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a-MAXIMAL FILTERS IN ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, the concepts of a-dense element and a-maximal filter in an ADL A (a, fixed arbitrary element in A) are introduced and discussed certain properties of these. Mainly, a-maximal filters are characterized in terms of relative a-annihilator ideals as well as a-pseudo complementations. A one-to-one correspondence is exhibited between the set of a-maximal filters of A and the set of all maximal filters of A/θ_a, where θ_a is a congruence on A corresponding to a. Furthermore, the concept of congruence relation on an a-pseudo complemented ADL A is introduced and exhibited a congruence ϕ_a on A for which A/ϕ_a is a Boolean algebra.

1. Introduction

The axiomatization of Boole's propositional two valued logic led to the concept of Boolean algebra. M. H. Stone [**2**] has proved that any Boolean algebra can be made into a Boolean ring and vice-versa. U. M. Swamy and G. C. Rao [**5**] have introduced a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebras and Boolean rings in the form of an Almost Distributive Lattice (ADL) as an algebra $(A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ which satisfies all the axioms of a distributive lattice with zero except the commutativity of the operations $\vee$ and $\wedge$ and the right distributivity of $\vee$ over $\wedge$. It is known that, in an ADL the commutativity of $\vee$ is equivalent to that of $\wedge$ and also to the right distributivity of $\vee$ over $\wedge$. In a lattice $(L, \wedge, \vee)$, interchanging the operations $\wedge$ and $\vee$ yields a lattice again, known as the dual of L. An ideal of the dual $(L, \vee, \wedge)$ is called as a filter of the lattice $(L, \wedge, \vee)$. Un like the case of a lattice, by interchanging

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the operations $\wedge$ and $\vee$ in an ADL $(A, \wedge, \vee, 0)$, we do not get an ADL again. The main reason is that in an ADL $(A, \wedge, \vee, 0)$, $\wedge$ distributes over $\vee$ both from left and right, where as $\vee$ distributes over $\wedge$ from left only. If $\vee$ distributes $\wedge$ from right also, then the ADL becomes a lattice and hence a distributive lattice. For all these reasons, we deal the case of filters in an ADL separately.

J.C Varlet [**7**] introduced the notions of a-dense element and a-maximal filter of a general semilattice S. An element x of a semilattice S is a-dense (a, fixed element of S) if $\langle x, a\rangle \subseteq (a]$, that is, for every $y \in S$, $xy \leqslant a$ implies $y \leqslant a$. It is the natural extension of the notion of dense element (that is, 0-dense element) in a semilattice S bounded bellow by 0. The properties of a-dense elements are closely linked to those of the filters maximal with respect to the property of not containing a, which are called a-maximal filters. In this paper, we extend these notions to the case of ADL's and initiate the study of the properties of these. In [**3**], we have introduced the notion of a-pseudo complementation on an ADL A for an arbitrary fixed element a of A. It is the generalization of the notion of pseudo-complementation on an ADL. It is proved that an ADL A is a-pseudo complemented if and only if the relative a-annihilator $\langle x, a\rangle$, that is $\{y \in A : x \wedge y \in (a]\}$ is a principal ideal, for each $x \in A$. Here, a-dense elements and a-maximal filters of an ADL A are characterized in terms of relative a-annihilator ideals and a-pseudo complementations. Finally, the concept of congruence relation on an ADL A with an a-pseudo complementation is introduced and obtained a congruence ϕ_a on A for which A/ϕ_a is a Boolean Algebra.

2. Preliminaries

In this article, we recall certain elementary definitions and results concerning Almost Distributive Lattices, that we need in sequel.

Definition 2.1. ([**5**]) An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following independent identities

(i) $0 \wedge a \approx 0$,
(ii) $a \vee 0 \approx a$,
(iii) $a \wedge (b \vee c) \approx (a \wedge b) \vee (a \wedge c)$,
(iv) $(a \vee b) \wedge c \approx (a \wedge c) \vee (b \wedge c)$,
(v) $a \vee (b \wedge c) \approx (a \vee b) \wedge (a \vee c)$,
(vi) $(a \vee b) \wedge b \approx b$.

Clearly any distributive lattice bounded below is an ADL, in which the smallest element is the zero element. Further any non-empty set X can be made into an ADL by fixing an arbitrarily chosen element x_0 in X and by defining the operations $\wedge$ and $\vee$ on X by

$$x \wedge y = \begin{cases} x_0 & \text{if } x = x_0 \\ y & \text{if } x \neq x_0 \end{cases} \quad \text{and} \quad x \vee y = \begin{cases} y & \text{if } x = x_0 \\ x & \text{if } x \neq x_0. \end{cases}$$

This ADL $(X, \wedge, \vee, x_0)$ is called a discrete ADL, in which x_0 is the zero element.

In the following we give an example of a non-trivial ADL.

EXAMPLE 2.1. Let $A = \{0, a, b, c, d, e\}$. Define $\wedge$ and $\vee$ on A as follows:

$\wedge$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	a	b	0	a	b
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	a	b	c	d	e
e	0	a	b	c	d	e

$\vee$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	a	a	d	d	d
b	b	b	b	e	e	e
c	c	d	e	c	d	e
d	d	d	d	d	d	d
e	e	e	e	e	e	e

Then $(A, \wedge, \vee, 0)$ is an ADL but it is not a lattice (since $a \wedge b \neq b \wedge a$) and not a discrete ADL (since $a \wedge c = 0 \neq c$)

DEFINITION 2.2. ([**5**]) Let $A = (A, \vee, \wedge, 0)$ be an ADL. For any a and $b \in A$, define

$$a \leqslant b \quad \text{if and only if } \; a = a \wedge b \; (\Leftrightarrow a \vee b = b).$$

Then $\leqslant$ is a partial order on A.

