu_A(x)\mu_B(x))^{1/q}, (\nu_A(x)\nu_B(x))^{1/q}\right\rangle |x \in X\right\}$$

$$(iii)\ A\#B = \left\{\left\langle x, \frac{(2)^{1/q}\,\mu_A(x)\mu_B(x)}{(\mu_A^q(x)+\mu_B^q(x))^{1/q}}, \frac{(2)^{1/q}\,\nu_A(x)\nu_B(x)}{(\nu_A^q(x)+\nu_B^q(x))^{1/q}}\right\rangle |x \in X\right\}$$

For which we shall accept that if $\mu_A(x) = \mu_B(x) = 0$ then $\dfrac{\mu_A(x)\mu_B(x)}{\mu_A(x)+\mu_B(x)} = 0$ and if $\nu_A(x) = \nu_B(x) = 0$, then $\dfrac{\nu_A(x)\nu_B(x)}{\nu_A(x)+\nu_B(x)} = 0$.

$$(iv)\ A*B = \left\{\left\langle x, \left(\frac{\mu_A^q(x)+\mu_B^q(x)}{2(\mu_A^q(x)+\mu_B^q(x)+1)}\right)^{\frac{1}{q}}, \left(\frac{\nu_A^q(x)+\nu_B^q(x)}{2(\nu_A^q(x)+\nu_B^q(x)+1)}\right)^{\frac{1}{q}}\right\rangle\right\}$$

$$(v)\ A \to B = \{\langle x, \max\{\nu_A(x), \mu_B(x)\}, \min\{\mu_A(x), \nu_B(x)\}\rangle \,|x \in X\}.$$

REMARK 2.1. Clearly, for each two q-ROFSs A and B, $(A@B)$,$(A\$B)$,$(A\#B)$, $(A*B)$,$(A \to B)$ are as yet an q-ROFS. Some basic representations are appear as follows:

For (i),

$$0 \leqslant \left(\left(\frac{\mu_A^q(x)+\mu_B^q(x)}{2}\right)^{1/q}\right)^q + \left(\left(\frac{\nu_A^q(x)+\nu_B^q(x)}{2}\right)^{1/q}\right)^q$$

$$= \frac{\mu_A^q(x)+\nu_A^q(x)}{2} + \frac{\mu_B^q(x)+\nu_B^q(x)}{2} \leqslant \frac{1}{2}+\frac{1}{2} = 1.$$

For (ii),

If $\nu_A(x) \geqslant \mu_B(x)$ and $\mu_A(x) \geqslant \nu_B(x)$, then
$0 \leqslant \max\{\nu_A^q(x), \mu_B^q(x)\} + \min\{\mu_A^q(x), \nu_B^q(x)\}$
$\leqslant \nu_A^q(x) + \mu_A^q(x) \leqslant 1.$

If $\nu_A(x) \geqslant \mu_B(x)$ and $\mu_A(x) \leqslant \nu_B(x)$, then
$0 \leqslant \max\{\nu_A^q(x), \mu_B^q(x)\} + \min\{\mu_A^q(x), \nu_B^q(x)\}$
$\leqslant \nu_A^q(x) + \mu_A^q(x) \leqslant 1.$

If $\nu_A(x) \leqslant \mu_B(x)$ and $\mu_A(x) \geqslant \nu_B(x)$, then
$0 \leqslant \max\{\nu_A^q(x), \mu_B^q(x)\} + \min\{\mu_A^q(x), \nu_B^q(x)\}$
$\leqslant \nu_B^q(x) + \mu_B^q(x) \leqslant 1.$

If $\nu_A(x) \leqslant \mu_B(x)$ and $\mu_A(x) \leqslant \nu_B(x)$, then
$0 \leqslant \max\{\nu_A^q(x), \mu_B^q(x)\} + \min\{\mu_A^q(x), \nu_B^q(x)\}$
$\leqslant \nu_B^q(x) + \mu_B^q(x) \leqslant 1.$

For (iii),

$$0 \leqslant \left((\mu_A(x)\mu_B(x))^{1/q}\right)^q + \left((\nu_A(x)\nu_B(x))^{1/q}\right)^q = \mu_A(x)\mu_B(x) + \nu_A(x)\nu_B(x)$$
$$= \frac{\mu_A^q(x) + \nu_A^q(x)}{2} + \frac{\mu_B^q(x) + \nu_B^q(x)}{2}$$
$$\leqslant \frac{1}{2} + \frac{1}{2} = 1.$$

For (iv),

$$0 \leqslant \frac{2\mu_A^q(x)\mu_B^q(x)}{\mu_A^q(x) + \mu_B^q(x)} + \frac{2\nu_A^q(x)\nu_B^q(x)}{\nu_A^q(x) + \nu_B^q(x)} \leqslant 1.$$

For (v),

$$0 \leqslant \left(\left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2(\mu_A^q(x) + \mu_B^q(x) + 1)}\right)^{1/q}\right)^q + \left(\left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2(\nu_A^q(x) + \nu_B^q(x) + 1)}\right)^{1/q}\right)^q$$
$$= \frac{\mu_A^q(x) + \mu_B^q(x)}{2(\mu_A^q(x) + \mu_B^q(x) + 1)}, \frac{\nu_A^q(x) + \nu_B^q(x)}{2(\nu_A^q(x) + \nu_B^q(x) + 1)} \leqslant 1.$$

Example 2.1. For understanding the q-ROFS better, we give an instance to illuminate the understandability of the q-ROFS: We can definitely get $0.9+0.6 > 1$, and, therefore, it does not follow the condition of intuitionistic fuzzy sets. Also, we can get $(0.9)^2+(0.6)^2 = 0.81+0.36 = 1.17 > 1$, which does not obey the constraint condition of Pythagorean fuzzy set. However, we can get $(0.9)^q+(0.6)^q \leqslant 1$ $(q \geqslant 1)$. which is good enough to apply the q-rung orthopair fuzzy set to control it.

Theorem 2.1. *For $A, B \in q - ROFS(X)$, the following holds*
(i) $A@B = B@A = (A^C@B^C)^C$,
(ii) $A\$B = B\$A = (A^C\$B^C)^C$,
(iii) $A\#B = B\#A = (A^C\#B^C)^C$,
(iv) $A * B = B * A = (A^C * B^C)^C$.

