JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOUTNAL
J. Int. Math. Virtual Inst., Vol. **10**(2)(2020), 287-307
DOI: 10.7251/JIMVI2002287S

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

POSITIVE SOLUTION FOR m-POINT IMPULSIVE TIME-SCALE BOUNDARY VALUE PROBLEMS ON THE HALF-LINE

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Abstract. This paper uses Leray-Schauder Nonlinear Alternative theorem to study the existence of at least one positive solution of m-point impulsive time-scale boundary value problems on the half-line. An example is given to demonstrate our main result.

1. Introduction

The theory of impulsive differential equations describe processes with experience a sudden change of their state at certain moments. Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For the introduction of the basic theory of impulsive differential equations see [**3, 4, 16, 19**] and the references therein. In the last few years boundary value problems for impulsive differential equations and impulsive differences equations have received much attention [**7, 8, 11, 12, 17, 24**] Especially, the study of impulsive dynamic equations on time scales has also attracted much attention since it proves an unifying structure for differential equations in the continuous cases and finite difference equations in the discrete cases, see [**2, 9, 10, 15, 18, 21**] and references therein. Some basic definitions and theorems on time scales can be found in the books [**5, 6**]. In recent years, there are a few authors studied the existence of positive solutions for time scale boundary value problems on infinite intervals. We refer the reader to [**13, 14, 22, 23**]. Due to the fact that an infinite interval is noncompact, the discussion about boundary value

2010 *Mathematics Subject Classification.* 34B18; 34B37; 34K10.
Key words and phrases. impulsive boundary value problems, positive solutions, time scales.
Communicated by Daniel A. Romano.

problem on the half-line is more complicated. There is not work on positive solutions for $m-$ point impulsive time-scale boundary value problem on the half line expect that in [**13, 20**]. Hence, these results can be considered as a contribution to this field.

solutions of the following second-order m-point impulsive boundary value problem (IBVP):

$$(1.1)\quad \begin{cases} y^{\Delta\Delta}(t) = f(t, y(t), y^{\Delta}(t), (Ty)(t), (Sy)(t)), \quad \text{for all } t \in \mathbb{R}^{'}_{++}, \\ y(0) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\xi_i), \quad y^{\Delta}(\infty) = \beta y^{\Delta}(0), \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1,\ 2,\ 3, \cdots, \\ y^{\Delta}(t_k^+) - y^{\Delta}(t_k^-) = \overline{I}_k(y^{\Delta}(t_k^-)), \quad k = 1,\ 2,\ 3, \cdots \end{cases}$$

where $\mathbb{T}$ is a time scale, $\xi_1, \xi_2, \cdots, \xi_{m-2} \in \mathbb{T}$, $\sigma(0) < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \infty$, $0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$, $\mathbb{R}^{'}_{++} = \mathbb{R}_{++} - \{t_1, t_2, ..., t_k, ...\}$, $f \in \mathcal{C}[\mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}_{+}, \mathbb{R}_{+}]$, $I_k, \overline{I}_k \in \mathcal{C}[\mathbb{R}_{++}, \mathbb{R}_{+}]$, $\beta \geqslant 0$, $\alpha_i \geqslant 0$, $(i = 1,\ 2,\ 3, ...,\ m-2)$,

$$(Ty)(t) = \int_0^t D(t,s) y(s) \Delta s, \quad (Sy)(t) = \int_0^{\infty} E(t,s) y(s) \Delta s.$$

$D \in \mathcal{C}[B, \mathbb{R}_{+}]$, $B = \{(t,s) \in \mathbb{R}_{+} \times \mathbb{R}_{+} : t \geqslant s\}$, $E \in \mathcal{C}[\mathbb{R}_{+} \times \mathbb{R}_{+}, \mathbb{R}_{+}]$, $\mathbb{R}_{+} = [0, \infty)$, *and* $\mathbb{R}_{++} = (0, \infty)$.

Throughout this paper we assume that following conditions hold:

$(H1)$ $\sup_{t \in \mathbb{R}_+} \int_0^t D(t,s) \Delta s < \infty$, $\sup_{t \in \mathbb{R}_+} \int_0^{\infty} E(t,s) \Delta s < \infty$ and

$$\lim_{t' \to t} \int_0^{\infty} |E(t', s) - E(t,s)|\, s \Delta s = 0, \quad \text{for all } t \in \mathbb{R}^{'}_{+}.$$

In this case, let

$$d^* = \sup_{t \in \mathbb{R}_+} \int_0^t D(t,s) \Delta s, \quad e^* = \sup_{t \in \mathbb{R}_+} \int_0^{\infty} E(t,s) \Delta s.$$

$(H2)$ There exist $a, b \in \mathcal{C}[\mathbb{R}_{++}, \mathbb{R}_{+}]$, $G \in \mathcal{C}[\mathbb{R}_{++}, \mathbb{R}_{+}]$, $H \in \mathcal{C}[\mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}_{+}, \mathbb{R}_{+}]$ and $r > 0$ such that

$$f(t, u, v, w, z) \leqslant a(t) G(u) + b(t) H(v, w, z) \quad \forall t, u, v \in \mathbb{R}_{++},\ \forall w, z \in \mathbb{R}_{+},$$

and

$$a_r^* = \int_0^{\infty} a(t) G_r(t) \Delta t < \infty, \quad b^* = \int_0^{\infty} b(t) \Delta t < \infty;$$

for any $r > 0$, where

$$G_r(t) = \max\left\{ G(u) \ :\ \beta^{-2} r \left(t + \sum_{i=1}^{m-2} \alpha_i \right) \leqslant u \leqslant r \left(t + \sum_{i=1}^{m-2} \alpha_i \right) \right\}.$$

$\eta_k \in \mathbb{R}_+ \ \ (k = 1, 2, 3, \cdots)$ such that

$$I_k(v) \leqslant t_k \overline{I}_k(v) \leqslant t_k \eta_k F(v), \quad \forall v \in \mathbb{R}_{++}$$

and

$$\overline{\eta} = \sum_{k=1}^{\infty} t_k \eta_k < \infty \text{ and, consequently, } \eta^* = \sum_{k=1}^{\infty} \eta_k \leqslant t_1^{-1} \overline{\eta} < \infty.$$

This paper is organized as follows: In section 2, we give some preliminaries lemmas. In section 3, we give the proof of necessary and sufficient conditions for existence of at least one positive solution of IBVP (1.1). In section 4, an example is also presented to illustrate our main result. The results are even new for the difference equations and differential equations as well as for dynamic equations on time scales.