THEOREM 2.1 ([**5**]). *The following holds for any a, b and c in an ADL $A = (A, \wedge, \vee, 0)$.*

(i) 0 *is the zero element for the operation* $\wedge$ *(that is; $a \wedge 0 = 0 = 0 \wedge a$)*
(ii) 0 *is the identity for the operation* $\vee$ *(that is; $a \vee 0 = a = 0 \vee a$)*
(iii) $a \wedge a = a = a \vee a$
(iv) $a \wedge b \leqslant b \leqslant b \vee a$
(v) $(a \wedge b) \vee b = b,\; a \vee (b \wedge a) = a$ *and* $a \wedge (a \vee b) = a$
(vi) $a \wedge b = a \Leftrightarrow a \vee b = b$ *and* $a \wedge b = b \Leftrightarrow a \vee b = a$
(vii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$(viii)$ $a \vee (b \vee a) = a \vee b$
(xi) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$
(x) $a \wedge b = b \wedge a \Leftrightarrow \inf\{a, b\} = a \wedge b \;\Leftrightarrow \sup\{a, b\} = a \vee b$
(xi) *If* $a \leqslant b$, *then* $a \wedge b = a = b \wedge a$ *and* $a \vee b = b = b \vee a$
(xii) $a \wedge b \wedge c = b \wedge a \wedge c$ *and* $(a \vee b) \wedge c = (b \vee a) \wedge c$

LEMMA 2.1 ([**5**]). *Let $A = (A, \wedge, \vee, 0)$ be an ADL. Then for any $m \in A$, the following are equivalent.*

(i) *m is a maximal element with respect to $\leqslant$;*
(ii) *$m \wedge x = x$ for all $x \in A$;*
(iii) *$m \vee x = m$ for all $x \in A$.*

In any discrete ADL A, every non-zero element is maximal; for, if $a \neq 0$, then $a \wedge x = x$ for all $x \in A$. An ADL $A = (A, \wedge, \vee, 0)$ is called an associative ADL if the operation $\vee$ is associative; that is $(a \vee b) \vee c = a \vee (b \vee c)$ for all a, b and $c \in A$. Throughout this paper, A stands for an associative ADL $(A, \wedge, \vee, 0)$ with maximal elements unless otherwise mentioned.

DEFINITION 2.3. ([**5**]) Let I be a non empty subset of an ADL A. Then I is called

(*i*) an ideal of A if $a \vee b \in I$ for all a and $b \in I$ and $x \wedge a \in I$ for all $x \in I$ and $a \in A$.
(*ii*) a filter of A if $a \wedge b \in I$ for all a and $b \in I$ and $a \vee x \in I$ for all $x \in I$ and $a \in A$.

As a consequence, for any ideal I of A, $a \wedge x \in I$ for all $x \in I$ and $a \in A$ and for any filter F of A, $x \vee a \in F$ for all $x \in F$ and $a \in A$. For any $x \in A$, the smallest ideal (filter) of A containing x is called the ideal (filter) generated by x in A and is denoted by (x] and [x) respectively. It is known that

$$(x] = \{x \wedge a : a \in A\} \text{ and } [x) = \{a \vee x : a \in A\}.$$

For any $x, y \in A$, we have

$$(x] \vee (y] = (x \vee y] = (y \vee x] \quad \text{and} \quad (x] \wedge (y] = (x \wedge y] = (y \wedge x].$$

A principal ideal in an ADL may have more than one generator, unlike the case of a lattice in which any principal ideal has a unique generator. However, for any x and y in an ADL, we have $(x] = (y] \Leftrightarrow x \wedge y = y$ and $y \wedge x = x \Leftrightarrow x \vee y = x$ and $y \vee x = y \Leftrightarrow [x) = [y)$ and we denote this situation by writing $x \sim y$ and calling x and y as associates to each other.

A proper ideal (filter) P of an ADL A is said to be prime if for any $x, y \in A$, $x \wedge y$ $(x \vee y) \in P$ implies either $x \in P$ or $y \in P$. A prime ideal (filter) P of A is said to be minimal if there exists no prime ideal (filter) Q of A such that $Q \subset P$. Let us recall from [4], for any elements x and a in an ADL A, the relative a-annihilator ideal of x is defined by

$$\langle x, a\rangle = \{y \in A : x \wedge y \in (a]\}.$$

Proposition 2.1 ([**4**]). *The following hold for any x, y and a in an ADL $A = (A, \wedge, \vee, 0)$.*
(*i*) $x \leqslant y \Rightarrow \langle y, a\rangle \subseteq \langle x, a\rangle$
(*ii*) $\langle x \wedge y,\ a\rangle = \langle y \wedge x,\ a\rangle$
(*iii*) $\langle x \vee y,\ a\rangle = \langle x, a\rangle \cap \langle y, a\rangle$
(*iv*) $\langle x, a\rangle \vee \langle y, a\rangle \subseteq \langle x \wedge y,\ a\rangle$
(*v*) $x \in (a] \Leftrightarrow \langle x, a\rangle = A$
(*vi*) $\langle a, a\rangle = A = \langle 0, a\rangle$
(*vii*) *a is maximal $\Leftrightarrow \langle x, a\rangle = A$ for all $x \in A$.*

Definition 2.4. ([**6**]) Let $A = (A, \wedge, \vee, 0)$ be an ADL. A unary operation $*$ on A is called a pseudo-complementation on A if, for any $a, b \in A$;
(*i*) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$;
(*ii*) $a \wedge a^* = 0$;
(*iii*) $(a \vee b)^* = a^* \wedge b^*$.

Definition 2.5. ([**3**]) Let $A = (A, \wedge, \vee, 0)$ be an ADL and a be an arbitrary fixed element in A. Then a unary operation $x \mapsto x * a$ on A is called an a-pseudo complementation on A if for any $x, y \in A$, it satisfies the following conditions:
(*i*) $\langle x, a\rangle = (x * a]$.
(*ii*) $(x \vee y) * a = (x * a) \wedge (y * a)$.

Clearly a 0-pseudo-complementation $x \mapsto x * 0$ on an ADL is a pseudo-complementation and denoted $x * 0$ by x^*.