PROOF. Let us (i) prove. Then other claims can be proved similarly.
(i) Let A and B be two given q-ROFSs, then

$$\begin{aligned} A@B &= \left\{ \left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q} \right\rangle | x \in X \right\} \\ &= \left\{ \left\langle x, \left(\frac{\mu_B^q(x) + \mu_A^q(x)}{2} \right)^{1/q}, \left(\frac{\nu_B^q(x) + \nu_A^q(x)}{2} \right)^{1/q} \right\rangle | x \in X \right\} \\ &= B@A. \end{aligned}$$

$$A^C@B^C = \left\{ \left\langle x, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q}, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q} \right\rangle | x \in X \right\}$$

$$\begin{aligned} (A^C@B^C)^C &= \left\{ \left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q} \right\rangle | x \in X \right\} \\ &= A@B. \end{aligned}$$

Hence, $A@B = B@A = (A^C@B^C)^C$. □

The following theorems are obvious.

THEOREM 2.2. *For $A, B, C \in q - ROFS(X)$, the following holds*
(i) $(A \cap B)@C = (A@C) \cap (B@C)$,
(ii) $(A \cup B)@C = (A@C) \cup (B@C)$,
(iii) $(A \cap B)\$C = (A\$C) \cap (B\$C)$,
(iv) $(A \cup B)\$C = (A\$C) \cup (B\$C)$,
(v) $(A \cap B)\#C = (A\#C) \cap (B\#C)$,
(vi) $(A \cup B)\#C = (A\#C) \cup (B\#C)$,
(vii) $(A \cap B) * C = (A * C) \cap (B * C)$,
(viii) $(A \cup B) * C = (A * C) \cup (B * C)$.

THEOREM 2.3. *For $A, B, C \in q - ROFS(X)$, the following holds*
(i) $(A \boxplus_q B)@C \subseteq (A@C) \boxplus_q (B@C)$,
(ii) $(A \boxtimes_q B)@C \supseteq (A@C) \boxtimes_q (B@C)$
(iii) $(A \boxplus_q B)\$C \subseteq (A\$C) \boxplus_q (B\$C)$,
(vi) $(A \boxtimes_q B)\$C \supseteq (A\$C) \boxtimes_q (B\$C)$,
(v) $(A \boxplus_q B) * C \subseteq (A * C) \boxplus_q (B * C)$,
(vi) $(A \boxtimes_q B) * C \supseteq (A * C) \boxtimes_q (B * C)$.

THEOREM 2.4. *For $A, B, C \in q - ROFS(X)$, the following holds*
(i) $(A@B) \boxplus_q C = (A \boxplus_q C)@(B \boxplus_q C)$,
(ii) $(A@B) \boxtimes_q C = (A \boxtimes_q C)@(B \boxtimes_q C)$,
(iii) $(A\$B) \boxplus_q C \subseteq (A \boxplus_q C)\$(B \boxplus_q C)$,
(iv) $(A\$B) \boxtimes_q C \supseteq (A \boxtimes_q C)\$(B \boxtimes_q C)$,

(v) $(A\#B) \boxplus_q C \subseteq (A \boxplus_q C)\#(B \boxplus_q C)$,

(vi) $(A\#B) \boxtimes_q C \supseteq (A \boxtimes_q C)\#(B \boxtimes_q C)$,

(vii) $(A * B) \boxplus_q C \subseteq (A \boxplus_q C) * (B \boxplus_q C)$,

(viii) $(A * B) \boxtimes_q C \supseteq (A \boxtimes_q C) * (B \boxtimes_q C)$.

3. Necessity and possibility operations on q-rung orthopair fuzzy sets

In this section, we prove the necessity and possibility operations of q-rung orthopair fuzzy sets. Then we compile some relevent properties of these operations are discussed.

Definition 3.1. ([**6**]) The necessity and possibility operations on a q-rung orthopair fuzzy set A is denoted by $\Box A, \Diamond A$ and is

(i) $\Box A = \left\{x, \left\langle \mu_A(x), (1-\mu_A^q(x))^{1/q} \right\rangle |x \in X\right\}$,

(ii) $\Diamond A = \left\{x, \left\langle (1-\nu_A^q(x))^{1/q}, \nu_A(x) \right\rangle |x \in X\right\}$.

Theorem 3.1. *For $A, B \in q-ROFS(X)$, the following holds*

(i) $\Box A @ \Box B = \Box(A@B) \subseteq \Diamond A @ \Diamond B = \Diamond(A@B)$,

(ii) $\Box(A\$B) \subseteq \Box A\$\Box B \subseteq \Diamond A\$\Diamond B \subseteq \Diamond(A\$B)$,

(iii) $\Box(A\#B) \subseteq \Box A\#\Box B \subseteq \Diamond A\#\Diamond B \subseteq \Diamond(A\#B)$,

(iv) $\Box(A * B) \subseteq \Box A * \Box B \subseteq \Diamond A * \Diamond B \subseteq \Diamond(A * B)$.

Proof. Let us prove (i) and (iii). The rest can be proved similarly.

(i) $\Box A @ \Box B$

$$= \left\{\left\langle x, \left(\frac{\mu_A^q(x)+\mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{1-\mu_A^q(x)+1-\mu_B^q(x)}{2}\right)^{1/q}\right\rangle |x \in X\right\}$$

$= \Box(A@B)$.

$\Diamond A @ \Diamond B$

$$= \left\{\left\langle x, \left(\frac{1-\nu_A^q(x)+1-\nu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x)+\nu_B^q(x)}{2}\right)^{1/q}\right\rangle |x \in X\right\}$$

$= \Diamond(A@B)$.

Hence, $\Box A @ \Box B = \Box(A@B) \subseteq \Diamond A @ \Diamond B = \Diamond(A@B)$.

(iii) Let $\Box A\#\Box B$

$$= \left\{\left\langle x, \frac{(2)^{1/q}\mu_A(x)\mu_B(x)}{(\mu_A^q(x)+\mu_B^q(x))^{1/q}}, \frac{(2)^{1/q}(1-\mu_A^q(x))^{1/q}(1-\mu_B^q(x))^{1/q}}{(1-\mu_A^q(x)+1-\mu_B^q(x))^{1/q}}\right\rangle |x \in X\right\}$$

$\subseteq \Box(A\#B)$.

$\Diamond A\#\Diamond B$

$$= \left\{\left\langle x, \frac{(2)^{1/q}(1-\nu_A^q(x))^{1/q}(1-\nu_B^q(x))^{1/q}}{(1-\nu_A^q(x)+1-\nu_B^q(x))^{1/q}}, \frac{(2)^{1/q}\nu_A(x)\nu_B(x)}{(\nu_A^q(x)+\nu_B^q(x))^{1/q}}\right\rangle |x \in X\right\}$$

$\subseteq \Diamond(A\#B)$.

