JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
J. Int. Math. Virtual Inst., Vol. **10**(2)(2020), 271-286
DOI: 10.7251/JIMVI2002271G

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

A DECOMPOSITION OF CONTINUITY AND CONTRA CONTINUITY IN IDEAL NANO TOPOLOGICAL SPACES

Carlos Granados

ABSTRACT. In this paper, we introduce the concepts of continuity and contra continuity on $nI^\star_C$-set, $\star$-nI_A-set, $\star$-nI_η-set and $\star$-nI_C-set. Besides, show some of their properties, as well as, we find some relations between them.

1. Introduction and preliminaries

The notion of contra continuity was introduced by Dontchev [**1**] in a topological space, this concept has been growing in several sub-areas of general topology such that soft topological space, fuzzy topological space, bitopological space, tritopological space, etc. On the other hand, many mathematicians have been interested in studying the behaviour of open and closed sets in ideal spaces, Premkumar. and Rameshpali [**3**] introduced the concepts of $nI^\star_C$-set, $\star$-nI_A-set, $\star$-nI_η-set and $\star$-nI_C-set. Besides, they showed and proved some properties.

In this paper, we use the notions of continuity, contra continuity and ideal spaces to define and study the $nI^\star_C$-continuous, $nI^\star_C$-irresolute, contra $nI^\star_C$-continuous, contra $nI^\star_C$-irresolute, $\star$-nI_A-continuous, $\star$-nI_A-irresolute, contra $\star$-nI_A-continuous, contra $\star$-nI_A-irresolute, $\star$-nI_η-continuous, $\star$-nI_η-irresolute, contra $\star$-nI_η-continuous, contra $\star$-nI_η-irresolute, $\star$-nI_C-continuous, $\star$-nI_C-irresolute, contra $\star$-nI_C-continuous and contra $\star$-nI_C-irresolute functions. Furthermore, we show some of their properties, as well as, we find some relations between them.

2010 *Mathematics Subject Classification.* Primary: 54C05; Secondary: 54C08.
Key words and phrases. Continuity, contra continuity, $nI^\star_C$-set, $\star$-nI_A-set, $\star$-nI_η-set and $\star$-nI_C-set.
Communicated by Daniel A. Romano.

DEFINITION 1.1. ([7]) An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions:

(1) $\emptyset \in I$.
(2) $A \in I$ and $B \subset A$, then $B \in I$.
(3) If $A, B \in I$, then $A \cup B \in I$.

DEFINITION 1.2. ([5]) A nano topological space $(U, \mathcal{N})$ with an ideal I is called an ideal nano topological space and is denoted by $(U, \mathcal{N}, I)$. $G_n(x) = \{G_n : x \in G_n, G_n \in \mathcal{N}\}$, denotes the family of nano open sets containing x.

DEFINITION 1.3. ([4]) A subset A of a space $(U, \mathcal{N}, I)$ is called $n\star$-closed if $A_n^\star \subseteq A$. The complement of a $n\star$-closed set is called $n\star$-open.

DEFINITION 1.4. ([3]) A subset A of a space $(U, \mathcal{N}, I)$ is said to be:

(1) α-nI-open if $A \subseteq n\text{-}Int(n\text{-}Cl^\star(n\text{-}Int(A)))$.
(2) Pre-nI-open if $A \subseteq n\text{-}Int(n\text{-}Cl^\star(A))$.

DEFINITION 1.5. ([2]) A subset A of a space $(U, \mathcal{N}, I)$ is said to be semi*-nI-open if $A \subseteq n\text{-}Cl(n\text{-}Int^\star(A))$.

DEFINITION 1.6. ([6]) A subset A of a space $(U, \mathcal{N}, I)$ is said to be $t^\#$-nI-set if $n\text{-}Int(A) = n\text{-}Cl^\star(n\text{-}Int(A))$.

DEFINITION 1.7. ([6]) A subset A of a space $(U, \mathcal{N}, I)$ is said to be pre-nI-regular if A is pre-nI-open and $t^\#$-nI-set.