THEOREM 2.2 ([**3**]). *Let $x \mapsto x * a$ be an a-pseudo complementation on an ADL A. Then for any $x, y \in A$, we have the following:*

(i) $m * a \leqslant x * a \leqslant 0 * a$ for any maximal element m of A.
(ii) $(x * a) \wedge (y * a) = (y * a) \wedge (x * a)$.
(iii) $x \leqslant y \Rightarrow y * a \leqslant x * a$.
(iv) $a * a$ is maximal and $a * a = 0 * a$.
(v) $m * a = a \wedge (0 * a)$ for any maximal m of A
(vi) $(x * a) \wedge a = a$
(vii) $((x * a) * a) \wedge x = x$
($viii$) $((x * a) * a) * a = x * a$.
(ix) $x \in (a] \Leftrightarrow x * a$ is maximal.
(x) $m * a = n * a$, for any maximal elements m and n of A.
(xi) if x is maximal then $x * a \sim a$
(xii) $(x \wedge a) * a = a * a$
($xiii$) $(x \wedge y) * a = (y \wedge x) * a$
(xiv) $(x \vee y) * a = (y \vee x) * a$
(xv) $((x \wedge y) * a) * a = ((x * a) * a) \wedge ((y * a) * a)$
(xvi) $x \sim y \Rightarrow x * a = y * a$.

3. a-dense elements and a-maximal filters.

In this section we give the definitions of a-dense element and a-maximal filter of an ADL A and provide example for these and discuss certain results.

DEFINITION 3.1. Let A be an ADL and a, fixed arbitrary element in A. Then an element x of A is said to be a-dense, if $\langle x, a\rangle \subseteq (a]$ (and hence $\langle x, a\rangle = (a]$). When $a = 0$, the expression 0-dense element is abbreviated to dense element.

EXAMPLE 3.1. Consider the ADL A defined in the Example 2.1. Then the elements c, d, e are a-dense elements in A. Note that the element b is not a-dense, since $\langle b, a\rangle = A \neq (a]$.

Let us denote the set of a-dense elements of an ADL A by D_a. Then we have the following those can be easily verified.

THEOREM 3.1. *The following hold for any ADL A.*

(i) *D_a is either empty or a filter of A, for every $a \in A$.*
(ii) *a is maximal $\Leftrightarrow a \in D_a \Leftrightarrow D_a = A$.*
(iii) *For any $x, y \in A$, if $x \in D_a$ and $x \wedge y = a$ then $y = a$.*
(iv) *If m is a maximal element in A then $m \in D_a$.*
(v) *For any a and $b \in A$, if $a \sim b$ then $D_a = D_b$*

DEFINITION 3.2. Let A be an ADL and a, fixed arbitrary non maximal element in A. Then a filter F of A is said to be a-maximal, if F is maximal with respect to the property of not containing a.

Example 3.2. Consider the ADL A defined in the Example 2.1. Then the set $\{c, d, e\}$ is an a-maximal filter of A. Note that the filter $\{d, e\}$ is not a-maximal, since $\{d, e\} \subsetneqq \{c, d, e\} \subsetneqq A$.

The following is an application of Zorn's lemma which allow us to denote the existence of a-maximal filters.

Theorem 3.2. *For any filter F of an ADL A and $a \notin F$, there exists an a-maximal filter of A containing F.*

In the following we furnish some characterizations of a-maximal filters.

Theorem 3.3. *A filter F of an ADL A is a-maximal if and only if $a \notin F$ and, for every $x \notin F$, $\langle x, a\rangle \cap F \neq \phi$.*

Proof. Suppose F is an a-maximal filter. Let $x \in A - F$. Then $F \subset [x) \vee F$ and hence, by the maximality of F, $a \in [x) \vee F$. Hence $a = (t \vee x) \wedge y$ for some $t \in A$ and $y \in F$. Now,

$$\begin{aligned} a \wedge x \wedge y &= (t \vee x) \wedge y \wedge x \wedge y \\ &= (t \vee x) \wedge x \wedge y \wedge y \text{ (by Theorem 2.1 } (xii)) \\ &= x \wedge y. \end{aligned}$$

Therefore $x \wedge y \in (a]$ and hence $y \in \langle x, a\rangle$ so that $\langle x, a\rangle \cap F \neq \phi$.

Conversely, suppose that $a \notin F$ and, for every $x \notin F$, $\langle x, a\rangle \cap F \neq \phi$. If F is not a-maximal, then there exists a filter G of A such that $F \subset G$ and $a \notin G$. Then select $x \in G$ such that $x \notin F$. By supposition, let $y \in \langle x, a\rangle \cap F$. Then $x \wedge y \in (a]$ and $y \in F$. Hence $a \wedge x \wedge y = x \wedge y$ so that $a \vee (x \wedge y) = a$ (by Theorem 2.1(vi)). As $x \wedge y \in G$, we get $a \in G$; a contradiction. Hence F is an a-maximal filter of A. □

We observe that $\langle x, 0\rangle = \{y \in A : x \wedge y = 0\} = \{x\}^*$, the annihilator ideal of x. 0-maximal filter will be called as maximal filter. Now we give a characterization of maximal filter of an ADL.

Corollary 3.1. *Let F be a proper filter of an ADL A. Then F is maximal if and only if for every $x \notin F, \{x\}^* \cap F \neq \phi$.*

It is kown that, in an ADL A, the intersection of an arbitrary family of filters is a filter if and only if A has a maximal element. A filter F of an ADL A is $\wedge-$irreducible if $F = \bigcap\limits_{i\in\Delta} F_i$ where $\{F_i : i \in \Delta\}$ is a an arbitrary family of filtrs of A then $F = F_i$ for some $i \in \Delta$.

Theorem 3.4. *The following statements are equivalent for any ADL A.*
- (i) *F is a-maximal filter for some $a \in A$.*
- (ii) *F is $\wedge-$irreducible.*
- (iii) *$F \subset G = \bigcap \{I : I$ is a filter of A and $F \subset I\}$.*
- (iv) *F is x-maximal filter of A for some $x \in G - F$.*

PROOF. $(i) \Longrightarrow (ii)$: Let $F = \bigcap_{i\in\Delta} F_i$, where $\{F_i : i \in \Delta\}$ is an arbitrary family of filters of A. Since F is a-maximal, $a \notin F$ and hence $a \notin F_i$ for some $i \in \Delta$. As $F \subseteq F_i$ and by the maximality of F we get $F = F_i$. Thus F is $\wedge-$ irreducible.

$(ii) \Longrightarrow (iii)$: If possible suppose that

$$F = G = \bigcap \{I : I \text{ is a filter of A and } F \subset I\}.$$

Then, since F is $\wedge-$ irreducible, F=I for some filter I of A contains F properly; a contradiction. Hence $F \subset G$.