Hence, $\Box(A\#B) \subseteq \Box A\#\Box B \subseteq \Diamond A\#\Diamond B \subseteq \Diamond(A\#B)$. $\Box$

The following theorems are obvious.

THEOREM 3.2. *For $A, B \in q-ROFS(X)$, the following holds*

(i) $\Box\Big[(\Diamond A@\Diamond B)^C\Big] = \Big[\Diamond(A@B)\Big]^C$,

(ii) $\Diamond\Big[(\Box A@\Box B)^C\Big] = \Big[\Box(A@B)\Big]^C$;

(iii) $\Box\Big[(\Diamond A\$\Diamond B)^C\Big] = \Big[\Diamond(A\$B)\Big]^C$,

(vi) $\Diamond\Big[(\Box A\$\Box B)^C\Big] = \Big[\Box(A\$B)\Big]^C$,

(v) $\Box\Big[(\Diamond A\#\Diamond B)^C\Big] = \Big[\Diamond(A\#B)\Big]^C$,

(vi) $\Diamond\Big[(\Box A\#\Box B)^C\Big] = \Big[\Box(A\#B)\Big]^C$,

(vii) $\Box\Big[(\Diamond A * \Diamond B)^C\Big] = \Big[\Diamond(A * B)\Big]^C$,

(viii) $\Diamond\Big[(\Box A * \Box B)^C\Big] = \Big[\Box(A * B)\Big]^C$.

THEOREM 3.3. *For $A, B \in q-ROFS(X)$, th following holds*

(i) $\Big[(\Box A \boxplus_q \Diamond B)^C @((\Box A)^C \boxtimes_q \Diamond B)\Big] \cup (\Box A)^C = (\Box A)^C$,

(ii) $\Big[(\Box A \boxtimes_q \Diamond B)^C @((\Box A)^C \boxplus_q \Diamond B)\Big] \cap (\Box A)^C = (\Box A)^C$,

(iii) $\Big[(\Box A \boxplus_q \Diamond B)^C \$((\Box A)^C \boxtimes_q \Diamond B)\Big] \cup (\Box A)^C = (\Box A)^C$,

(iv) $\Big[(\Box A \boxtimes_q \Diamond B)^C \$((\Box A)^C \boxplus_q \Diamond B)\Big] \cap (\Box A)^C = (\Box A)^C$,

(v) $\Big[(\Box A \boxplus_q \Diamond B)^C \#((\Box A)^C \boxtimes_q \Diamond B)\Big] \cup (\Box A)^C = (\Box A)^C$,

(vi) $\Big[(\Box A \boxtimes_q \Diamond B)^C \#((\Box A)^C \boxplus_q \Diamond B)\Big] \cap (\Box A)^C = (\Box A)^C$,

(vii) $\Big[(\Diamond A \boxplus_q \Box B)^C @((\Box A)^C \boxtimes_q \Diamond B)\Big] \cup (\Diamond A)^C = (\Diamond A)^C$,

(viii) $\Big[(\Diamond A \boxplus_q \Box B)^C \$((\Box A)^C \boxtimes_q \Diamond B)\Big] \cup (\Diamond A)^C = (\Diamond A)^C$,

(ix) $\Big[(\Diamond A \boxplus_q \Box B)^C \#((\Box A)^C \boxtimes_q \Diamond B)\Big] \cup (\Diamond A)^C = (\Diamond A)^C$,

(x) $\Big[(\Diamond A \boxplus_q \Box B)^C @((\Diamond A)^C \boxtimes_q \Box B)\Big] \cup (\Diamond A)^C = (\Diamond A)$,

(xi) $\Big[(\Diamond A \boxplus_q \Box B)^C \$((\Diamond A)^C \boxtimes_q \Box B)\Big] \cup (\Diamond A)^C = (\Diamond A)$,

(xii) $\Big[(\Diamond A \boxplus_q \Box B)^C \#((\Diamond A)^C \boxtimes_q \Box B)\Big] \cup (\Diamond A)^C = (\Diamond A)$.

In the next section, we state and prove some new results involving implication operator with other q-ROFS operations.

4. Some properties of q-rung orthopair fuzzy implication operator

In this section, the proofs of the following theorems and corollaries follows from the Definitions 1.2 & 2.1 and Lemma 1.1.

THEOREM 4.1. *For $A, B \in q-ROFS(X)$, the following holds*

(i) $\left(A^C \to B\right) @ \left(A \to B^C\right)^C = (A@B)$,

(ii) $\left(A^C \to B\right) \boxplus_q \left(A \to B^C\right)^C = (A \boxplus_q B)$,

(iii) $\left(A^C \to B\right) \boxtimes_q \left(A \to B^C\right)^C = (A \boxtimes_q B)$,

(iv) $\left(A^C \to B\right) \$ \left(A \to B^C\right)^C = (A\$B)$,

(v) $\left(A^C \to B\right) \# \left(A \to B^C\right)^C = (A\#B)$,

(vi) $(A \to B)^C \boxplus_q (B \to A) = \left(A \boxplus_q B^C\right)$,

(vii) $(A \to B)^C @ (B \to A) = \left(A@B^C\right)$,

(viii) $(A \to B)^C \boxtimes_q (B \to A) = \left(A \boxtimes_q B^C\right)$,

(ix) $(A \to B)^C \$ (B \to A) = \left(A\$B^C\right)$,

(x) $(A \to B)^C \# (B \to A) = \left(A\#B^C\right)$.

PROOF. We will prove (i) and (vi). Results (iii), (iv), (v), (vii), (viii), (ix) and (x) can be proved analogously.

(*i*) Let $\left(A^C \to B\right) @ \left(A \to B^C\right)^C$

$$= \left[x, \left(\frac{\max\{\mu_A^q(x), \mu_B^q(x)\} + \min\{\mu_A^q(x), \mu_B^q(x)\}}{2}\right)^{1/q}, \left(\frac{\min\{\nu_A^q(x), \nu_B^q(x)\} + \max\{\nu_A^q(x), \nu_B^q(x)\}}{2}\right)^{1/q} | x \in X\right]$$

$$= \left\{x, \left\langle \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q} \right\rangle | x \in X\right\}$$

$= (A@B)$.