2. Preliminaries

In this section, to state the main result of this paper, we need the following lemmas. Let

$$PC[\mathbb{R}_+, \mathbb{R}] = \{y : \ y \ \text{ is a real function on } \mathbb{R}_+ \text{ such that } y(t) \ \text{ is continuous at } t \neq t_k, \text{ left continuous at } t = t_k, \text{ and } y(t_k^+) \text{ exists}, \ k = 1, 2, 3, ...\},$$

$$PC^{\Delta}[\mathbb{R}_+, \mathbb{R}] = \{y \in PC[\mathbb{R}_+, \mathbb{R}] : \ y^{\Delta}(t) \text{ is continuous at } t \neq t_k, \ y^{\Delta}(t_k^+) \text{ and } y^{\Delta}(t_k^-) \text{ exist for } k = 1, 2, 3, ...\} \text{ and}$$

$$BPC^{\Delta}[\mathbb{R}_+, \mathbb{R}] = \{y \in PC^{\Delta}[\mathbb{R}_+, \mathbb{R}] : \sup_{t \in R_+} \frac{| \ y(t) \ |}{t + \sum_{i=1}^{m-2} \alpha_i} < \infty, \ \ \sup_{t \in \mathbb{R}_+} | \ y^{\Delta}(t) \ | < \infty\}.$$

It is clear that $BPC^{\Delta}[\mathbb{R}_+, \mathbb{R}]$ is a Banach space with the norm

$$||y|| = \max\{||y||_1, ||y||_2\}$$

where

$$||y||_1 = \sup_{t \in \mathbb{R}_+} \frac{| \ y(t) \ |}{t + \sum_{i=1}^{m-2} \alpha_i}, \quad ||y||_2 = \sup_{t \in \mathbb{R}_+} | \ y^{\Delta}(t) \ | \ .$$

Let

$$W = \{y \in BPC^1[\mathbb{R}_+, \mathbb{R}] : y(t) \geqslant 0, \ y^{\Delta}(t) \geqslant 0, \forall t \in \mathbb{R}_+\}$$

and

$$Q = \left\{ y \in BPC^1[\mathbb{R}_+, \mathbb{R}] : \inf_{t \in \mathbb{R}_+} \frac{y(t)}{t + \sum_{i=1}^{m-2} \alpha_i} \geqslant \beta^{-1} ||y||_1, \ \inf_{t \in \mathbb{R}_+} y^{\Delta}(t) \geqslant \beta^{-1} ||y||_2 \right\}.$$

Obviously, W and Q are two cones in the space $BPC^{\Delta}[\mathbb{R}_+, \mathbb{R}]$ and $Q \subset W$. Let

$$Q_+ = \{y \in Q : ||y|| > 0\}, \quad Q_{pq} = \{u \in Q : p \leqslant ||y|| \leqslant q\}$$

for $q > p > 0$.

LEMMA 2.1. *For $y \in Q$, we have*

$$\|y\|_1 \geqslant \beta^{-1}\|y\|_2, \quad \|y\|_2 \geqslant \beta^{-1}\|y\|_1, \tag{2.1}$$

$$\beta^{-1}\|y\| \leqslant \|y\|_1 \leqslant \|y\|, \quad \beta^{-1}\|y\| \leqslant \|y\|_2 \leqslant \|y\|, \tag{2.2}$$

$$\beta^{-2}\|y\| \leqslant \frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \|y\|, \quad \beta^{-2}\|y\| \leqslant y^{\Delta}(t) \leqslant \|y\|. \tag{2.3}$$

PROOF. Since

$$\begin{aligned} y(t) &= \int_0^t y^{\Delta}(s)\Delta s + y(0) = \int_0^t y^{\Delta}(s)\Delta s + \sum_{i=1}^{m-2}\alpha_i y^{\Delta}(\xi_i) \\ &\geqslant y^{\Delta}(0)\left[t+\sum_{i=1}^{m-2}\alpha_i\right], \end{aligned}$$

we get

$$\frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \geqslant y^{\Delta}(0).$$

By definition of Q and the fact that y^{Δ} is nondecreasing, we have

$$\beta^{-1}\|y\|_2 \leqslant \inf_{t\in\mathbb{R}_+}\Delta(t) = y^{\Delta}(0) \leqslant \frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \|y\|_1,$$

i.e.,

$$\beta^{-1}\|y\|_2 \leqslant \|y\|_1.$$

Since

$$\begin{aligned} y(t) &= \int_0^t y^{\Delta}(s)\Delta s + y(0) \leqslant t y^{\Delta}(t) + \sum_{i=1}^{m-2}\alpha_i y^{\Delta}(\xi_i) \\ &\leqslant \left(t+\sum_{i=1}^{m-2}\alpha_i\right)\sup_{t\in\mathbb{R}_+} y^{\Delta}(t). \end{aligned}$$

we obtain

$$\frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \sup_{t\in\mathbb{R}_+} y^{\Delta}(t).$$

By using definition of Q, we get

$$\beta^{-1}\|y\|_1 \leqslant \inf_{t\in\mathbb{R}_+}\frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \sup_{t\in\mathbb{R}_+} y^{\Delta}(t).$$

So,

$$\beta^{-1}\|y\|_1 \leqslant \|y\|_2.$$

Thus (2.1) inequality is shown.

By (2.1), we have

$$\|y\| \leqslant \max\left\{\|y\|_1, \beta\|y\|_1\right\} = \beta\|y\|_1,$$

and

$$\|y\| \leqslant \max\Big\{\beta\|y\|_2, |y\|_2\Big\} = \beta\|y\|_2,$$

i.e.,

$$\beta^{-1}\|y\| \leqslant \|y\|_1 \leqslant \|y\|, \quad \beta^{-1}\|y\| \leqslant \|y\|_2 \leqslant \|y\|.$$

Hence (2.2) inequality is shown.

Finally, we show (2.3) inequality. By definition of Q and (2.2), we get

$$\beta^{-2}\|y\| \leqslant \beta^{-1} \sup_{t\in\mathbb{R}_+} \frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \|y\|,$$

and

$$\beta^{-2}\|y\| \leqslant \beta^{-1} \sup_{t\in\mathbb{R}_+} y^{\Delta}(t) \leqslant y^{\Delta}(t) \leqslant \|y\|,$$

i.e.,

$$\beta^{-2}\|y\| \leqslant \frac{y(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \|y\|, \quad \beta^{-2}\|y\| \leqslant y^{\Delta}(t) \leqslant \|y\|,,$$

This completes the proof. □

We consider operator A defined by

(2.4)

$$\begin{aligned}
(Ay)(t) &= \sum_{i=1}^{m-2}\alpha_i[\frac{1}{\beta-1}\{\int_0^\infty f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))ds \\
&+ \sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))\} + \int_0^{\xi_i} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))ds \\
&+ \sum_{t_k<\xi_i}\overline{I}_k(y^{\Delta}(t_k^-))] + \frac{t}{\beta-1}[\int_0^\infty f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))ds \\
&+ \sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))] + \int_0^t (t-s)f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))ds \\
&+ \sum_{t_k<t}[(t-t_k)\overline{I}_k(y^{\Delta}(t_k^-)) + I_k(y^{\Delta}(t_k^-))].
\end{aligned}$$

In what follows, we write $J_1 = [0,t_1]$ and $J_k = (t_{k-1},t_k]$ $(k = 2,3,4,...)$.