$(iii) \Longrightarrow (iv)$: Assume that $F \subset G = \bigcap \{I : I \text{ is a filter of A and } F \subset I\}$. Then there exists $x \in G$ such that $x \notin F$. If F is not x-maximal filter of A, then there exists a filter J of A such that $x \notin J$ and $F \subset J$. Therefore $J \in \{I : I \text{ is a filter of A and } F \subset I\}$. So that $x \in J$, a contradiction. Hence F is x-maximal filter of A for any $x \in G - F$.

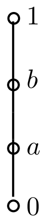
$(iv) \Longrightarrow (i)$: It is clear. □

Let us recall that, a proper filter F of an ADL A is prime filter if and only if for any filters F_1 and F_2 of A, $F = F_1 \cap F_2$ implies either $F = F_1$ or $F = F_2$.

COROLLARY 3.2. *Every a-maximal filter is prime filter.*

REMARK 3.1. The converse of the above Corollary is not true; that is, there are prime filters of ADL's which are not a-maximal; even when ADL is a lattice. For, consider the following example.

EXAMPLE 3.3. Let L be a bounded chain (totally ordered set). Then L is a distributive lattice in which every proper filter is a prime filter (since $x \vee y = y$ or x depending up on $x \leqslant y$ or $y \leqslant x$) . For example, let L be the four element chain represented by the Hassee diagram given bellow. Then $\{1\}, \{b, 1\}, \{a, b, 1\}$ are all prime filters of L but $\{1\}$ is not an a-maximal filter of L, since $\{1\} \subsetneqq \{b, 1\} \subsetneqq L$.



Now we prove some properties of a-dense elements which are closely linked to those of a-maximal filters.

THEOREM 3.5. *In an ADL A, the intersection of the a-maximal filters is equal to D_a.*

PROOF. Let $\{F_i : i \in \Delta\}$ be the family of a-maximal filters of A. Let $x \in D_a$ such that $x \notin F_i$ for some $i \in \Delta$. Since F_i is a-maximal A, we have $\langle x, a\rangle \cap F_i \neq \phi$ (by Theorem 3.3). Hence we can find $y \in F_i$ such that $y \in \langle x, a\rangle$. Since x is

a-dense, $\langle x, a\rangle = (a]$ and therefore $y \in (a]$ and hence $a \wedge y = y$. Since $y \in F_i$ we get $a \in F_i$ which is contradiction . Therefore $x \in F_i$. Thus $D_a \subseteq F_i$ for all $i \in \Delta$. On the other hand, let $x \in F_i$ for all $i \in \Delta$ and $x \notin D_a$. Then $\langle x, a\rangle \nsubseteq (a]$ and hence there exists $y \in A$ such that $x \wedge y \in (a]$ and $y \notin (a]$. Now consider the class.

$$\mathcal{C} = \{F : F \text{ is a filter of } A,\ y \in F \text{ and } a\ \notin F\}.$$

Now $a \notin [y)$, for; $a \in [y) \Rightarrow a = a \vee y \Rightarrow y = a \wedge y \Rightarrow y \in (a]$, which is contradiction. Therefore $[y) \in \mathcal{C}$ so that $\mathcal{C}$ is non-empty. It can be easily prove that $\mathcal{C}$ is closed under union of chains in $\mathcal{C}$. By the Zorn's lemma, $\mathcal{C}$ has a maximal member, say F. We have $y \in F$ and $a \notin F$. Clearly $F = F_i$ for some $i \in \Delta$. Therefore $x \in F$ and $y \in F$ and hence $x \wedge y \in F$. This implies $a \in F$, which is a contradiction. Therefore $x \in D_a$. Thus $D_a = \cap\{F_i : i \in \Delta\}$. □

THEOREM 3.6. *The following statements* (i) *and* (ii) *are equivalent for any ADL A and any one of them implies* (iii).

(i) *A has unique a-maximal filter.*

(ii) *D_a is an a-maximal filter of A.*

(iii) *D_a is a prime filter of A.*

PROOF. It follows from the above theorem and the fact that the class of all a-maximal filters of A is not a chain. Therefore (i), (ii) are equivalent. (iii) is a consequence of Corollary 3.2. □

Let us recall from [**8**] that an element p in an ADL A is said to be $\wedge-$irreducible if, for any x and $y \in A$,

$$p = x \wedge y = y \wedge x \Rightarrow \text{either } p = x \text{ or } p = y$$

LEMMA 3.1. *Let a be non-maximal element in an ADL A. Then a is $\wedge-$irreducible if and only if the principle ideal* $(a]$ *is prime.*

PROOF. Suppose that a is $\wedge-$irreducible. Suppose $x \wedge y \in (a]$. Then $y \wedge x \in (a]$. So that $x \wedge y = a \wedge s$ and $y \wedge x = a \wedge t$ for some $s, t \in A$. Now, we have

$$\begin{aligned} a &= a \vee (a \wedge s) = a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \\ \text{and } a &= a \vee (a \wedge t) = a \vee (y \wedge x) = (a \vee y) \wedge (a \vee x). \end{aligned}$$

Since a is $\wedge-$irreducible, either $a = a \vee x$ or $a = a \vee y$

$$\begin{aligned} a = a \vee x &\Rightarrow x = a \wedge x \text{ (by Theorem 2.2 } (vi)) \\ &\Rightarrow x \in (a] \\ \text{and } a = a \vee y &\Rightarrow y = a \wedge y \\ &\Rightarrow y \in (a]. \end{aligned}$$

Hence $x \in (a]$ or $y \in (a]$. Thus $(a]$ is a prime ideal of A. Also, note that $(a] \neq A$, since a is not maximal.

Conversely, suppose $(a]$ is a prime ideal of A. Let x as $y \in A$ such that $x \wedge y = y \wedge x = a$. Since $x \wedge y \leqslant y$ and $y \wedge x \leqslant x$, we get that $a \leqslant y$ and $a \leqslant x$.