DEFINITION 1.8. ([3]) A subset A of a space $(U, \mathcal{N}, I)$ is said to be:

(1) $\star$-nI_C-set if $A = S \cap P$, where S is $n\star$-open set and P is pre-nI-set in U.
(2) $\star$-nI_η-set if $A = S \cap P$, where S is $n\star$-open set and P is α-nI-closed in U.
(3) $\star$-nI_A-set if $A = S \cap P$, where S is $n\star$-open set and $P = n\text{-}Cl(n\text{-}Int^\star(P))$.
(4) $\star$-$nI_C^\star$-set if $A = S \cap P$, where S is $n\star$-open set and P is pre-nI-regular set.

REMARK 1.1. ([3]) Let $(U, \mathcal{N}, I)$ be an ideal nano topological space and $A \subseteq U$. Then, the following statements hold:

(1) Every $nI_C^\star$-set is $\star$-nI_C-set.
(2) Every $\star$-nI_A-set is $\star$-nI_η-set.
(3) Every $\star$-nI_η-set is $\star$-nI_C-set.
(4) Every $\star$-nI_A-set is $\star$-nI_C-set.
(5) Every $n\star$-open set is $nI_C^\star$-open-set, $\star$-nI_C-open-set, $\star$-nI_A-open-set and $\star$-nI_η-open-set.

2. Continuity and contra continuity via $nI_C^\star$-set

In this section, we introduce the notion of $nI_C^\star$-continuous, $nI_C^\star$-irresolute, contra $nI_C^\star$-continuous and contra $nI_C^\star$-irresolute functions. Besides, we show some of their properties.

DEFINITION 2.1. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called $nI_C^\star$-continuous if $f^{-1}(V)$ is a $nI_C^\star$-open-set of $(O, \mathcal{N}, I)$ for every n-open set V of $(P, \mathcal{N}', J)$.

Definition 2.2. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called $nI^{\star}_{C}$-irresolute if $f^{-1}(V)$ is a $nI^{\star}_{C}$-open-set of $(O, \mathcal{N}, I)$ for every $nI^{\star}_{C}$-open-set V of $(P, \mathcal{N}', J)$.

Theorem 2.1. *Every $n\star$-continuous function is $nI^{\star}_{C}$-continuous.*

Proof. Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a $n\star$-continuous function. Let V be a n-open set of $(P, \mathcal{N}', J)$. V is n-open-set. Since f is $n\star$-continuous, then $f^{-1}(V)$ is a n-open-set of $(O, \mathcal{N}, I)$. By the Remark 1.1, $f^{-1}(V)$ is $nI^{\star}_{C}$-open. Therefore f is $nI^{\star}_{C}$-continuous. □

Theorem 2.2. *Every $nI^{\star}_{C}$-irresolute function is $nI^{\star}_{C}$-continuous.*

Proof. Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a $nI^{\star}_{C}$-irresolute function. Let V be a n-open set of $(P, \mathcal{N}', J)$. V is $nI^{\star}_{C}$-open-set. Since f is $nI^{\star}_{C}$-irresolute, then $f^{-1}(V)$ is a $nI^{\star}_{C}$-open-set of $(O, \mathcal{N}, I)$. Therefore f is $nI^{\star}_{C}$-continuous. □

Theorem 2.3. *Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \to (Q, \mathcal{N}_{*}, \mathcal{K})$ be any two functions. Then, the following statements hold:*

1. *$g \circ f$ is $nI^{\star}_{C}$-irresolute if both f and g are $nI^{\star}_{C}$-irresolute.*
2. *$g \circ f$ is $nI^{\star}_{C}$-continuous if g is $nI^{\star}_{C}$-continuous and f is $nI^{\star}_{C}$-irresolute.*

Proof. Consider two $nI^{\star}_{C}$-irresolute functions, $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \to (Q, \mathcal{N}'_{*}, K))$ are $nI^{\star}_{C}$-irresolute functions. As g is consider to be a $nI^{\star}_{C}$-irresolute function, by Definition 2.2, for every $nI^{\star}_{C}$-open-set $q \subseteq (Q, \mathcal{N}'_{*}, K))$, $g^{-1}(q) = G$ which is $nI^{\star}_{C}$-open-set in $(P, \mathcal{N}', J)$. Since f is $nI^{\star}_{C}$-open-irresolute, $(g \circ f)^{-1}(q) = f^{-1}(g^{-1}(q)) = f^{-1}(G) = S$ and S is a $nI^{\star}_{C}$-open-set in $(O, \mathcal{N}, I)$. Hence $(g \circ f)$ is $nI^{\star}_{C}$-irresolute.