(*vi*) Let $(A \to B)^C \boxplus_q (B \to A)$

$$= \Big[\min\{\mu_A^q(x), \nu_B^q(x)\} + \max\{\nu_B^q(x), \mu_A^q(x)\} - \min\{\mu_A^q(x), \nu_B^q(x)\} \max\{\nu_B^q(x), \mu_A^q(x)\}\Big]^{1/q}, \max\{\nu_A(x), \mu_B(x)\} \min\{\mu_B(x), \nu_A(x)\}$$

$$= \left\{x, \left\langle \left(\mu_A^q(x) + \nu_B^q(x) - \mu_A^q(x)\nu_B^q(x)\right)^{1/q}, \nu_A(x)\mu_B(x) \right\rangle | x \in X\right\}$$

$= (A \boxplus_q B^C)$. □

THEOREM 4.2. *For $A, B \in q-ROFS(X)$, the following holds*
(i) $\big((A \boxplus_q B) \to (A@B)^C\big)^C = \big((A@B) \to (A \boxplus_q B)^C\big)^C = (A@B)$,
(ii) $\big((A \boxplus_q B)^C \to (A@B)\big) = \big((A@B)^C \to (A \boxplus_q B)\big) = (A \boxplus_q B)$,
(iii) $\big((A \boxtimes_q B) \to (A@B)^C\big)^C = \big((A@B) \to (A \boxtimes_q B)^C\big)^C = (A \boxtimes_q B)$,
(iv) $\big((A \boxtimes_q B)^C \to (A@B)\big) = \big((A@B)^C \to (A \boxtimes_q B)\big) = (A@B)$,
(v) $\big((A \boxplus_q B) \to (A\#B)^C\big)^C = \big((A\#B) \to (A \boxplus_q B)^C\big)^C = (A\#B)$,
(vi) $\big((A \boxplus_q B)^C \to (A\#B)\big) = \big((A\#B)^C \to (A \boxplus_q B)\big) = (A \boxplus_q B)$,
(vii) $\big((A \boxtimes_q B) \to (A\#B)^C\big)^C = \big((A\#B) \to (A \boxtimes_q B)^C\big)^C = (A \boxtimes_q B)$,
$(viii)$ $\big((A \boxtimes_q B)^C \to (A\#B)\big) = \big((A\#B)^C \to (A \boxtimes_q B)\big) = (A\#B)$,
(ix) $\big((A \boxplus_q B) \to (A\$B)^C\big)^C = \big((A\$B) \to (A \boxplus_q B)^C\big)^C = (A\$B)$,
(x) $\big((A \boxplus_q B)^C \to (A\$B)\big) = \big((A\$B)^C \to (A \boxplus_q B)\big) = (A \boxplus_q B)$,
(xi) $\big((A \boxtimes_q B) \to (A\$B)^C\big)^C = \big((A\$B) \to (A \boxtimes_q B)^C\big)^C = (A \boxtimes_q B)$,
(xii) $\big((A \boxtimes_q B)^C \to (A\$B)\big) = \big((A\$B)^C \to (A \boxtimes_q B)\big) = (A\$B)$,
$(xiii)$ $\big((A \boxplus_q B) \to (A \boxtimes_q B)^C\big)^C = \big((A \boxtimes_q B) \to (A \boxplus_q B)^C\big)^C = (A \boxtimes_q B)$,
(xiv) $\big((A \boxtimes_q B)^C \to (A \boxplus_q B)\big) = \big((A \boxplus_q B)^C \to (A \boxtimes_q B)\big) = (A \boxplus_q B)$.

PROOF. We prove (i), (iii), (v), (vii), (ix) and (xiii). Other results can be proved analogously.

(i) Let $\big((A \boxplus_q B) \to (A@B)^C\big)^C$

$$
\begin{aligned}
&= \Bigg[x, \min\left\{(\mu_A^q(x) + \mu_A^q(x) - \mu_A^q(x)\mu_A^q(x))^{1/q}, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}\right\},\\
&\qquad \max\left\{\nu_A(x)\nu_B(x), \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q}\right\} |x \in X\Bigg]\\
&= \left\{\left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q}\right\rangle |x \in X\right\}\\
&= A@B
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
&\big((A@B) \to (A \boxplus_q B)^C\big)^C\\
&= \Bigg[x, \min\left\{\left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}, (\mu_A^q(x) + \mu_A^q(x) - \mu_A^q(x)\mu_A^q(x))^{1/q}\right\},\\
&\qquad \max\left\{\left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q}, \nu_A(x)\nu_B(x)\right\} |x \in X\Bigg]\\
&= \left\{\left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q}\right\rangle |x \in X\right\}
\end{aligned}
$$

$$= A@B \tag{4.2}$$

From (4.1) and (4.2)$\Rightarrow$ (i) holds. Thus,

$\big((A \boxplus_q B) \to (A@B)^C\big)^C = \big((A@B) \to (A \boxplus_q B)^C\big)^C = (A@B).$

(iii) Let $\big((A \boxtimes_q B) \to (A@B)^C\big)^C$

$$= \min\left\{\mu_A(x)\mu_B(x), \left(\frac{\mu_A^q(x)+\mu_B^q(x)}{2}\right)^{1/q}\right\},$$

$$\max\left\{(\nu_A^q(x)+\nu_B^q(x)-\nu_A^q(x)\nu_B^q(x))^{1/q}, \left(\frac{\nu_A^q(x)+\nu_B^q(x)}{2}\right)^{1/q}\right\}$$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x)+\nu_B^q(x)-\nu_A^q(x)\nu_B^q(x))^{1/q}\right\rangle \ |x \in X\right\}$$

$$= A \boxtimes_q B \tag{4.3}$$

and

$\big((A@B) \to (A \boxtimes_q B)^C\big)^C$

$$= \left[x, \min\left\{\left(\frac{\mu_A^q(x)+\mu_B^q(x)}{2}\right)^{1/q}, \mu_A(x)\mu_B(x)\right\},\right.$$

$$\left.\max\left\{\left(\frac{\nu_A^q(x)+\nu_B^q(x)}{2}\right)^{1/q}, (\nu_A^q(x)+\nu_B^q(x)-\nu_A^q(x)\nu_B^q(x))^{1/q}\right\} |x \in X\right]$$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x)+\nu_A^q(x)-\nu_A^q(x)\nu_A^q(x))^{1/q}\right\rangle |x \in X\right\}$$

$$= A \boxtimes_q B \tag{4.4}$$

From (4.3) and (4.4) $\Rightarrow$ (iii) holds. Thus,

$\big((A \boxtimes_q B) \to (A@B)^C\big)^C = \big((A@B) \to (A \boxtimes_q B)^C\big)^C = (A \boxtimes_q B).$