Now, since $x \wedge y = a \in (a]$ and $(a]$ is prime it follows that either $x \in (a]$ or $y \in (a]$.

$$\begin{aligned} x \in (a] &\Rightarrow x = a \wedge s \text{ for some } s \in A \\ &\Rightarrow x = x \wedge x = a \wedge s \wedge x = s \wedge a \wedge x = s \wedge a \leqslant a \\ &\Rightarrow a = x. \end{aligned}$$

Similarly, we can show that $y \in (a] \Rightarrow a = y$. Therefore $a = x$ or $a = y$. Thus a is $\wedge-$irreducible element in A. □

Now, we prove the following

Theorem 3.7. *Let a be a non-maximal element in an ADL A. Then a is $\wedge-$irreducible if and only if $D_a = A - (a]$*

Proof. Suppose a is $\wedge-$ irreducible. Then, by Lemma 3.1, $(a]$ is a prime ideal of A. For any maximal element m of A and $x \in A$,

$$\begin{aligned} x \notin A - (a] &\Rightarrow x \in (a] \\ &\Rightarrow x \wedge m \in (a] \text{ and } m \notin (a] \\ &\Rightarrow m \in \langle x, a\rangle \text{ and } m \notin (a] \\ &\Rightarrow \langle x, a\rangle \nsubseteq (a] \\ &\Rightarrow x \text{ is not } a\text{-dense element in } A. \end{aligned}$$

Therefore $x \notin D_a$ and hence $D_a \subseteq A-(a]$. On the other hand, let $x \in A-(a]$. Then $x \notin (a]$ and for any $y \in A$, $x \wedge y \in (a] \Rightarrow y \in (a]$ (since (a] is prime ideal). Therefore $\langle x, a\rangle \subseteq (a]$. So that x is a-dense and hence $x \in D_a$. Therefore $A-(a] \subseteq D_a$. Thus $D_a = A-(a]$.

Conversely, suppose $D_a = A-(a]$. Let x and $y \in A$ such that $x \wedge y = y \wedge x = a$ and $a \neq x$. Then $x \notin (a]$. For,

$$\begin{aligned} x \in (a] \Rightarrow\ & a \wedge x = x \\ \Rightarrow\ & x \wedge a = a \wedge x = x \text{ (since } a = y \wedge x \leqslant x) \\ \Rightarrow\ & x \leqslant a \text{ and hence } x = a;\ a \text{ contradiction.} \end{aligned}$$

Therefore $x \in D_a$ and hence $\langle x, a\rangle = (a]$. Since $x \wedge y = a$, we have $y \in \langle x, a\rangle$ so that $a \wedge y = y$. Since $a = x \wedge y \leqslant y$, $y \wedge a = a \wedge y = y$ and hence $y \leqslant a$. Therefore $a = y$. Thus a is $\wedge-$ irreducible. □

Let us re call that an equivalence relation θ on an ADL $A = (A, \wedge, \vee, 0)$ is said to be a congruence if θ is compatible with $\wedge$ and $\vee$ on A; that is

$$(p,\ q) \text{ and } (r,\ s) \in \theta \Rightarrow (p \wedge r,\ q \wedge s) \in \theta \text{ and } (p \vee r,\ q \vee s) \in \theta$$

If θ is a congruence on A, then for any $x \in A$, the set $x/\theta = \{y \in A : (x, y) \in \theta\}$ is called the congruence class of x corresponding to θ. If $x = 0$, then $0/\theta = \{y \in A : (0, y) \in A\}$ is called the kernal of θ and is denoted $ker\theta$. $ker\theta$ is a unique congruence class which is an ideal of A. In general, for any ADL A and θ a congruence on A, the Quotient $A/\theta = \{x/\theta : x \in A\}$ is an ADL under the operation $\wedge$ and $\vee$ on A/θ defined by

$$x/\theta \wedge y/\theta = (x \wedge y)/\theta \textit{ and } x/\theta \vee y/\theta = (x \vee y)/\theta$$

and its zero element is $0/\theta$.

Finally, in this section we consider the congruence θ_a on an ADL A corresponding to $a \in A$.

Definition 3.3. ([**5**]) For any $a \in A$, define

$$\theta_a = \{(x, y) \in A \times A : a \vee x = a \vee y\}.$$

Then θ_a is a congruence on A.

Theorem 3.8. *Let A be an ADL and $a \in A$. For any a-maximal filter F of A, let $\overline{F} = \{x/\theta_a : x \in F\}$. Then $\overline{F}$ is a maximal filter of A/θ_a and the map $F \mapsto \overline{F}$ is an one-to-one correspondence between the set of a-maximal filters of A on to the set of all maximal filters of A/θ_a.*

Proof. Let F be an a- maximal filter of A. Put $\overline{F} = \{x/\theta_a : x \in F\}$ then clearly $\overline{F}$ is a filter of A/θ_a as F is a filter of A. First observe that, for any $x \in A$, $x/\theta_a \in \overline{F}$ if and only if $x \in F$. For, $x/\theta_a \in \overline{F}$ implies that $x/\theta_a = y/\theta_a$ for some $y \in F$. So that $(x, y) \in \theta_a$ and hence $a \vee x = a \vee y$. Now $a \vee y \in F$ since $y \in F$. Hence $a \vee x \in F$ and $a \notin F$. Thus $x \in F$ since F is prime filter of A (by Corollary 3.2).

Converse is trivial.

Now, we prove that $\overline{F}$ is a maximal filter of A/θ_a. Since $a \vee a = a \vee 0$, $(a, 0) \in \theta_a$ and hence $a/\theta_a = 0/\theta_a$. If $0/\theta_a \in \overline{F}$ then $a/\theta_a \in \overline{F}$ so that $a \in F$ contradicting the fact F is a- maximal. Thus $0/\theta_a \notin \overline{F}$ so that $\overline{F}$ is proper. Let if possible there exists a proper filter G of A/θ_a such that $\overline{F} \subset G$. Select $x/\theta_a \in G - \overline{F}$ so that $x \notin F$. As F is a-maximal and $x \notin F$, we get $\langle x, a\rangle \cap F \neq \theta$ (by Theorem 3.3). Hence there exists $y \in F$ such that $x \wedge y \in (a]$. So that $a \wedge x \wedge y = x \wedge y$ and hence $a \vee (x \wedge y) = a$ (by Theorem 2.1(vi)). As $y \in F$, $y/\theta_a \in \overline{F}$ and hence $y/\theta_a \in G$. Now, $(x \wedge y)/\theta_a = x/\theta_a \wedge y/\theta_a \in G$ and hence, $a/\theta_a \vee (x \wedge y)/\theta_a \in G$. This implies $(a \vee (x \wedge y))/\theta_a \in G$. Hence $a/\theta_a \in G$; a contradiction since G is proper. Hence $\overline{F}$ is a maximal filter of A/θ_a. Clearly the mapping $F \mapsto \overline{F}$ is one-one.