Lemma 2.2. *If conditions (H1)-(H3) are satisfied, then operator defined by* (2.4) *is continuous operator from* Q_+ *into* Q; *moreover, for any* $q > p > 0$, $A(Q_{pq})$ *relatively compact.*

Proof. Let $y \in Q$, and $\|y\| = r$. Then $r > 0$ and by (2.3),

(2.5)
$$\beta^{-2} r\left(t+\sum_{i=1}^{m-2}\alpha_i\right) \leqslant y(t) \leqslant r\left(t+\sum_{i=1}^{m-2}\alpha_i\right), \quad \beta^{-2} r \leqslant y^{\Delta}(t) \leqslant r \quad \forall t \in [0,\infty).$$

By conditions $(H1), (H2)$ and (2.5), we have

$$f(t, y(t), y^{\Delta}(t), (Ty)(t), (Sy)(t)) \leqslant a(t)G_r(t) + H_r b(t), \quad \forall t \in [0,\infty) \tag{2.6}$$

where

$$H_r = \max\Big\{G(v,w,z) \mid \beta^{-2} r \leqslant v \leqslant r,\ 0 \leqslant w \leqslant d^* r,\ 0 \leqslant z \leqslant e^* r\Big\}, \tag{2.7}$$

which implies the convergence of the infinite integral

$$\int_0^{\infty} f(t, y(t), y^{\Delta}(t), (Ty)(t), (Sy)(t))\Delta t \tag{2.8}$$

and

$$\int_0^{\infty} f(t, y(t), y^{\Delta}(t), (Ty)(t), (Sy)(t))\Delta t \leqslant a_r^* + H_r b^*. \tag{2.9}$$

On the other hand, $(H3)$ and (2.5), we have

$$\overline{I}_k(y^{\Delta}(t_k^-)) \leqslant \eta_k F(y^{\Delta}(t_k^-)) \leqslant \eta_k M_r \tag{2.10}$$

where

$$M_r = \max\Big\{F(v) \mid \beta^{-2} r \leqslant v \leqslant r\Big\} \tag{2.11}$$

which implies the convergence of the infinite series

$$\sum_{k=1}^{\infty} \overline{I}_k(y^{\Delta}(t_k^-)) \tag{2.12}$$

and

$$\sum_{k=1}^{\infty} \overline{I}_k(y^{\Delta}(t_k^-)) \leqslant \sum_{k=1}^{\infty} M_r \eta_k = M_r \sum_{k=1}^{\infty} \eta_k = M_r \eta^*. \tag{2.13}$$

In addition, from (2.4) we get

(2.14)
$$\frac{(Ay)(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \geqslant \frac{1}{\beta-1}\left[\int_0^{\infty} f(t, y(t), y^{\Delta}(t), (Ty)(t), (Sy)(t))\Delta t + \sum_{k=1}^{\infty} \overline{I}_k(y^{\Delta}(t_k^-))\right].$$

Hence, by (2.4), we have

$$\begin{aligned}
(Ay)(t) &\leqslant \frac{1}{\beta-1}\Bigg[\beta\sum_{i=1}^{m-2}\alpha_i\int_0^{\infty} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))\Delta s \\
&+ \beta\sum_{i=1}^{m-2}\alpha_i\sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-)) + \beta t\int_0^{\infty} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))\Delta s \\
&+ \beta t\sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))\Bigg] \\
&= \frac{\beta}{\beta-1}\Bigg[\left(\sum_{i=1}^{m-2}\alpha_i+t\right)\int_0^{\infty} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))\Delta s \\
&+ \left(\sum_{i=1}^{m-2}\alpha_i+t\right)\sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))\Bigg]
\end{aligned}$$

so,

(2.15)

$$\frac{(Ay)(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant \frac{\beta}{\beta-1}\left[\int_0^{\infty} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))\Delta s + \sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))\right].$$

On the other hand, by (2.4), we have

(2.16)

$$\begin{aligned}
(Ay)^{\Delta}(t) &= \frac{1}{\beta-1}\left[\int_0^{\infty} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))\Delta s + \sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))\right] \\
&+ \int_0^{t} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))\Delta s + \sum_{t_k<t}\overline{I}_k(y^{\Delta}(t_k^-)),
\end{aligned}$$

so,

(2.17)

$$(Ay)^{\Delta}(t) \geqslant \frac{1}{\beta-1}\left[\int_0^{\infty} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))\Delta s + \sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))\right],$$

and

(2.18)

$$(Ay)^{\Delta}(t) \leqslant \frac{\beta}{\beta-1}\left[\int_0^{\infty} f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))\Delta s + \sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))\right].$$

It follows from $(2.4), (2.14) - (2.18)$, we get

$$\beta^{-1}\|Ay\|_1 \leqslant \inf_{t\in[0,\infty)} \frac{(Ay)(t)}{t+\sum_{i=1}^{m-2}\alpha_i}$$

$$\beta^{-1}\|(Ay)^{\Delta}\|_2 \leqslant \inf_{t\in[0,\infty)} (Ay)^{\Delta}(t)$$

and, by $(2.9), (2.13), (2.15)$ and (2.18),

$$\|Ay\|_1 \leqslant \frac{\beta}{\beta-1}(a_r^* + H_r b^* + M_r\eta^*) < \infty \tag{2.19}$$

$$\|Ay\|_2 \leqslant \frac{\beta}{\beta-1}(a_r^* + H_r b^* + M_r\eta^*) < \infty. \tag{2.20}$$

Thus, we have proved that A maps Q_+ into Q. Now, we are going to show that A is continuous. Let $y_n, \overline{y} \in Q_+,\ \|y_n - \overline{y}\| \to 0\ (n \to \infty)$. Write $\|\overline{y}\| = 2\overline{r}, (\overline{r} > 0)$ and we may assume that

$$\overline{r} \leqslant \|y_n\| \leqslant 3\overline{r} \quad (n = 1, 2, 3, ...).$$

So, (2.3) implies

$$\beta^{-2}\overline{r} \leqslant \frac{y_n(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant 3\overline{r}, \quad \beta^{-2}\overline{r} \leqslant \frac{\overline{y}(t)}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant 3\overline{r}, \tag{2.21}$$

and

$$\beta^{-2}\overline{r} \leqslant y_n^{\Delta}(t) \leqslant 3\overline{r}, \quad \beta^{-2}\overline{r} \leqslant \overline{y}^{\Delta}(t) \leqslant 3\overline{r}. \tag{2.22}$$

By (2.4), we have

(2.23)

$$\begin{aligned}\frac{|(Ay_n)(t) - (A\overline{y})(t)|}{t+\sum_{i=1}^{m-2}\alpha_i} \leqslant\ & \frac{\beta}{\beta-1}\Bigg[\int_0^{\infty} |f(s, y_n(s), y_n^{\Delta}(s), (Ty_n)(s), (Sy_n)(s)) \\ -\ & f(s, \overline{y}(s), \overline{y}^{\Delta}(s), (T\overline{y})(s), (S\overline{y})(s))|\Delta s \\ +\ & \sum_{k=1}^{\infty} |\overline{I}_k(y_n^{\Delta}(t_k^-)) - \overline{I}_k(\overline{y}^{\Delta}(t_k^-))|\Bigg] \\ +\ & \frac{1}{t+\sum_{i=1}^{m-2}\alpha_i} \sum_{t_k<t} |I_k(y_n^{\Delta}(t_k^-)) - I_k(\overline{y}^{\Delta}(t_k^-))|.\end{aligned}$$

It follows from (2.23) that

(2.24)

$$
\begin{aligned}
\|Ay_n - A\overline{y}\|_1 \quad &\leqslant \quad \frac{1}{\sum_{i=1}^{m-2}\alpha_i}\sum_{k=1}^{\infty}|I_k(y_n^{\Delta}(t_k^-)) - I_k(\overline{y}^{\Delta}(t_k^-))| \\
&+ \quad \frac{\beta}{\beta-1}\Bigg[\int_0^{\infty}|f(s, y_n(s), y_n^{\Delta}(s), (Ty_n)(s), (Sy_n)(s)) \\
&\quad\quad f(s, \overline{y}(s), \overline{y}^{\Delta}(s), (T\overline{y})(s), (S\overline{y})(s))|\Delta s \\
&+ \quad \sum_{k=1}^{\infty}|\overline{I}_k(y_n^{\Delta}(t_k^-)) - \overline{I}_k(\overline{y}^{\Delta}(t_k^-))|\Bigg].
\end{aligned}
$$