(v) Let $\big((A \boxplus_q B) \to (A\#B)^C\big)^C$

$$= \left[x, \min\left\{(\mu_A^q(x)+\mu_B^q(x)-\mu_A^q(x)\mu_B^q(x))^{1/q}, \frac{(2)^{1/q}\,\mu_A(x)\mu_B(x)}{(\mu_A^q(x)+\mu_B^q(x))^{1/q}}\right\},\right.$$

$$\left.\max\left\{\nu_A(x)\nu_B(x), \frac{(2)^{1/q}\,\nu_A(x)\nu_B(x)}{(\nu_A^q(x)+\nu_B^q(x))^{1/q}}\right\} |x \in X\right]$$

$$= \left\{\left\langle x, \frac{(2)^{1/q}\,\mu_A(x)\mu_B(x)}{(\mu_A^q(x)+\mu_B^q(x))^{1/q}}, \frac{(2)^{1/q}\,\nu_A(x)\nu_B(x)}{(\nu_A^q(x)+\nu_B^q(x))^{1/q}}\right\rangle |x \in X\right\}$$

$$= A\#B \tag{4.5}$$

and

$\big((A\#B) \to (A \boxplus_q B)^C\big)^C$

$$= \left[x, \min\left\{\frac{(2)^{1/q}\,\mu_A(x)\mu_B(x)}{(\mu_A^q(x)+\mu_B^q(x))^{1/q}}, (\mu_A^q(x)+\mu_B^q(x)-\mu_A^q(x)\mu_B^q(x))^{1/q}\right\},\right.$$

$$\left.\max\left\{\frac{(2)^{1/q}\,\nu_A(x)\nu_B(x)}{(\nu_A^q(x)+\nu_B^q(x))^{1/q}}, \nu_A(x)\nu_B(x)\right\} |x \in X\right]$$

$$= \left\{ \left\langle x, \frac{(2)^{1/q}\,\mu_A(x)\mu_B(x)}{\left(\mu_A^q(x)+\mu_B^q(x)\right)^{1/q}}, \frac{(2)^{1/q}\,\nu_A(x)\nu_B(x)}{\left(\nu_A^q(x)+\nu_B^q(x)\right)^{1/q}} \right\rangle |x \in X \right\}$$

$$= A\#B \tag{4.6}$$

From (4.5) and (4.6)⇒ (v) holds. Thus,

$$\left((A \boxplus_q B) \to (A\#B)^C\right)^C = \left((A\#B) \to (A \boxplus_q B)^C\right)^C = (A\#B).$$

(vii) Let $\left((A \boxtimes_q B) \to (A\#B)^C\right)^C$

$$= \min\left\{ \mu_A(x)\mu_B(x), \frac{(2)^{1/q}\,\mu_A(x)\mu_B(x)}{\left(\nu_A^q(x)+\nu_B^q(x)\right)^{1/q}} \right\},$$

$$\max\left\{ \left(\nu_A^q(x)+\nu_B^q(x)-\nu_A^q(x)\nu_B^q(x)\right)^{1/q}, \frac{(2)^{1/q}\,\nu_A(x)\nu_B(x)}{\left(\nu_A^q(x)+\nu_B^q(x)\right)^{1/q}} \right\}$$

$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), \left(\nu_A^q(x)+\nu_B^q(x)-\nu_A^q(x)\nu_B^q(x)\right)^{1/q} \right\rangle |x \in X \right\}$$

$$= A \boxtimes_q B \tag{4.7}$$

and

$$\left((A\#B) \to (A \boxtimes_q B)^C\right)^C$$

$$= \min\left\{ \frac{(2)^{1/q}\,\mu_A(x)\mu_B(x)}{\left(\nu_A^q(x)+\nu_B^q(x)\right)^{1/q}}, \mu_A(x)\mu_B(x) \right\},$$

$$\max\left\{ \frac{(2)^{1/q}\,\nu_A(x)\nu_B(x)}{\left(\nu_A^q(x)+\nu_B^q(x)\right)^{1/q}}, \left(\nu_A^q(x)+\nu_B^q(x)-\nu_A^q(x)\nu_B^q(x)\right)^{1/q} \right\}$$

$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), \left(\nu_A^q(x)+\nu_B^q(x)-\nu_A^q(x)\nu_B^q(x)\right)^{1/q} \right\rangle |x \in X \right\}$$

$$= A \boxtimes_q B \tag{4.8}$$

From (4.7) and (4.8) ⇒ (vii) holds. Thus,

$$\left((A \boxtimes_q B) \to (A\#B)^C\right)^C = \left((A\#B) \to (A \boxtimes_q B)^C\right)^C = (A \boxtimes_q B).$$

(ix) Let $\left((A \boxplus_q B) \to (A\$B)^C\right)^C$

$$= \left[x, \min\left\{ \left(\mu_A^q(x)+\mu_B^q(x)-\mu_A^q(x)\mu_B^q(x)\right)^{1/q}, \left(\mu_A(x)\mu_B(x)\right)^{1/q} \right\}, \right.$$

$$\left. \max\left\{ \nu_A(x)\nu_B(x), \left(\nu_A(x)\nu_B(x)\right)^{1/q} \right\} \ |x \in X \right]$$

$$= \left\{ \left\langle x, \left(\mu_A(x)\mu_B(x)\right)^{1/q}, \left(\nu_A(x)\nu_B(x)\right)^{1/q} \right\rangle |x \in X \right\}$$

$$= A\$B \tag{4.9}$$

and

$$\left((A\$B) \to (A \boxplus_q B)^C\right)^C$$

$$= \left[x, \min\left\{ \left(\mu_A(x)\mu_B(x)\right)^{1/q}, \left(\mu_A^q(x)+\mu_A^q(x)-\mu_A^q(x)\mu_A^q(x)\right)^{1/q} \right\}, \right.$$

$$\left. \max\left\{ \left(\nu_A(x)\nu_B(x)\right)^{1/q}, \nu_A(x)\nu_B(x) \right\} \ |x \in X \right]$$

$$= \left\{ \left\langle x, (\mu_A(x)\mu_B(x))^{1/q}, (\nu_A(x)\nu_B(x))^{1/q} \right\rangle |x \in X \right\}$$
$$= A\$B \tag{4.10}$$