Finally we prove that $F \mapsto \overline{F}$ is onto. Let M be a maximal filter of A/θ_a. Put $F = \{x \in A : x/\theta_a \in M\}$. Then, as M is proper filter of A, F is a filter of A not containing a. Suppose $x \notin F$ then $x/\theta_a \notin M$. As M is maximal in A/θ_a and $x/\theta_a \notin M$, we get that $\{x/\theta_a\}^* \cap M \neq \theta$ (by Corollary 3.1). Hence there exists $y/\theta_a \in M$ such that $x/\theta_a \wedge y/\theta_a = 0/\theta_a$. So that $(x \wedge y)/\theta_a = 0/\theta_a$ and hence $a \vee (x \wedge y) = a \vee 0 = a$. This in turn implies that F is an a- maximal filter of A and clearly $\overline{F} = M$. Hence $F \mapsto \overline{F}$ is onto. $\square$

4. a-pseudo complemented ADL

In this section, we describe a-dense elements and a-maximal filters in an a-pseudo complemented ADL A and exhibit the congruence ϕ_a on A such that the quotient A/ϕ_a is a Boolean algebra.

First we recall from 3 that an ADL A is said to be a-pseudo complemented, if there exists an a-pseudo complementation $x \mapsto x * a$ on A. In the following we exhibit an a-pseudo complementation $x \mapsto x * a$ on a non-trivial discrete ADL as follows.

EXAMPLE 4.1. Let $(X, \wedge, \vee, 0)$ be a non-trivial discrete ADL. Fix $y \neq 0$ and $a \neq 0$ in X. Define for any $x \in X$,

$$x * 0 = \begin{cases} 0 & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases} \qquad \text{and } x * a = y.$$

Then $x \mapsto x * 0$ is 0-pseudo complementation on X and $x \mapsto x * a$ is an a-pseudo complementation on X. Thus X is a-pseudo complemented for every $a \in X$.

LEMMA 4.1. *The following are equivalent:*
(i) $x \in D_a$.
(ii) $x * a \sim a$.
(iii) $(x * a) * a = a * a$.

PROOF. $(i) \Longrightarrow (ii) : x \in D_a \Rightarrow x$ is $a -$ dense

$$\begin{aligned} &\Rightarrow \langle x, a\rangle = (a] \\ &\Rightarrow (x * a] = (a] \\ &\Rightarrow x * a \sim a. \end{aligned}$$

$(ii) \Longrightarrow (iii) : x * a \sim a \Rightarrow (x * a) \wedge a = a$ and $a \wedge (x * a) = x * a$.

$$\begin{aligned} \text{Now, } (x * a) * a &= (a \wedge (x * a)) * a \\ &= ((x * a) \wedge a) * a) \text{ (by Theorem 2.2}(xiii)) \\ &= a * a \text{ (by 2.1}(vi)) \end{aligned}$$

$$\begin{aligned} (iii) \Longrightarrow (i) : (x * a) * a &= a * a \\ &\Rightarrow ((x * a) * a) * a = (a * a) * a \\ &\Rightarrow x * a = (a * a) * a \text{ (by Theorem 2.2 } (viii)) \\ &\Rightarrow \langle x, a\rangle = (x * a] = ((a * a) * a] = (a] \\ &\text{(since } a * a \text{ is maximal and (by Theorem 2.2 } (xi)) \\ &\Rightarrow x \text{ is } a - \text{dense} \\ &\Rightarrow x \in D_a. \end{aligned}$$

□

LEMMA 4.2. *The following holds*
(i) *For every $x \in A$, $x \vee (x * a) \in D_a$.*
(ii) *If a is $\wedge$-irreducible and $x \notin (a]$ then $x \in D_a$.*
(iii) *The interval $[a, \infty) = \{x \in A : a \leqslant x < \infty\}$ is pseudo-complemented ADL.*

PROOF. (i). Consider

$$\begin{aligned}((x \vee (x * a)) * a) * a &= ((x * a) \wedge ((x * a) * a)) * a \text{ (by definition 2.5}(ii))) \\ &= (a \wedge (x * a) \wedge ((x * a) * a)) * a \text{ (by definition 2.5}(i)) \\ &= ((x * a) \wedge a \wedge ((x * a) * a)) * a \text{ (by Theorem 2.2}(xii)) \\ &= (a \wedge (x * a) * a)) * a \text{ (by Theorem 2.2}(vi)) \\ &= (((x * a) * a) \wedge a) * a \text{ (by Theorem 2.2}(xiii)) \\ &= a * a. \text{ (by Theorem 2.2}(vi))\end{aligned}$$

Hence by Lemma 4.2(iii), $x \vee (x * a)$ is a-dense. So that $x \vee (x * a) \in D_a$.

(ii). Suppose a is $\wedge$-irreducible. Then the principal ideal $(a]$ is prime. Let $x \in A - (a]$. Then for any $y \in A$, $x \wedge y \in (a] \Rightarrow y \in (a] \Rightarrow \langle x, a\rangle \subseteq (a]$. Hence x is a-dense, so that $x \in D_a$.

(iii). Clearly the interval $[a, \infty) = \{x \in A : a \leqslant x < \infty\}$ is an ADL under the induced operations $\wedge$ and $\vee$ with a as its smallest element. Since $x \mapsto x * a$ is an a-pseudo complementation on A, its restriction on the interval $[a, \infty)$ is a pseudo-complementation on the ADL $[a, \infty)$ and hence $[a, \infty)$ is pseudo-complemented ADL. □

THEOREM 4.1. *Let A be an a-pseudo complemented ADL and F a filter of A. Then F is a-maximal if and only if F contains exactly one of the elements x and $x * a$, for every $x \in A$.*

PROOF. We suppose for every $x \in A$, F contains exactly one of x and $x * a$. Since $a * a$ is maximal, $a * a \in F$ and hence $a \notin F$. Let $x \in A - F$. Then $x * a \in F$. Since $\langle x, a\rangle = (x * a]$, we have $x * a \in \langle x, a\rangle \cap F$ so that $\langle x, a\rangle \cap F \neq \phi$. Hence, (by Theorem 3.3) F is a-maximal.