It is clear that

(2.25)

$$
\begin{aligned}
f(t, y_n(t), y_n^{\Delta}(t), (Ty_n)(t), (Sy_n)(t)) \quad &\to f(t, \overline{y}(t), \overline{y}^{\Delta}(t), (T\overline{y})(t), (S\overline{y})(t)) \\
&as \quad n \to \infty, \ \forall t \in \mathbb{R}_{++},
\end{aligned}
$$

and similar to (2.6) and observing (2.21), we have for $\forall t \in \mathbb{R}_{++}$,

(2.26)

$$
\begin{aligned}
|f(t, y_n(t), y_n^{\Delta}(t), (Ty_n)(t), (Sy_n)(t)) \quad &- \quad f(t, \overline{y}(t), \overline{y}^{\Delta}(t), (T\overline{y})(t), (S\overline{y})(t))| \\
&\leqslant \quad 2[a(t)\overline{G}(t) + \overline{H}b(t)], \ n = 1, 2, \cdots
\end{aligned}
$$

where

$$
\overline{G}(t) = \max\left\{G(u) \ : \ \beta^{-2}\overline{r}\left(t + \sum_{i=1}^{m-2}\alpha_i\right) \leqslant u \leqslant 3\overline{r}\left(t + \sum_{i=1}^{m-2}\alpha_i\right)\right\},
$$

$$
\overline{H} = \max\left\{G(v, w, z) \ : \ \beta^{-2}\overline{r} \leqslant v \leqslant 3\overline{r}, \ 0 \leqslant w \leqslant 3d^*\overline{r}, \ 0 \leqslant z \leqslant 3e^*\overline{r}\right\}.
$$

It is easy to see that condition $(H2)$ implies

$$
a_{pq}^* = \int_0^{\infty} a(t)G_{pq}(t)\Delta t < \infty \tag{2.27}
$$

for any $q > p > 0$, where

$$
G_{pq}(t) = \max\left\{G(u) \ : \ \beta^{-2}p\left(t + \sum_{i=1}^{m-2}\alpha_i\right) \leqslant u \leqslant q\left(t + \sum_{i=1}^{m-2}\alpha_i\right)\right\} \tag{2.28}
$$

So,

$$
\int_0^{\infty} a(t)\overline{G}(t)\Delta t < \infty
$$

and therefore

$$
2\int_0^{\infty}[a(t)\overline{G}(t) + \overline{H}b(t)]\Delta t < \infty. \tag{2.29}
$$

It follows from (2.25), (2.26), (2.29) and the dominated convergence theorem that

(2.30)

$$\lim_{n\to\infty}\int_0^\infty |f(t,y_n(t),y_n^\Delta(t),(Ty_n)(t),(Sy_n)(t))-f(t,\overline{y}(t),\overline{y}^\Delta(t),(T\overline{y})(t),(S\overline{y})(t))|\Delta t=0.$$

On the other hand, similar to (2.10) and observing (2.22), we have

$$\overline{I}_k(y_n^\Delta(t_k^-))\leqslant \overline{M}_r\eta_k,\ \ \overline{I}_k(\overline{y}^\Delta(t_k^-))\leqslant \overline{M}_r\eta_k\ \ (k,n=1,2,3,\cdots) \tag{2.31}$$

where

$$\overline{M}_r=\max\{F(v)\ :\ \beta^{-2}\overline{r}\leqslant v\leqslant 3\overline{r}\}.$$

For any given $\epsilon>0$, by (2.31) and condition $(H3)$, we can choose a positive integer k_0 such that

$$\sum_{k=k_0+1}^{\infty} t_k\overline{I}_k(y_n^\Delta(t_k^-))<\epsilon \quad (n=1,2,3,\cdots)$$

and

$$\sum_{k=k_0+1}^{\infty} t_k\overline{I}_k(\overline{y}^\Delta(t_k^-))<\epsilon$$

so,

$$\sum_{k=k_0+1}^{\infty} I_k(y_n^\Delta(t_k^-))<\epsilon \quad (n=1,2,3,\cdots), \tag{2.32}$$

$$\sum_{k=k_0+1}^{\infty} I_k(\overline{y}^\Delta(t_k^-))<\epsilon, \tag{2.33}$$

$$\sum_{k=k_0+1}^{\infty} \overline{I}_k(y_n^\Delta(t_k^-))\leqslant \frac{1}{t_1}\sum_{k=k_0+1}^{\infty} t_k\overline{I}_k(y_n^\Delta(t_k^-))<t_1^{-1}\epsilon \tag{2.34}$$

and

$$\sum_{k=k_0+1}^{\infty} \overline{I}_k(\overline{y}^\Delta(t_k^-))\leqslant \frac{1}{t_1}\sum_{k=k_0+1}^{\infty} t_k\overline{I}_k(\overline{y}^\Delta(t_k^-))<t_1^{-1}\epsilon. \tag{2.35}$$

It is clear that

$$I_k(y_n^\Delta(t_k^-))\to I_k(\overline{y}^\Delta(t_k^-)),\quad n\to\infty$$

and

$$\overline{I}_k(y_n^\Delta(t_k^-))\to \overline{I}_k(\overline{y}^\Delta(t_k^-)),\quad n\to\infty,$$

so, we can choose a positive integer n_0 such that

$$\sum_{k=1}^{k_0}|I_k(y_n^\Delta(t_k^-))-I_k(\overline{y}^\Delta(t_k^-))|<\epsilon\quad \forall n>n_0 \tag{2.36}$$

and

$$\sum_{k=1}^{k_0} |\overline{I}_k(y_n^{\Delta}(t_k^-)) - \overline{I}_k(\overline{y}^{\Delta}(t_k^-))| < \epsilon, \quad \forall n > n_0. \tag{2.37}$$

From (2.32) – (2.37), we get

$$\sum_{k=1}^{\infty} |I_k(y_n^{\Delta}(t_k^-)) - I_k(\overline{y}^{\Delta}(t_k^-))| < 3\epsilon, \quad \forall n > n_0$$

and

$$\sum_{k=1}^{\infty} |\overline{I}_k(y_n^{\Delta}(t_k^-)) - \overline{I}_k(\overline{y}^{\Delta}(t_k^-))| < (1 + 2t_1^{-1})\epsilon, \quad \forall n > n_0,$$

hence

$$\lim_{n\to\infty} \sum |I_k(y_n^{\Delta}(t_k^-)) - I_k(\overline{y}^{\Delta}(t_k^-))| = 0 \tag{2.38}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\overline{I}_k(y_n^{\Delta}(t_k^-)) - \overline{I}_k(\overline{y}^{\Delta}(t_k^-))| = 0. \tag{2.39}$$

It follows from $(2.24), (2.30), (2.38)$ and (2.39),

$$\lim_{n\to\infty} \|Ay_n - A\overline{y}\|_1 = 0. \tag{2.40}$$

On the other hand, from (2.4) it is easy to get

(2.41)

$$\begin{aligned} \|Ay_n - A\overline{y}\|_2 \leqslant & \frac{\beta}{\beta-1}\Bigg[\int_0^{\infty} |f(s, y_n(s), y_n^{\Delta}(s), (Ty_n)(s), (Sy_n)(s)) \\ - & f(s, \overline{y}(s), \overline{y}^{\Delta}(s), (T\overline{y})(s), (S\overline{y})(s))|\Delta s \\ + & \sum_{k=1}^{\infty} |\overline{I}_k(y_n^{\Delta}(t_k^-)) - \overline{I}_k(\overline{y}^{\Delta}(t_k^-))|\Bigg]. \end{aligned}$$

So, $(2.30), (2.39)$ and (2.41) imply

$$\lim_{n\to\infty} \|Ay_n - A\overline{y}\|_2 = 0. \tag{2.42}$$

It follows from (2.40) and (2.42) that $\lim_{n\to\infty} \|Ay_n - A\overline{y}\| = 0 \quad$ as $n \to \infty$, and the continuity of A is proved.