The proof of (2) is similar to (1) □

Definition 2.3. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called contra $nI^{\star}_{C}$-continuous if $f^{-1}(V)$ is a $nI^{\star}_{C}$-closed-set of $(O, \mathcal{N}, I)$ for every n-open set V of $(P, \mathcal{N}', J)$.

Definition 2.4. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called contra $nI^{\star}_{C}$-irresolute if $f^{-1}(V)$ is a $nI^{\star}_{C}$-closed-set of $(O, \mathcal{N}, I)$ for every $nI^{\star}_{C}$-open-set V of $(P, \mathcal{N}', J)$.

Theorem 2.4. *Let f: $(O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a function. Then the following conditions are equivalent*

1. *f is contra $nI^{\star}_{C}$-continuous.*
2. *The inverse image of each n-open set in P is $nI^{\star}_{C}$-closed-set in O.*
3. *The inverse image of each n-closed set in P is $nI^{\star}_{C}$-open-set in O.*
4. *For each point o in O and each n-closed set G in P with f(o) $\in$ G, there is an $nI^{\star}_{C}$-open-set U in O containing o such that f(U) $\subset$ G.*

Proof. (1) $\Rightarrow$ (2). Let G be n-open in P. Then P $-$ G is n-closed in P. By definition of contra $nI^{\star}_{C}$-continuous, f^{-1}(P $-$ G) is $nI^{\star}_{C}$-open-set in O. But f^{-1}(P $-$ G) = O - f^{-1}(G). This implies f^{-1}(G) is $nI^{\star}_{C}$-closed-set in O.

(2) ⇒ (3) Let G be any n-closed set in P. Then P−G is n-open set in P. By the assumption of (2), f^{-1}(P − G) is $nI^{\star}_{C}$-closed-set in O. But f^{-1}(P − G) = O - f^{-1}(G). This implies f^{-1}(G) is $nI^{\star}_{C}$-open-set in O.

(3) ⇒ (4). Let o ∈ O and G be any n-closed set in P with f(o) ∈ G. By (3), f^{-1}(G) is $nI^{\star}_{C}$-open-set in O. Set U= f^{-1}(G). Then there is a $nI^{\star}_{C}$-open-set U in O containing o such that f(U) ⊂ G.

(4) ⇒ (1). Let o ∈ O and G be any n-closed set in P with f(o) ∈ G. Then P − G is n-open in P with f(o) ∈ G. By (4), there is a $nI^{\star}_{C}$-open-set U in O containing o such that f(U) ⊂ G. This implies U= f^{-1}(G) . Therefore, O − U = O − f^{-1}(G) = f^{-1}(P−G) which is $nI^{\star}_{C}$-closed-set in O. □

THEOREM 2.5. *Let f: (O, $\mathcal{N}$, I) → (P, $\mathcal{N}'$, J) be a function. Then the following conditions are equivalent*

(1) *f is contra $nI^{\star}_{C}$-irresolute.*
(2) *The inverse image of each $nI^{\star}_{C}$-open-set in P is $nI^{\star}_{C}$-closed-set in O.*
(3) *The inverse image of each $nI^{\star}_{C}$-closed-set in P is $nI^{\star}_{C}$-open-set in O.*
(4) *For each point o in O and each $nI^{\star}_{C}$-closed-set set G in P with f(o) ∈ G, there is an $nI^{\star}_{C}$-open-set U in O containing o such that f(U) ⊂ G.*

PROOF. The proof is similar to the Theorem 2.4 □

THEOREM 2.6. *Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \to (Q, \mathcal{N}'_*, K)$. Then the following properties hold:*