Conversely suppose F is a-maximal, then $a \notin F$. For any $x \in A$, if x and $x * a \in F$, Then $x \wedge (x * a) \in F$ and hence $a \in F$ (since $a \wedge x \wedge (x * a) = x \wedge (x * a)$ and hence by 2.1(vi), $a \vee (x \wedge (x * a)) = a$) which is contradiction. Let $x \in A$ such that $x \notin F$ and $x*a \notin F$. Since F is a-maximal and $x \notin F$, again (by Theorem 3.3), $\langle x, a\rangle \cap F \neq \phi$. Hence these exists $y \in A$ such that $y \in F$ and $y \in \langle x, a\rangle = (x * a]$ and hence $(x * a) \wedge y = y$. As $y \in F$, $x * a \in F$ which is contradiction. Therefore, for every $x \in A$, F contains exactly one of x and $x * a$. □

COROLLARY 4.1. *Let $x \mapsto x^*$ be a pseudo-complementation on an ADL A. Then a filter F of A is maximal if and only if F contains exactly one of x and x^*, for every $x \in A$.*

THEOREM 4.2. *Let A be an a-pseudo complemented ADL and P a prime filter of A such that $D_a \subseteq P$ and $a \notin P$. Then P is a-maximal filter of A.*

PROOF. For every $x \in A$, $x \vee (x * a) \in D_a$. Therefore $x \vee (x * a) \in P$. Since P is prime, $x \in P$ or $x * a \in P$ but not both as $a \notin P$. Therefore P contains exactly one of x and $x * a$. Hence P is a-maximal (by Theorem 4.1). □

COROLLARY 4.2. *D_a is a prime filter of A if and only if it is a-maximal.*

COROLLARY 4.3. *Let A be a pseudo complemented ADL. Then every prime filter of A which contains the dense set D is maximal.*

Note that an ADL can have more than one a-pseudo complementations. Infact, it is observed that the a-pseudo complementations on an ADL are in one-to-one correspondence with the maximal elements (refer [3]). Finally, we conclude this paper by introducing the concept of congruence on a-pseudo complemented ADL and exhibiting a congruence on an a-pseudo complemented ADL with respect to which the Quotient is a Boolean algebra . First we need the following.

LEMMA 4.3. *Let θ be an equivalence relation on an ADL A and suppose that $x \mapsto x * a$ and $x \mapsto x + a$ be a-pseudo complementation on A. Then, for any x and $y \in A$, $(x * a,\ y * a) \in \theta$ if and only if $(x + a,\ y + a) \in \theta$.*

PROOF. It follows from the fact that, for any $x \in A$, we have

$$x * a = (x + a) \wedge (0 * a) \text{ and } x + a = (x * a) \wedge (0 + a).$$

□

DEFINITION 4.1. Let $A = (A, \wedge, \vee, 0)$ be an ADL and $x \mapsto x * a$ an a-pseudo complementation on A. An equivalence relation θ on A is said to be a congruence relation if

(*i*) θ is compatible with $\wedge$ and $\vee$ on A.

(*ii*) $(x, y) \in \theta \Rightarrow (x * a,\ y * a) \in \theta$ (this condition is independent of any a-pseudo complementatin on A (by Lemma 4.3))

THEOREM 4.3. *Let A be an ADL and $x \mapsto x * a$ an a-pseudo complementation on A. Define a relation ϕ_a on A by*

$$(x, y) \in \phi_a \text{ if and only if } x * a = y * a.$$

Then ϕ_a is a congruence on the a-pseudo complemented ADL A.

PROOF. Clearly ϕ_a is an equivalence relation on A. Let (p, q), $(r, s) \in \phi_a$. Then $p * a = q * a$ and $r * a = s * a$. Therefore

$$\begin{aligned} ((p \wedge r) * a) * a &= ((p * a) * a) \wedge ((r * a) * a) \text{ (by Theorem 2.2 } (xv)) \\ &= ((q * a) * a) \wedge ((s * a) * a) \\ &= ((q \wedge s) * a) * a. \end{aligned}$$

This implies $(((p \wedge r) * a) * a) * a = (((q \wedge s) * a) * a) * a$ and hence $(p \wedge r) * a = (q \wedge s) * a$ (by Theorem 2.2 $(viii)$). Therefore $(p \wedge r,\ q \wedge s) \in \phi_a$. Again $(p \vee r) * a = (p * a) \wedge (r * a) = (q * a) \wedge (s * a) = (q \vee s) * a$. Therefore $(p \vee r,\ q \vee s) \in \phi_a$. Clearly $(p, q) \in \phi_a \Rightarrow (p * a,\ q * a) \in \phi_a$. Thus ϕ_a is a congruence on A. □

In the following we prove certain properties of the congruence classes of ϕ_a.

LEMMA 4.4. *We have the following.*

(*i*) *$0/\phi_a = a/\phi_a = (a]$ and is the smallest element in A/ϕ_a.*

(*ii*) *For any maximal elements m and n in A, $m/\phi_a = n/\phi_a = D_a$ and is the greatest element in A/ϕ_a.*

(*iii*) *The restriction of* ϕ_a *on the ADL* $[a, \infty)$ *is the congruence on the pseudo complemented ADL* $[a, \infty)$.

(*iv*) *For every* $x \in A$, $(x * a) * a \in x/\phi_a$.

(*v*) *For every* $x, y \in A$, *if there exists* $d \in D_a$ *such that* $x \wedge d = y \wedge d$, *then* $(x, y) \in \phi_a$.