Finally, we prove that $A(Q_{pq})$ is relatively compact, where $q > p > 0$ are arbitrarily given. Let $\overline{y}_n \in Q_{pq}, \quad (n = 1, 2, 3, ...)$. Then, by (2.3),

$$\beta^{-2}p\left(t + \sum_{i=1}^{m-2} \alpha_i\right) \leqslant \overline{y}_n(t) \leqslant q\left(t + \sum_{i=1}^{m-2} \alpha_i\right), \ \beta^{-2}p \leqslant \overline{y}_n^{\Delta}(t) \leqslant q. \tag{2.43}$$

Similar to (2.6), (2.10), (2.19) and observing (2.43), we have

$$f(t,\overline{y}_n(t),\overline{y}_n^{\Delta}(t),(T\overline{y}_n)(t),(S\overline{y}_n)(t)) \leqslant a(t)G_{pq}(t)+H_{pq}b(t) \tag{2.44}$$

$$\overline{I}_k(\overline{y}_n^{\Delta}(t_k)) \leqslant M_{pq}\eta_k \quad (k,n=1,2,3,...) \tag{2.45}$$

and

$$\|A\overline{y}_n\|_1 \leqslant \frac{\beta}{\beta-1}(a^*_{pq}+H_{pq}b^*+M_{pq}\eta^*) \tag{2.46}$$

where $G_{pq}(t)$ ve a^*_{pq} are defined by (2.27) and (2.28), respectively, and

$$H_{pq}=\max\{G(v,w,z) \mid \beta^{-2}p\leqslant v\leqslant q,\ \ 0\leqslant w\leqslant d^*q,\ \ 0\leqslant z\leqslant e^*q\}$$

$$M_{pq}=\max\{F(v) \mid \beta^{-2}p\leqslant v\leqslant q\}.$$

From (2.46) we see that functions $\{(A\overline{y}_n)(t)\}$ $n=1,2,3,...$ are uniformly bounded on $[0,r]$ for any $r\geqslant 0$. On the other hand, by (2.4) and (2.44) – (2.46) we have

$$\begin{aligned}
0 &\leqslant (A\overline{y}_n)(t')-(A\overline{y}_n)(t)\\
&=\frac{t'-t}{\beta-1}\left[\int_0^\infty f(s,\overline{y}_n(s),\overline{y}_n^{\Delta}(s),(T\overline{y}_n)(s),(S\overline{y}_n)(s))\Delta s+\sum_{k=1}^{\infty}\overline{I}_k(\overline{y}_n^{\Delta}(t_k))\right]\\
&+(t'-t)\int_0^t f(s,\overline{y}_n(s),\overline{y}_n^{\Delta}(s),(T\overline{y}_n)(s),(S\overline{y}_n)(s))\Delta s\\
&+\int_t^{t'}(t'-s)f(s,\overline{y}_n(s),\overline{y}_n^{\Delta}(s),(T\overline{y}_n)(s),(S\overline{y}_n)(s))\Delta s\\
&\leqslant\frac{t'-t}{\beta-1}(a^*_{pq}+H_{pq}b^*+M_{pq}\eta^*)+(t'-t)(a^*_{pq}+H_{pq}b^*)\\
&+(t_k-t_{k-1})\int_t^{t'}[a(s)G_{pq}(s)+H_{pq}b(s)]\Delta s,\ \forall t,t'\in J_k,\ t'>t\ (k,n=1,2,3,\cdots),
\end{aligned}$$

which implies that functions $\{\gamma_n(t)\}$ $(n=1,2,3,...)$ defined by (for any fixed k)

$$\gamma_n(t)=\begin{cases}(A\overline{y}_n)(t), & \forall t\in J_k=(t_{k-1},t_k],\\ (A\overline{y}_n)(t^+_{k-1}), & \forall t=t_{k-1}.\end{cases}$$

are equicontinuous on $\overline{J}_k=[t_{k-1},t_k]$ $(k=1,2,3,...)$. Consequently, by the Ascoli-Arzela theorem, $\{\gamma_n(t)\}$ has a subsequence which is convergent uniformly on J_k. So, funvtions $\{A\overline{y}_n(t)\}$ $(n=1,2,3,...)$ have a subsequence which is convergent uniformly on J_k. Now, by the diagonal method, we can choose a subsequence $\{A\overline{y}_{n_i}(t)\}$ $(i=1,2,3,...)$ of $\{A\overline{y}_n(t)\}$ $(n=1,2,3,...)$ such that $\{A\overline{y}_{n_i}(t)\}$ $(i=1,2,3,...)$ is convergent uniformly on each J_k $(k=1,2,3,...)$.

Let

$$\lim_{i\to\infty}(A\overline{y}_{n_i})(t)=\overline{\gamma}(t),\quad \forall t\in\mathbb{R}_+. \tag{2.47}$$

Similarly, we can discuss $\{(A\overline{y}_n)^\Delta(t)\}\quad (n=1,2,3,...)$. Similar to (2.20) and by (2.16), we have

$$\|A\overline{y}_n\|_2\leqslant\frac{\beta}{\beta-1}(a^*_{pq}+H_{pq}b^*+M_{pq}\eta^*),\ (n=1,2,3,...) \tag{2.48}$$

and

$$\begin{aligned}(A\overline{y}_n)^\Delta(t')-(A\overline{y}_n)^\Delta(t) &= \int_t^{t'} f(s,\overline{y}_n(s),\overline{y}_n^\Delta(s),(T\overline{y}_n)(s),(S\overline{y}_n)(s))\Delta s\\ &\leqslant \int_t^{t'}[a(s)G_{pq}(s)+H_{pq}b(s)]\Delta s\\ &\quad \forall t,t'\in J_k,\ t'>t\ (n=1,2,3,...)\end{aligned}$$

and by a similar method, we can prove that $\{(A\overline{y}_n)^\Delta(t)\}\quad (n=1,2,3,...)$ has a subsequence which is convergent uniformly on each $J_k\quad (k=1,2,3,...)$. For the sake of simplicity of notation, we may assume that $\{(A\overline{y}_{n_i})^\Delta(t)\}(i=1,2,3,...)$ itself converges uniformly on each $J_k\quad (k=1,2,3,...)$. Let

$$\lim_{i\to\infty}(A\overline{y}_{n_i})^\Delta(t)=\tau(t),\quad \forall t\in\mathbb{R}_+. \tag{2.49}$$

By (2.47), (2.49) and the uniformly convergence, we have

$$\overline{\gamma}^\Delta(t)=\tau(t),\quad \forall t\in\mathbb{R}_+, \tag{2.50}$$

and so, $\overline{\gamma}\in PC^\Delta[\mathbb{R}_+,\mathbb{R}]$. From (2.46) and (2.48), we get

$$\|\overline{\gamma}\|_1\leqslant\frac{\beta}{\beta-1}(a^*_{pq}+H_{pq}b^*+M_{pq}\eta^*)$$

and

$$\|\overline{\gamma}\|_2\leqslant\frac{\beta}{\beta-1}(a^*_{pq}+H_{pq}b^*+M_{pq}\eta^*).$$

Consequently, $\overline{\gamma}\in BPC^\Delta[\mathbb{R}_+,\mathbb{R}]$ and $\|\overline{\gamma}\|\leqslant\frac{\beta}{\beta-1}(a^*_{pq}+H_{pq}b^*+M_{pq}\eta^*)$.