From (4.9) and (4.10) $\Rightarrow$ (ix) holds. Thus,

$$\left((A \boxplus_q B) \to (A\$B)^C\right)^C = \left((A\$B) \to (A \boxplus_q B)^C\right)^C = (A\$B).$$

(xi) Let $\left((A \boxtimes_q B) \to (A\$B)^C\right)^C$

$$= \Bigg[x, \min\left\{ \mu_A(x)\mu_B(x), (\mu_A(x)\mu_B(x))^{1/q} \right\},$$
$$\max\left\{ (\nu_A^q(x) + \nu_A^q(x) - \nu_A^q(x)\nu_A^q(x))^{1/q}, (\nu_A(x)\nu_B(x))^{1/q} \right\} \ |x \in X \Bigg]$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_A^q(x) - \nu_A^q(x)\nu_A^q(x))^{1/q} \right\rangle |x \in X \right\}$$
$$= A \boxtimes_q B \tag{4.11}$$

and

$$\left((A\$B) \to (A \boxtimes_q B)^C\right)^C$$
$$= \Bigg[x, \min\left\{ (\mu_A(x)\mu_B(x))^{1/q}, \mu_A(x)\mu_B(x) \right\},$$
$$\max\left\{ (\nu_A(x)\nu_B(x))^{1/q}, (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\} |x \in X \Bigg]$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\rangle |x \in X \right\}$$
$$= A \boxtimes_q B \tag{4.12}$$

From (4.11) and (4.12) $\Rightarrow$ (xi) holds. Thus,

$$\left((A \boxtimes_q B) \to (A\$B)^C\right)^C = \left((A\$B) \to (A \boxtimes_q B)^C\right)^C = (A \boxtimes_q B).$$

$(xiii)$ Let $\left((A \boxplus_q B) \to (A \boxtimes_q B)^C\right)^C$

$$= \Bigg[x, \min\left\{ (\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}, \mu_A(x)\mu_B(x) \right\},$$
$$\max\left\{ \nu_A(x)\nu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\} \ |x \in X \Bigg]$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\rangle |x \in X \right\}$$
$$= A \boxtimes_q B \tag{4.13}$$

and

$$\left((A \boxtimes_q B) \to (A \boxplus_q B)^C\right)^C$$
$$= \Bigg[x, \min\left\{ \mu_A(x)\mu_B(x), (\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q} \right\},$$
$$\max\left\{ (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}, \nu_A(x)\nu_B(x) \right\} \ |x \in X \Bigg]$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\rangle |x \in X \right\}$$

$$= A \boxtimes_q B \tag{4.14}$$

From (4.13) and (4.14)$\Rightarrow$ (xiii) holds. Thus,

$$\big((A \boxplus_q B) \to (A \boxtimes_q B)^C\big)^C = \big((A \boxtimes_q B) \to (A \boxplus_q B)^C\big)^C = (A \boxtimes_q B). \qquad \square$$

The proof of the following Corollaries follows from Theorem 4.2.

Corollary 4.1. *For $A, B \in q - ROFS(X)$, the following holds*

$$\begin{aligned}
&\big((A \boxtimes_q B) \to (A@B)^C\big)^C = \big((A@B) \to (A \boxtimes_q B)^C\big)^C \\
&= \big((A \boxtimes_q B) \to (A\#B)^C\big)^C = \big((A\#B) \to (A \boxtimes_q B)^C\big)^C \\
&= \big((A \boxtimes_q B) \to (A\$B)^C\big)^C = \big((A\$B) \to (A \boxtimes_q B)^C\big)^C \\
&= \big((A \boxplus_q B) \to (A \boxtimes_q B)^C\big)^C = \big((A \boxtimes_q B) \to (A \boxplus_q B)^C\big)^C \\
&= (A \boxtimes_q B).
\end{aligned}$$

Corollary 4.2. *For $A, B \in q - ROFS(X)$, the following holds*

$$\begin{aligned}
&\big((A \boxplus_q B)^C \to (A@B)\big) = \big((A@B)^C \to (A \boxplus_q B)\big) \\
&= \big((A \boxplus_q B)^C \to (A\#B)\big) = \big((A\#B)^C \to (A \boxplus_q B)\big) \\
&= \big((A \boxplus_q B)^C \to (A\$B)\big) = \big((A\$B)^C \to (A \boxplus_q B)\big) \\
&= \big((A \boxtimes_q B)^C \to (A \boxplus_q B)\big) = \big((A \boxplus_q B)^C \to (A \boxtimes_q B)\big) \\
&= (A \boxplus_q B).
\end{aligned}$$

Theorem 4.3. *For $A, B \in q - ROFS(X)$, the following holds*

$$\Big[\big(A^C \to B\big) \boxplus_q \big(A \to B^C\big)^C\Big] @ \Big[\big(A^C \to B\big) \boxtimes_q \big(A \to B^C\big)^C\Big] = (A@B)\,.$$

Proof. Let $\Big[\big(A^C \to B\big) \boxplus_q \big(A \to B^C\big)^C\Big]$

$$= \Big\{\Big\langle x, (\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}, \nu_A(x)\nu_B(x)\Big\rangle \,|x \in X\Big\} \tag{4.15}$$

and

$$\Big[\big(A^C \to B\big) \boxtimes_q \big(A \to B^C\big)^C\Big]$$

$$= \Big\{\Big\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\Big\rangle \,|x \in X\Big\} \tag{4.16}$$

Now with @ of (4.15) and (4.16),

$$\Big[\big(A^C \to B\big) \boxplus_q \big(A \to B^C\big)^C\Big] @ \Big[\big(A^C \to B\big) \boxtimes_q \big(A \to B^C\big)^C\Big]$$

$$\begin{aligned}
&= \Bigg[x, \left(\frac{\Big((\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}\Big)^q + \mu_A^q(x)\mu_B^q(x)}{2}\right)^{1/q}, \\
&\qquad \left(\frac{\nu_A^q(x)\nu_B^q(x) + \Big((\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\Big)^q}{2}\right)^{1/q} \,|x \in X\Bigg] \\
&= \left\{\left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q}\right\rangle |x \in X\right\} \\
&= (A@B).
\end{aligned}$$