(1) *If f is contra $nI^{\star}_{C}$-irresolute and g is contra $nI^{\star}_{C}$-irresolute then g ∘ f is $nI^{\star}_{C}$-irresolute.*
(2) *If f is contra $nI^{\star}_{C}$-irresolute and g is contra $nI^{\star}_{C}$-continuous then g ∘ f is $nI^{\star}_{C}$-continuous.*
(3) *If f is contra $nI^{\star}_{C}$-continuous and g is contra n⋆-continuous then g ∘ f is $nI^{\star}_{C}$-continuous.*
(4) *If f is $nI^{\star}_{C}$-irresolute and g is contra $nI^{\star}_{C}$-irresolute then g ∘ f is contra $nI^{\star}_{C}$-irresolute.*
(5) *If f is contra $nI^{\star}_{C}$-irresolute and g is $nI^{\star}_{C}$-irresolute then g ∘ f is contra $nI^{\star}_{C}$-irresolute.*
(6) *If f is contra $nI^{\star}_{C}$-irresolute and g is contra n⋆-continuous then g ∘ f is $nI^{\star}_{C}$-continuous.*

PROOF. (1) Let G be $nI^{\star}_{C}$-open-set in Q. Since g is contra $nI^{\star}_{C}$-irresolute, g^{-1}(G) is $nI^{\star}_{C}$-closed-set in P. Since f is contra $nI^{\star}_{C}$-irresolute, $(g \circ f)^{-1}$(G) = $f^{-1}(g^{-1}$(G)) is $nI^{\star}_{C}$-open-set in O. Therefore g ∘ f is $nI^{\star}_{C}$-irresolute.

The proofs of (2), (3), (4), (5) and (6) are similar to (1). □

THEOREM 2.7. *Let f: (O, $\mathcal{N}$, I) → (P, $\mathcal{N}'$, J) be a function and g: (O, $\mathcal{N}$, I) → ((O, $\mathcal{N}$, I) × (P, $\mathcal{N}'$, J)) the graph function of f, defined by g(o) = (o, f(o)) for every o ∈ O. If g is contra $nI^{\star}_{C}$-continuous, then f is contra $nI^{\star}_{C}$-continuous.*

PROOF. Let G be an n-open set in (P, $\mathcal{N}'$, J). Then ((O, $\mathcal{N}$, I) × G) is an n-open set in ((O, $\mathcal{N}$, I) × (P, $\mathcal{N}'$, J)). It follows from Theorem 2.4, that

$f^{-1}(G) = g^{-1}((O, \mathcal{N}, I) \times G)$ is $nI^{\star}_{C}$-closed-set in $(O, \mathcal{N}, \mathcal{I})$. Thus, f is contra $nI^{\star}_{C}$-continuous. □

Definition 2.5. A space $(O, \mathcal{N}, I)$ is said to be an $nI^{\star}_{C}$-space if every $nI^{\star}_{C}$-open set is n-open in $(O, \mathcal{N}, I)$.

Theorem 2.8. *A function f : $(O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is contra $nI^{\star}_{C}$-continuous and O is $nI^{\star}_{C}$-space, then f is contra n⋆-continuous.*

Proof. Let G be n-closed set in P. Since f is contra $nI^{\star}_{C}$-continuous, $f^{-1}(G)$ is $nI^{\star}_{C}$-open in O. Since O is a $nI^{\star}_{C}$-space, $f^{-1}(G)$ is n-open in O. Therefore f is contra n⋆-continuous. □

Definition 2.6. An nano ideal topological space $(O, \mathcal{N}, I)$ is said to be $nI^{\star}_{C}$-connected if $(O, \mathcal{N}, I)$ cannot be expressed as the union of two disjoint non empty $nI^{\star}_{C}$-open-set subsets of $(O, \mathcal{N}, I)$.

Theorem 2.9. *A contra $nI^{\star}_{C}$-continuous image of a $nI^{\star}_{C}$-connected space is nano connected.*

Proof. Let f : $(O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a contra $nI^{\star}_{C}$-continuous function of a $nI^{\star}_{C}$-connected space $(O, \mathcal{N}, I)$ onto a nano topological space $(P, \mathcal{N}')$. If possible, let P be nano disconnected. Let G and S form a nano disconnection of P. Then G and S are nano clopen and $P = G \cup S$ where $G \cap S = \phi$. Since f is contra $nI^{\star}_{C}$-continuous, $O = f^{-1}(P) = f^{-1}(G \cup S) = f^{-1}(G) \cup f^{-1}(S)$, where $f^{-1}(G)$ and $f^{-1}(S)$ are non empty $nI^{\star}_{C}$-open sets in O. Also $f^{-1}(G) \cap f^{-1}(S) = \phi$. Hence O is not $nI^{\star}_{C}$-connected. This is a contradiction. Therefore P is nano connected. □