Let $\epsilon>0$ be arbitrarily given. Choose a sufficiently large positive number μ such that

$$\int_\mu^\infty a(t)G_{pq}(t)\Delta t+H_{pq}\int_\mu^\infty b(t)\Delta t+M_{pq}\sum_{t_k\geqslant\mu}\eta_k<\epsilon. \tag{2.51}$$

For any $\mu < t < \infty$, we have, by (2.16), (2.44) and (2.45),

$$\begin{aligned} 0 &\leqslant (A\overline{y}_{n_i})^{\Delta}(t) - (A\overline{y}_{n_i})^{\Delta}(\mu) \\ &= \int_{\mu}^{t} f(s, \overline{y}_{n_i}(s), \overline{y}_{n_i}^{\Delta}(s), (T\overline{y}_{n_i})(s), (S\overline{y}_{n_i})(s))\Delta s + \sum_{\mu \leqslant t_k < t} \overline{I}_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) \\ &\leqslant \int_{\mu}^{\infty} a(s)G_{pq}(s)\Delta s + H_{pq}\int_{\mu}^{\infty} b(s)\Delta s + M_{pq}\sum_{\mu \leqslant t_k < t} \eta_k \qquad (i = 1, 2, 3, ...) \end{aligned}$$

which implies by virtue of (2.51) that

$$(A\overline{y}_{n_i})^{\Delta}(t) - (A\overline{y}_{n_i})^{\Delta}(\mu) < \epsilon, \quad \forall t > \mu \quad (i = 1, 2, 3, ...) \tag{2.52}$$

Letting $i \to \infty$ in (2.52) and obseving (2.49) and (2.50), we get

$$0 \leqslant \overline{\gamma}^{\Delta}(t) - \overline{\gamma}^{\Delta}(\mu) \leqslant \epsilon, \quad \forall t > \mu \tag{2.53}$$

On the other hand, since $\{(A\overline{y}_{n_i})^{\Delta}(t)\}$ converges unifomly to $\overline{\gamma}^{\Delta}(t)$ on $[0, \mu]$ as $i \to \infty$, there exists a positive integer i_0 such that

$$\left|(A\overline{y}_{n_i})^{\Delta}(t) - \overline{\gamma}^{\Delta}(t)\right| < \epsilon, \quad \forall t \in [0, \mu], \quad i > i_0 \tag{2.54}$$

It follows from (2.52) – (2.54) that

(2.55)

$$\begin{aligned} \left|(A\overline{y}_{n_i})^{\Delta}(t) - \overline{\gamma}^{\Delta}(t)\right| &\leqslant \left|(A\overline{y}_{n_i})^{\Delta}(t) - (A\overline{y}_{n_i})^{\Delta}(\mu)\right| + \left|(A\overline{y}_{n_i})^{\Delta}(\mu) - \overline{\gamma}^{\Delta}(\mu)\right| \\ &+ \left|\overline{\gamma}^{\Delta}(\mu) - \overline{\gamma}^{\Delta}(t)\right| < 3\epsilon, \quad \forall t > \mu, \quad i > i_0. \end{aligned}$$

By (2.54) and (2.55), we have

$$\left|(A\overline{y}_{n_i})^{\Delta}(t) - \overline{\gamma}^{\Delta}(t)\right| \leqslant 3\epsilon, \quad \forall i > i_0,$$

hence

$$\lim_{i\to\infty} ||\, A\overline{y}_{n_i} - \overline{\gamma}\, ||_2 = 0 \tag{2.56}$$

It is clear that (2.4) implies

$$(A\overline{y}_{n_i})(t_k^+) - (A\overline{y}_{n_i})(t_k^-) = I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) \quad (k, i = 1, 2, 3, ...). \tag{2.57}$$

By virtue of the uniformity of convergence of $\{(A\overline{y}_{n_i})(t)\}$, we see that

$$\lim_{i\to\infty}(A\overline{y}_{n_i})(t_k^-) = \overline{\gamma}(t_k^-) \quad \lim_{i\to\infty}(A\overline{y}_{n_i})(t_k^+) = \overline{\gamma}(t_k^+)$$

so, (2.57) implies that

$$\lim_{i\to\infty} I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) \quad (k = 1, 2, 3, ...).$$

exist and

$$\overline{\gamma}(t_k^+) - \overline{\gamma}(t_k^-) = \lim_{i\to\infty} I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) \quad (k = 1, 2, 3, ...).$$

Let

$$\lim_{i\to\infty} I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) = \psi_k, \quad (k = 1, 2, 3, ...).$$

Then $\psi_k \geqslant 0 \quad (k = 1, 2, 3, ...)$ and

$$\overline{\gamma}(t_k^+) - \overline{\gamma}(t_k^-) = \psi_k, \quad (k = 1, 2, 3, ...). \tag{2.58}$$

By (2.45) and condition $(H3)$, we have

$$I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) \leqslant M_{pq} t_k \eta_k \quad (k, i = 1, 2, 3, ...). \tag{2.59}$$

so,

$$\psi_k \leqslant M_{pq} t_k \eta_k \quad (k = 1, 2, 3, ...). \tag{2.60}$$

For any given $\epsilon > 0$, choose a sufficiently large positive integer k_0 such that

$$\sum_{k=k_0+1}^{\infty} \psi_k \leqslant M_{pq} \sum_{k=k_0+1}^{\infty} t_k \eta_k < \epsilon \tag{2.61}$$

and then, choose another sufficiently large integer i_1 such that

$$\left| I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) - \psi_k \right| < \frac{\epsilon}{k_0}, \quad \forall i > i_1 \ \ (k = 1, 2, ..., k_0). \tag{2.62}$$

It follows from (2.59) – (2.62) that

$$\begin{aligned} \sum_{k=1}^{\infty} \left| I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) - \psi_k \right| &\leqslant \sum_{k=1}^{k_0} \left| I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) - \psi_k \right| + \sum_{k=k_0+1}^{\infty} I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) \\ &+ \sum_{k=k_0+1}^{\infty} \psi_k \\ &\leqslant \epsilon + \epsilon + \epsilon = 3\epsilon, \quad \forall i > i_1, \end{aligned}$$

hence

$$\lim_{i\to\infty} \sum_{k=1}^{\infty} \left| I_k(\overline{y}_{n_i}^{\Delta}(t_k^-)) - \psi_k \right| = 0. \tag{2.63}$$