$\square$

Theorem 4.4. *For $A, B \in q - ROFS(X)$, the following holds*

$$\left[\left(\left(A^C \to B\right) \boxplus_q \left(A \to B^C\right)^C\right) \bigcap \left(\left(A^C \to B\right) \boxtimes_q \left(A \to B^C\right)^C\right)\right]$$
$$@\left[\left(\left(A^C \to B\right) \boxplus_q \left(A \to B^C\right)^C\right) \bigcup \left(\left(A^C \to B\right) \boxtimes_q \left(A \to B^C\right)^C\right)\right] = (A@B).$$

PROOF. Taking with $\bigcap$ of (4.15) and (4.16), we get

$$\left[\left(\left(A^C \to B\right) \boxplus_q \left(A \to B^C\right)^C\right) \bigcap \left(\left(A^C \to B\right) \boxtimes_q \left(A \to B^C\right)^C\right)\right]$$
$$= \left[x, \min\left\{\left(\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x)\right)^{1/q}, \mu_A(x)\mu_B(x)\right\},\right.$$
$$\left.\max\left\{\nu_A(x)\nu_A(x), \left(\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x)\right)^{1/q}\right\} \ |x \in X\right]$$
$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), \left(\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x)\right)^{1/q}\right\rangle |x \in X\right\} \tag{4.17}$$

Again taking with $\bigcup$ of (4.15) and (4.16),

$$\left[\left(\left(A^C \to B\right) \boxplus_q \left(A \to B^C\right)^C\right) \bigcup \left(\left(A^C \to B\right) \boxtimes_q \left(A \to B^C\right)^C\right)\right]$$
$$= \left[x, \max\left\{\left(\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x)\right)^{1/q}, \mu_A(x)\mu_B(x)\right\},\right.$$
$$\left.\min\left\{\nu_A(x)\nu_B(x), \left(\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x)\right)^{1/q}\right\} \ |x \in X\right]$$
$$= \left\{\left\langle x, \left(\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x)\right)^{1/q}, \nu_A(x)\nu_B(x)\right\rangle |x \in X\right\} \tag{4.18}$$

Now wih @ of (4.17) and (4.18),

$$\left[\left(\left(A^C \to B\right) \boxplus_q \left(A \to B^C\right)^C\right) \bigcap \left(\left(A^C \to B\right) \boxtimes_q \left(A \to B^C\right)^C\right)\right]$$
$$@\left[\left(\left(A^C \to B\right) \boxplus_q \left(A \to B^C\right)^C\right) \bigcup \left(\left(A^C \to B\right) \boxtimes_q \left(A \to B^C\right)^C\right)\right]$$
$$= \left[x, \left(\frac{\left(\left(\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x)\right)^{1/q}\right)^q + \mu_A^q(x)\mu_B^q(x)}{2}\right)^{1/q},\right.$$
$$\left.\left(\frac{\nu_A^q(x)\nu_B^q(x) + \left(\left(\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x)\right)^{1/q}\right)^q}{2}\right)^{1/q} \ |x \in X\right]$$
$$= \left\{\left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q}\right\rangle |x \in X\right\}$$
$$= A@B. \qquad \square$$

THEOREM 4.5. *For $A, B \in q - ROFS(X)$, the following holds*

$$\left[\left((A \boxplus_q B) \to (A@B)^C\right)^C \bigcup \left((A \boxtimes_q B) \to (A@B)^C\right)^C\right]$$
$$\bigcup \left[\left((A \boxplus_q B) \to (A@B)^C\right)^C \bigcap \left((A \boxtimes_q B) \to (A@B)^C\right)^C\right] = A@B.$$

PROOF. From Theorem 4.2, we have

$$\left((A \boxplus_q B) \to (A@B)^C\right)^C$$

$$= \left\{ \left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q} \right\rangle | x \in X \right\} \tag{4.19}$$

and

$$\left((A \boxtimes_q B) \to (A@B)^C \right)^C$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_A^q(x) - \nu_A^q(x)\nu_A^q(x))^{1/q} \right\rangle | x \in X \right\} \tag{4.20}$$

Now with $\bigcup$ of (4.19) and (4.20),

$$\left[\left((A \boxplus_q B) \to (A@B)^C \right)^C \bigcup \left((A \boxtimes_q B) \to (A@B)^C \right)^C \right]$$
$$= \max \left\{ \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, (\mu_A(x)\mu_B(x)) \right\},$$
$$\min \left\{ \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q}, (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\}$$
$$= \left\{ \left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q} \right\rangle | x \in X \right\} \tag{4.21}$$

and with $\bigcap$ of (4.19) and (4.20),

$$\left[\left((A \boxplus_q B) \to (A@B)^C \right)^C \bigcap \left((A \boxtimes_q B) \to (A@B)^C \right)^C \right]$$
$$= \min \left\{ \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, (\mu_A(x)\mu_B(x)) \right\},$$
$$\max \left\{ \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q}, (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\}$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\rangle | x \in X \right\} \tag{4.22}$$

Now we consider

$$\left[\left((A \boxplus_q B) \to (A@B)^C \right)^C \bigcup \left((A \boxtimes_q B) \to (A@B)^C \right)^C \right]$$
$$\bigcup \left[\left((A \boxplus_q B) \to (A@B)^C \right)^C \bigcap \left((A \boxtimes_q B) \to (A@B)^C \right)^C \right]$$
$$= \max \left\{ \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, (\mu_A(x)\mu_B(x)) \right\},$$
$$\min \left\{ \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q}, (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\}$$
$$= \left\{ \left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q} \right\rangle | x \in X \right\}$$
$$= A@B.$$

□

THEOREM 4.6. *For $A, B \in q - ROFS(X)$, the following holds*

$$\left[\left((A \boxplus_q B) \to (A@B)^C \right)^C \bigcup \left((A \boxtimes_q B) \to (A@B)^C \right)^C \right]$$
$$\bigcap \left[\left((A \boxplus_q B) \to (A@B)^C \right)^C \bigcap \left((A \boxtimes_q B) \to (A@B)^C \right)^C \right] = A \boxtimes_q B.$$

Proof. The proof is similar to that of Theorem 4.5. $\square$

Theorem 4.7. *For $A, B \in q-ROFS(X)$, the following holds* $\big((A \boxplus_q B)^C \to (A@B)\big) @ \big((A \boxtimes_q B) \to (A@B)^C\big)^C = (A@B)$.