Lemma 2.1. *For an nano ideal topological space $(O, \mathcal{N}, \mathcal{I})$, the following statements are equivalent.*

(1) *O is $nI^{\star}_{C}$-connected.*
(2) *The only subset of O which are both $nI^{\star}_{C}$-open-set and $nI^{\star}_{C}$-closed-set are the empty set ϕ and O.*

Proof. (1) ⇒ (2) Let G be a $nI^{\star}_{C}$-open-set and $nI^{\star}_{C}$-closed-set subset of O. Then O − G is both $nI^{\star}_{C}$-open-set and $nI^{\star}_{C}$-closed-set. Since O is $nI^{\star}_{C}$-connected, O can be expressed as union of two disjoint non empty $nI^{\star}_{C}$-open-set sets O and O − G, which implies O − G is empty.

(2) ⇒ (1) Suppose $O = G \cup S$ where G and S are disjoint non empty $nI^{\star}_{C}$-open-set subsets of O. Then G is both $nI^{\star}_{C}$-open-set and $nI^{\star}_{C}$-closed-set. By assumption either $G = \phi$ or O which contradicts the assumption G and S are disjoint non empty $nI^{\star}_{C}$-open-set subsets of O. Therefore O is $nI^{\star}_{C}$-connected. □

3. Continuity and contra continuity via ⋆-nI_A-set

In this section, we introduce the notion of ⋆-nI_A-continuous, ⋆-nI_A-irresolute, contra ⋆-nI_A-continuous and contra ⋆-nI_A-irresolute functions Besides, we show some of their properties.

DEFINITION 3.1. A function $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ is called $\star$-nI_A-continuous if $f^{-1}(V)$ is a $\star$-nI_A-open-set of $(O, \mathcal{N}, I)$ for every n-open set V of $(P, \mathcal{N}', J)$.

DEFINITION 3.2. A function $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ is called $\star$-nI_A-irresolute if $f^{-1}(V)$ is a $\star$-nI_A-open-set of $(O, \mathcal{N}, I)$ for every $\star$-nI_A-open-set V of $(P, \mathcal{N}', J)$.

THEOREM 3.1. *Every $n\star$-continuous function is $\star$-nI_A-continuous.*

PROOF. Let $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ be a $n\star$-continuous function. Let V be a n-open set of $(P, \mathcal{N}', J)$. V is n-open-set. Since f is $n\star$-continuous, then $f^{-1}(V)$ is a n-open-set of $(O, \mathcal{N}, I)$. By the Remark 1.1, $f^{-1}(V)$ is $\star$-nI_A-open-set. Therefore f is $\star$-nI_A-continuous. □

THEOREM 3.2. *Every $\star$-nI_A-irresolute function is $\star$-nI_A-continuous.*

PROOF. Let $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ be a $\star$-nI_A-irresolute function. Let V be a n-open set of $(P, \mathcal{N}', J)$. V is $\star$-nI_A-open-set. Since f is $\star$-nI_A-irresolute, then $f^{-1}(V)$ is a $\star$-nI_A-open-set of $(O, \mathcal{N}, I)$. Therefore f is $\star$-nI_A-continuous. □

THEOREM 3.3. *Let $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \rightarrow (Q, \mathcal{N}_*, \mathcal{K})$ be any two functions. Then, the following statements hold:*

(1) *$g \circ f$ is $\star$-nI_A-irresolute if both f and g are $\star$-nI_A-irresolute.*
(2) *$g \circ f$ is $\star$-nI_A-continuous if g is $\star$-nI_A-continuous and f is $\star$-nI_A-irresolute.*

PROOF. Consider two $\star$-nI_A-irresolute functions, $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \rightarrow (Q, \mathcal{N}'_*, K))$ are $\star$-nI_A-irresolute functions. As g is consider to be a $\star$-nI_A-irresolute function, by Definition 3.2, for every $\star$-nI_A-open-set $q \subseteq (Q, \mathcal{N}'_*, K))$, $g^{-1}(q) = G$ which is $\star$-nI_A-open-set in $(P, \mathcal{N}', J)$. Since f is $\star$-nI_A-irresolute, $(g \circ f)^{-1}(q) = f^{-1}(g^{-1}(q)) = f^{-1}(G) = S$ and S is a $\star$-nI_A-open-set in $(O, \mathcal{N}, I)$. Hence $(g \circ f)$ is $\star$-nI_A-irresolute.