By (2.57) and (2.58), we have

$$(A\overline{y}_{n_i})(t) = (A\overline{y}_{n_i})(0) + \int_0^t (A\overline{y}_{n_i})^{\Delta}(s)\Delta s + \sum_{0<t_k<t} I_k(\overline{y}_{n_i}^{\Delta}(t_k^-))$$

and

$$\overline{\gamma}(t) = \overline{\gamma}(0) + \int_0^t \overline{\gamma}^{\Delta}(s)\Delta s + \sum_{0<t_k<t} \psi_k$$

which imply

(2.64)

$$\begin{aligned}\left|(A\overline{y}_{n_i})(t)-\overline{\gamma}(t)\right| &\leqslant \left(t+\sum_{i=1}^{m-2}\alpha_i\right)\|A\overline{y}_{n_i}-\overline{\gamma}\|_2\\ &+ \left(t+\sum_{i=1}^{m-2}\alpha_i\right)\sum_{k=1}^{\infty}\left|I_k(\overline{y}_{n_i}^{\Delta}(t_k^-))-\psi_k\right|\\ &+ \frac{t+\sum\limits_{i=1}^{m-2}\alpha_i}{\sum\limits_{i=1}^{m-2}\alpha_i}\left|(A\overline{y}_{n_i})(0)-\overline{\gamma}(0)\right|\end{aligned}$$

(2.64) implies

$$\begin{aligned}\sup_{t\in\mathbb{R}_+}\frac{\left|(A\overline{y}_{n_i})(t)-\overline{\gamma}(t)\right|}{t+\sum_{i=1}^{m-2}\alpha_i} &\leqslant \|A\overline{y}_{n_i}-\overline{\gamma}\|_2+\sum_{k=1}^{\infty}\left|I_k(\overline{y}_{n_i}^{\Delta}(t_k^-))-\psi_k\right|\\ &+ \frac{\left|(A\overline{y}_{n_i})(0)-\overline{\gamma}(0)\right|}{\sum_{i=1}^{m-2}\alpha_i}.\end{aligned}$$

By $(2.56), (2.63)$ and (2.64), we have

$$\lim_{i\to\infty}\|A\overline{y}_{n_i}-\overline{\gamma}\|_1=0. \tag{2.65}$$

It follows from (2.56) and (2.65) that $\|A\overline{y}_{n_i}-\overline{\gamma}\|\to 0, \quad i\to\infty$, and the relative compactness of $A(Q_{pq})$ is proved. □

LEMMA 2.3. *Let (H1)-(H3) be satisfied. Then $y\in Q_+\cap C^{\Delta^2}[\mathbb{R}_{++},\mathbb{R}]$ is a positive solution of IBVP* (1.1) *if and only if $y\in Q_+$ is a solution of the following impulsive integral equation*

(2.66)

$$\begin{aligned}y(t) &= \sum_{i=1}^{m-2}\alpha_i[\frac{1}{\beta-1}\{\int_0^{\infty}f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))ds+\sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))\}\\ &+ \int_0^{\xi_i}f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))ds+\sum_{t_k<\xi_i}\overline{I}_k(y^{\Delta}(t_k^-))]\\ &+ \frac{t}{\beta-1}[\int_0^{\infty}f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))ds+\sum_{k=1}^{\infty}\overline{I}_k(y^{\Delta}(t_k^-))]\\ &+ \int_0^t(t-s)f(s,y(s),y^{\Delta}(s),(Ty)(s),(Sy)(s))ds\\ &+ \sum_{t_k<t}[(t-t_k)\overline{I}_k(y^{\Delta}(t_k^-))+I_k(y^{\Delta}(t_k^-))].\end{aligned}$$

PROOF. First, suppose that $y \in Q_+ \cap C^{\Delta^2}[\mathbb{R}_{++}, \mathbb{R}]$ is a solution of IBVP (1.1). It is easy to see by integration of (1.1) that

(2.67)

$$y^{\Delta}(t) = y^{\Delta}(0) + \int_0^t f(s, y(s), y^{\Delta}(s), (Ty)(s), (Sy)(s))\Delta s + \sum_{t_k<t} \overline{I}_k(y^{\Delta}(t_k^-)).$$

Under conditions $(H1)-(H3)$, we have shown in the proof of Lemma 2.2 that the infinite integral (2.8) and the infinite series (2.12) are convergent. So, by taking limits as $t \to \infty$ in both sides of (2.67) and using the relation $y^{\Delta}(\infty) = \beta y^{\Delta}(0)$, we get

(2.68)

$$y^{\Delta}(0) = \frac{1}{\beta-1}\left[\int_0^{\infty} f(s, y(s), y^{\Delta}(s), (Ty)(s), (Sy)(s))\Delta s + \sum_{k=1}^{\infty} \overline{I}_k(y^{\Delta}(t_k^-))\right]\Delta s$$

Integrating (2.67) from 0 to t, we obtain

(2.69)

$$\begin{aligned} y(t) &= y(0) + ty^{\Delta}(0) + \int_0^t \int_0^s f(\tau, y(\tau), y^{\Delta}(\tau), (Ty)(\tau), (Sy)(\tau))\Delta r \Delta s \\ &+ \sum_{t_k<t} I_k(y^{\Delta}(t_k^-)) + \int_0^t \sum_{t_k<s} \overline{I}_k(y^{\Delta}(t_k^-))\Delta s. \end{aligned}$$

Now, substituting (2.68) into (2.67) and using the relation $y(0) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\xi_i)$ we see that $y(t)$ satisfies equation (2.66).

Conversely, if $y \in Q_+$ is a solution of equation (2.66), then direct differention of (2.66) gives

(2.70)

$$\begin{aligned} y^{\Delta}(t) &= \frac{1}{\beta-1}\left[\int_0^{\infty} f(s, y(s), y^{\Delta}(s), (Ty)(s), (Sy)(s))\Delta s + \sum_{k=1}^{\infty} \overline{I}_k(y^{\Delta}(t_k^-))\right] \\ &+ \int_0^t f(s, y(s), y^{\Delta}(s), (Ty)(s), (Sy)(s))\Delta s + \sum_{t_k<t} \overline{I}_k(y^{\Delta}(t_k^-)), \end{aligned}$$

and

$$y^{\Delta\Delta}(t) = f(t, y(t), y^{\Delta}(t), (Ty)(t), (Sy)(t)), \quad \forall t \in \mathbb{R}'_{++}.$$

So, $y \in Q_+ \cap C^{\Delta^2}[\mathbb{R}_{++}, \mathbb{R}]$ and

$$y(t_k^+) - y(t_k^-) = I_k(y^{\Delta}(t_k^-)), \quad y^{\Delta}(t_k^+) - y^{\Delta}(t_k^-) = \overline{I}_k(y^{\Delta}(t_k^-)), \quad (k = 1,\ 2,\ 3, \cdots).$$

By (2.66) and (2.70), we have $y(0) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\xi_i)$. Moreover, taking limits as $t \to \infty$ in (2.70), we see that $y^{\Delta}(\infty) = \beta y^{\Delta}(0)$.

Hence, $y(t)$ is a positive solution of IBVP (1.1). □

3. Main Results

In this section, we show that IBVP (1.1) has at least one position solution by using fixed point theorem which is given below.