Proof. Let $\big((A \boxplus_q B)^C \to (A@B)\big)$

$$= \left\{\left\langle x, (\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}, \nu_A(x)\nu_B(x)\right\rangle | x \in X\right\} \tag{4.23}$$

and

$\big((A \boxtimes_q B) \to (A@B)^C\big)^C$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\right\rangle | x \in X\right\} \tag{4.24}$$

Now with @ of (4.23) and (4.24)

$$\big((A \boxplus_q B)^C \to (A@B)\big) @ \big((A \boxtimes_q B) \to (A@B)^C\big)^C$$

$$= \left(\frac{\left((\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}\right)^q + (\mu_A(x)\mu_B(x))^q}{2}\right)^{1/q},$$

$$\left(\frac{(\nu_A(x)\nu_B(x))^q + \left((\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\right)^q}{2}\right)^{1/q}$$

$$= \left\{\left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q}\right\rangle | x \in X\right\}$$

$= A@B$. $\square$

Theorem 4.8. *For $A, B \in q-ROFS(X)$, the following holds* $\big((A \boxplus_q B)^C \to (A\#B)\big) @ \big((A \boxtimes_q B) \to (A\#B)^C\big)^C = (A@B)$.

Proof. Let $\big((A \boxplus_q B)^C \to (A\#B)\big)$

$$= \left\{\left\langle x, (\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}, \nu_A(x)\nu_B(x)\right\rangle | x \in X\right\} \tag{4.25}$$

and

$\big((A \boxtimes_q B) \to (A\#B)^C\big)^C$

$$= \left\{\left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\right\rangle | x \in X\right\} \tag{4.26}$$

Now with @ of (4.25) and (4.26),

$$\big((A \boxplus_q B)^C \to (A\#B)\big) @ \big((A \boxtimes_q B) \to (A\#B)^C\big)^C$$

$$= \left(\frac{\left((\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}\right)^q + (\mu_A(x)\mu_B(x))^q}{2}\right)^{1/q},$$

$$\left(\frac{(\nu_A(x)\nu_B(x))^q + \left((\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\right)^q}{2}\right)^{1/q}$$

$$= \left\{ \left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q} \right\rangle |x \in X \right\}$$
$$= A@B.$$ □

THEOREM 4.9. *For $A, B \in q - ROFS(X)$, the following holds* $\left((A \boxplus_q B)^C \to (A\$B)\right) @ \left((A \boxtimes_q B) \to (A\$B)^C\right)^C = (A@B)$.

PROOF. Let $\left((A \boxplus_q B)^C \to (A\$B)\right)$

$$= \left\{ \left\langle x, (\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}, \nu_A(x)\nu_B(x) \right\rangle |x \in X \right\} \tag{4.27}$$

and

$$\left((A \boxtimes_q B) \to (A\$B)^C\right)^C$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\rangle |x \in X \right\} \tag{4.28}$$

Now with @ of (4.27) and (4.28),

$$\left((A \boxplus_q B)^C \to (A\$B)\right) @ \left((A \boxtimes_q B) \to (A\$B)^C\right)^C$$
$$= \left(\frac{\left((\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}\right)^q + (\mu_A(x)\mu_B(x))^q}{2} \right)^{1/q},$$
$$\left(\frac{(\nu_A(x)\nu_B(x))^q + \left((\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\right)^q}{2} \right)^{1/q}$$
$$= \left\{ \left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2} \right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2} \right)^{1/q} \right\rangle |x \in X \right\}$$
$$= A@B.$$ □

THEOREM 4.10. *For $A, B \in q - ROFS(X)$, the following holds* $\left((A \boxtimes_q B)^C \to (A \boxplus_q B)\right) @ \left((A \boxplus_q B) \to (A \boxtimes_q B)^C\right)^C = (A@B)$

PROOF. Let

$$\left((A \boxtimes_q B)^C \to (A \boxplus_q B)\right)$$
$$= \left\{ \left\langle x, (\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}, \nu_A(x)\nu_B(x) \right\rangle |x \in X \right\} \tag{4.29}$$

and

$$\left((A \boxplus_q B) \to (A \boxtimes_q B)^C\right)^C$$
$$= \left\{ \left\langle x, \mu_A(x)\mu_B(x), (\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q} \right\rangle |x \in X \right\} \tag{4.30}$$

Now with @ of (4.29) and (4.30),

$$\left((A \boxtimes_q B)^C \to (A \boxplus_q B)\right) @ \left((A \boxplus_q B) \to (A \boxtimes_q B)^C\right)^C$$
$$= \left(\frac{\left((\mu_A^q(x) + \mu_B^q(x) - \mu_A^q(x)\mu_B^q(x))^{1/q}\right)^q + (\mu_A(x)\mu_B(x))^q}{2} \right)^{1/q},$$

$$\left(\frac{(\nu_A(x)\nu_B(x))^q + \left((\nu_A^q(x) + \nu_B^q(x) - \nu_A^q(x)\nu_B^q(x))^{1/q}\right)^q}{2}\right)^{1/q}$$

$$= \left\{\left\langle x, \left(\frac{\mu_A^q(x) + \mu_B^q(x)}{2}\right)^{1/q}, \left(\frac{\nu_A^q(x) + \nu_B^q(x)}{2}\right)^{1/q}\right\rangle |x \in X\right\}$$

$$= A@B. \qquad \square$$

5. Results and discussion

More importantly, in this paper we have proposed some new operations

$$@ ,\$,\# ,* ,\rightarrow$$

for q-ROFS and discussed many interesting properties not limit to novel operations $(\boxplus_q, \boxtimes_q, \Box, \Diamond, \cap, \cup)$, which can enrich the operation theory.

6. Conclusion remarks

In this paper, we defined some new operations

$$[(A@B), (A\$B), (A\#B), (A * B), (A \rightarrow B)]$$

of q-rung orthopair fuzzy sets. Then we discussed several properties of these operations. Further we proved necessity and possibility operations of q-rung orthopair fuzzy sets. Finally, we have identified and proved several of these properties, particularly those involving the operator $A \rightarrow B$ defined as q-rung orthopair fuzzy implication with other operations. Our study prompts for further properties as also for defining possibly new operations.

7. Future scope

Thus there remains scope for studying more properties of these sets arising from those other defining set operations that may be thought of using other ways of combining the functions μ, ν. In further research, we may apply these operations in the field of different areas, for example, dynamic decision and consensus , business and marketing management, design, engineering and manufacturing, information technology and networking applications, human resources management, military applications, energy management, geographic information system applications etc.

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Department of Mathematics, Annamalai University, Annamalainagar, India
E-mail address: sksimbuking@gmail.com