PROOF. (*i*): Since $0 * a = a * a$, $(0, a) \in \phi_a$ and hence $0/\phi_a = a/\phi_a$. Also, $0/\phi_a = ker\phi_a$ and hence an ideal of A containing a. Therefore $(a] \subseteq 0/\phi_a$. Let $x \in 0/\phi_a$. Then $x * a = 0 * a$ which is maximal. Therefore, by Theorem 2.2 (*ix*), $x \in (a]$. Therefore $0/\phi_a \subseteq (a]$. Thus $0/\phi_a = (a]$. Clearly $0/\phi_a$ is the smallest element in A/ϕ_a.

(*ii*): For any maximal elements m and n in A, by 2.2 (*x*), $m * a = n * a$ so that $m/\phi_a = n/\phi_a$.

$$\begin{aligned}
\text{Now, } x \in D_a &\Leftrightarrow x * a \sim a \\
&\Leftrightarrow x * a \sim m * a \text{ (since } a \sim m * a) \\
&\Leftrightarrow (x * a) * a = (m * a) * a \text{ (by Theorem 2.2}(xvi)) \\
&\Leftrightarrow ((x * a) * a) * a = ((m * a) * a) * a \\
&\Leftrightarrow x * a = m * a \text{ (by Theorem 2.2}(viii)) \\
&\Leftrightarrow (x, m) \in \phi_a \\
&\Leftrightarrow x \in m/\phi_a.
\end{aligned}$$

Therefore $m/\phi_a = D_a$. Also for any $x \in A$, we have $(x \wedge m) * a = (m \wedge x) * a = x * a$ (by Theorem 2.2 (*xiii*)), it follows that $x/\phi_a \wedge m/\phi_a = m/\phi_a \wedge x/\phi_a$. Therefore $x/\phi_a \subseteq m/\phi_a$ so that m/ϕ_a is the largest element in A/ϕ_a.

(*iii*) It is clear obviously.

(*iv*) It follows by Theorem 2.2 (*viii*).

(*v*) Let $x, y \in A$ such that $x \wedge d = y \wedge d$ for some $d \in D_a$. Then

$$\begin{aligned}
((x \wedge d) * a) * a = ((y \wedge d) * a) * a &\text{and } (d * a) * a = a * a \text{ (by Theorem 2.2}(xv)) \\
&\Rightarrow ((x * a) * a) \wedge ((d * a) * a) = ((y * a) * a) \wedge ((d * a) * a) \\
&\Rightarrow ((x * a) * a) \wedge (a * a) = ((y * a) * a) \wedge (a * a) \\
&\Rightarrow (a * a) \wedge ((x * a) * a) = (a * a) \wedge ((y * a) * a) \text{ (by Thm 2.2}(i)) \\
&\Rightarrow (x * a) * a) = (y * a) * a) \text{ (since } a * a \text{ is maximal)} \\
&\Rightarrow ((x * a) * a) * a = ((y * a) * a) * a \\
&\Rightarrow x * a = y * a \text{ (by Theorem 2.2 } (viii)) \\
&\Rightarrow (x, y) \in \phi_a.
\end{aligned}$$

□

THEOREM 4.4. *Let* A *be an ADL with a maximal element* m *and* $x \mapsto x * a$ *an* a*-pseudo complementation on* A. *Then the quotient* A/ϕ_a *is a Boolean algebra.*

PROOF. For any x and $y \in A$, we have that, $(x \wedge y) * a = (y \wedge x) * a$ and $(x \vee y) * a = (y \vee x) * a$, it follows that, $x/\phi_a \wedge y/\phi_a = (x \wedge y)/\phi_a = (y \wedge x)/\phi_a =$

$y/\phi_a \wedge x/\phi_a$ and $x/\phi_a \vee y/\phi_a = (x \vee y)/\phi_a = (y \vee x)/\phi_a = y/\phi_a \vee x/\phi_a$. Therefore the induced operations $\wedge$ and $\vee$ on the quotient A/ϕ_a are commutative and hence A/ϕ_a is a lattice. The distributivity of A/ϕ_a follows from that of A. Hence A/ϕ_a is a bounded distributive lattice in which a/ϕ_a is the smallest element and m/ϕ_a is the largest element (by Lemma 4.4). Finally, let $x/\phi_a \in A/\phi_a$ with $x \in A$. Since $x \wedge (x*a) \in (a]$, $x \wedge (x*a) = a \wedge x \wedge (x*a)$ and it follows that $(x \wedge (x*a))*a = (x \wedge a)*a$. Also, since $a/\phi_a \subseteq x/\phi_a$, $(x \wedge a) * a = a * a$. Hence $(x \wedge (x * a)) * a = a * a$. This implies $(x \wedge (x * a),\ a) \in \phi_a$ and hence $x/\phi_a \wedge (x * a)/\phi_a = (x \wedge (x * a))/\phi_a = a/\phi_a$. From Lemma 4.2 (i), $x \vee (x * a) \in D_a$, it follows that $x \vee (x * a) \in m/\phi_a$. Therefore $x/\phi_a \vee (x*a)/\phi_a = (x \vee (x*a))/\phi_a = m/\phi_a$. Therefore $(x*a)/\phi_a$ is the complement of x/ϕ_a in A/ϕ_a. Thus A/ϕ_a is a Boolean algebra. $\square$

REMARK 4.1. The converse of above theorem need not be true. For, see the following example.

EXAMPLE 4.2. Let $A = \{0, a, b\}$ be a discrete ADL. Define

$$0 * 0 = b \text{ and } a * 0 = a = b * 0.$$

Then $\phi_0 = \{(0, 0), (a, a), (b, b), (a, b), (b, a)\}$ and it is a congruence on A. Now the quotient $A/\phi_0 = \{0/\phi_0,\ a/\phi_0\}$ which is a two-element Boolean algebra but the unary operation $x \mapsto x * 0$ on A is not a 0-pseudo complementation on A, since $a \wedge (a * 0) = a \wedge a = a \neq 0$.

5. Conclusions

In this work, the notion of a-dense elements and a-maximal filters of semilattices introduced by J. C. Varlet is extended to the case of ADL's and obtained certain results of these. In our future of work, we will focus on to investigate a-minimal prime ideals of ADL's and their characterizations in terms of relative a-annihilator ideals and a-pseudo complementations. Also, we will study the space of a-minimal prime ideals with the Hull-Kernel topology and characterize a-Stone ADL's with respect to this space.

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