THEOREM 3.1. (**Leray – Schauder Nonlinear Alternative Theorem**) [**1**] *Let C be a convex subset of a Banach space, U be a open subset of C with $0 \in U$. Then every completely continuouns map $T : \bar{U} \to C$ has at least one of the two following properties:*

(E_1) *There exist an $u \in \bar{U}$ such that $Tu = u$.*
(E_2) *There exist an $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u = \lambda T u$.*

THEOREM 3.2. *Assume Let that conditions $(H1)-(H3)$ hold anad the following condition is satisfied: there exist positive constant r such that*

$$\frac{\beta}{\beta - 1}\Big[a_r^* + H_r b^* + M_r \eta^*\Big] \leqslant r. \tag{3.1}$$

where a_r^*. b^* and η^* are defined $(H2)$ and $(H3)$, and H_r and M_r are two equalities below (2.7) *and* (2.11), *respectively. Then the IBVP* (1.1) *has a positive solution $y = y(t)$ such that*

$$0 < \frac{y(t)}{t + \sum_{i=1}^{m-2} \alpha_i} \leqslant r, \quad 0 < y^{\Delta}(t) \leqslant r, \quad t \in \mathbb{R}_{++}.$$

PROOF. Let us consider the following IBVP:

$$\begin{cases} y^{\Delta\Delta}(t) = \lambda(t) f(t, y(t), y^{\Delta}(t), (Ty)(t), (Sy)(t)), \quad \forall t \in \mathbb{R}'_{++}, \\ y(0) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\xi_i), \quad y^{\Delta}(\infty) = \beta y^{\Delta}(0), \\ y(t_k^+) - y(t_k^-) = I_k(y^{\Delta}(t_k^-)), \quad k = 1,\ 2,\ 3, \cdots, \\ y^{\Delta}(t_k^+) - y^{\Delta}(t_k^-) = \overline{I}_k(y^{\Delta}(t_k^-)), \quad k = 1,\ 2,\ 3, \cdots \end{cases} \tag{3.2}$$

We know that solving (3.2) is equivalent to solving the fixed point problem $y = \lambda A y$. Assume that

$$\Omega_r = \{y \in Q : \|y\| < r\}.$$

We claim that there is no $y \in \partial\Omega_r$ such that $y = \lambda A y$ for $\lambda \in (0, 1)$. The proof is immediate, because if there exist $y \in \partial\Omega_r$ with $y = \lambda A y$, then by (2.19), (2.20), we have for $y \in Q \cap \partial\Omega_r$ and $\lambda \in (0,1)$,

$$\begin{aligned} \|y(t)\| = \|\lambda(Ay)(t)\| &\leqslant \lambda \frac{\beta}{\beta-1}\Big[a_r^* + H_r b^* + M_r \eta^*\Big] \\ &< \frac{\beta}{\beta-1}\Big[a_r^* + H_r b^* + M_r \eta^*\Big], \end{aligned}$$

Therefore, we conclude that $\|y\| = \|\lambda Ay\| < \dfrac{\beta}{\beta-1}\Big[a_r^* + H_r b^* + M_r \eta^*\Big]$. This yields that

$$r < \frac{\beta}{\beta-1}\Big[a_r^* + H_r b^* + M_r \eta^*\Big],$$

which is contradiction with (3.1). Then by means of Theorem 3.1, the IBVP (1.1) has a positive solution $y = y(t)$ such that

$$0 < \frac{y(t)}{t + \sum_{i=1}^{m-2} \alpha_i} \leqslant r, \quad 0 < y^{\Delta}(t) \leqslant r, \quad t \in \mathbb{R}_{++}.$$

□

4. Example

To illustrate how our main result can be used in practice we present an example.

EXAMPLE 4.1. In IBVP (1.1), suppose that $\mathbb{T} = \mathbb{R}$, $\beta = 2$, $m = 4$, $\alpha_1 = \alpha_2 = \dfrac{1}{2}$, $\mu_1 = \dfrac{1}{5}$, $\mu_2 = \dfrac{1}{6}$, $t_k = 2k$, i.e.,

(4.1)

$$\begin{cases} y''(t) = \dfrac{3^{-2t}}{100\sqrt{t+1}}\left(\dfrac{1}{\sqrt{y(t)+1}} + \ln(1+y'(t))\right) \\ +\dfrac{3^{-3t}}{90\sqrt{t+1}}\left(\displaystyle\int_0^t (1+ts+s^2)^{-1} y(s)ds + \int_0^\infty e^{-s} sin^2(t-s) y(s)ds\right) \\ \Delta y\mid_{t=t_k} = I_k(v), \qquad \Delta y'\mid_{t=t_k} = \overline{I}_k(v), \\ y(0) = \dfrac{1}{2} y'\Big(\dfrac{1}{5}\Big) + \dfrac{1}{2} y'\Big(\dfrac{1}{6}\Big), \\ y'(\infty) = 2y'(0) \end{cases}$$

where

$$D(t,s) = (1+ts+s^2)^{-1}, \quad E(t,s) = e^{-s} sin^2(t-s)$$

$$f(t,u,v,w,z) = \frac{3^{-2t}}{100\sqrt{t+1}}\Big(\frac{1}{\sqrt{u+1}} + \ln(1+v)\Big) + \frac{3^{-3t}}{90\sqrt{t+1}}(w+z),$$

$$I_k(v) = \frac{2.3^{-k}}{\sqrt{v+2}} 2k, \quad \overline{I}_k(v) = \frac{e^{-k} + 3^{-k}}{\sqrt{v+2}}.$$

It is easy to see that condition $(H1)$ is satisfied and $d^* \leqslant \frac{\pi}{2}, \quad e^* \leqslant 1$

$$\begin{aligned} f(t,u,v,w.z) &\leqslant a(t)G(u) + b(t)H(v,w,z) \\ &= \frac{3^{-2t}}{100\sqrt{t+1}} \frac{1}{\sqrt{u+1}} + \frac{3^{-2t}}{\sqrt{t+1}} \left[\frac{1}{100} \ln(1+v) + \frac{1}{90}(w+z) \right], \end{aligned}$$

so, condition $(H2)$ is satisfied for

$$a(t) = \frac{3^{-2t}}{100\sqrt{t+1}}, \quad G(u) = \frac{1}{\sqrt{u+1}}, \quad b(t) = \frac{3^{-2t}}{\sqrt{t+1}}$$

with

$$H_r(t) = \frac{2}{\sqrt{r(t+1)+4}}$$

$$a_r^* = \int_0^\infty a(t)G_r(t)dt < \frac{1}{50}\frac{1}{\sqrt{r}} \int_0^\infty \frac{3^{-2t}}{t+1} dt < \frac{1}{\sqrt{r}}.0,01487 < \infty$$

and

$$b^* = \int_0^\infty \frac{e^{-3t}}{\sqrt{t+1}} dt < \int_0^1 \frac{dt}{(1+t)^{\frac{1}{2}}} + \int_1^\infty 3^{-2t} dt = 0,850 < \infty.$$

It is obvious that condition $(H3)$ is satisfied for $\eta_k = e^{-k} + 3^{-k}$ $(\eta^* = \frac{1}{e-1} + \frac{1}{2} = 1,0819)$ and $F(v) = \frac{1}{\sqrt{v+1}}$.

Since

$$\frac{\beta}{\beta - 1} \left[a_r^* + G_r b^* + M_r \gamma^* \right] < 1,7661 < r = 4,$$

the condition (3.1) is satisfied. Then all conditions of Theorem 3.2 hold. Hence, we find that the IBVP (4.1) has at least one positive solution $y = y(t)$ such that

$$0 < \frac{y(t)}{t + \sum_{i=1}^{m-2} \alpha_i} \leqslant 4, \quad 0 < y^\Delta(t) \leqslant 4, \quad t \in \mathbb{R}_{++}.$$

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Received by editors 23.04.2020; Revised version 22.08.2020; Available online 31.08.2020.

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