The proof of (2) is similar to (1). □

DEFINITION 3.3. A function $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ is called contra $\star$-nI_A-continuous if $f^{-1}(V)$ is a $\star$-nI_A-closed-set of $(O, \mathcal{N}, I)$ for every n-open set V of $(P, \mathcal{N}', J)$.

DEFINITION 3.4. A function $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ is called contra $\star$-nI_A-irresolute if $f^{-1}(V)$ is a $\star$-nI_A-closed-set of $(O, \mathcal{N}, I)$ for every $\star$-nI_A-open-set V of $(P, \mathcal{N}', J)$.

THEOREM 3.4. *Let f: $(O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ be a function. Then the following conditions are equivalent*

(1) *f is contra $\star$-nI_A-continuous.*
(2) *The inverse image of each n-open set in P is $\star$-nI_A-closed-set in O.*
(3) *The inverse image of each n-closed set in P is $\star$-nI_A-open-set in O.*
(4) *For each point o in O and each n-closed set G in P with f(o) $\in$ G, there is a $\star$-nI_A-open-set U in O containing o such that f(U) $\subset$ G.*

PROOF. (1) ⇒ (2). Let G be n-open in P. Then P − G is n-closed in P. By definition of contra $\star$-nI_A-continuous, f^{-1}(P − G) is $\star$-nI_A-open-set in O. But f^{-1}(P − G) = O - f^{-1}(G). This implies f^{-1}(G) is $\star$-nI_A-closed-set in O.

(2) ⇒ (3) Let G be any n-closed set in P. Then P−G is n-open set in P. By the assumption of (2), f^{-1}(P − G) is $\star$-nI_A-closed-set in O. But f^{-1}(P − G) = O - f^{-1}(G). This implies f^{-1}(G) is $\star$-nI_A-open-set in O.

(3) ⇒ (4). Let o ∈ O and G be any n-closed set in P with f(o) ∈ G. By (3), f^{-1}(G) is $\star$-nI_A-open-set in O. Set U= f^{-1}(G). Then there is a $\star$-nI_A-open-set U in O containing o such that f(U) ⊂ G.

(4) ⇒ (1). Let o ∈ O and G be any n-closed set in P with f(o) ∈ G. Then P − G is n-open in P with f(o) ∈ G. By (4), there is a $\star$-nI_A-open-set U in O containing o such that f(U) ⊂ G. This implies U= f^{-1}(G) . Therefore, O − U = O − f^{-1}(G) = f^{-1}(P−G) which is $\star$-nI_A-closed-set in O. □

THEOREM 3.5. *Let f: (O, $\mathcal{N}$, I) → (P, $\mathcal{N}'$, J) be a function. Then the following conditions are equivalent*

1. *f is contra $\star$-nI_A-irresolute.*
2. *The inverse image of each $\star$-nI_A-open-set in P is $\star$-nI_A-closed-set in O.*
3. *The inverse image of each $\star$-nI_A-closed-set in P is $\star$-nI_A-open-set in O.*
4. *For each point o in O and each $\star$-nI_A-closed-set set G in P with f(o) ∈ G, there is a $\star$-nI_A-open-set U in O containing o such that f(U) ⊂ G.*

PROOF. The proof is similar to the Theorem 3.4 □

THEOREM 3.6. *Let $f : (O,\mathcal{N}, I) \to (P,\mathcal{N}', J)$ and $g : (P,\mathcal{N}', J) \to (Q,\mathcal{N}'_*, K)$. Then the following properties hold:*

1. *If f is contra $\star$-nI_A-irresolute and g is contra $\star$-nI_A-irresolute then $g \circ f$ is $\star$-nI_A-irresolute.*
2. *If f is contra $\star$-nI_A-irresolute and g is contra $\star$-nI_A-continuous then $g \circ f$ is $\star$-nI_A-continuous.*
3. *If f is contra $\star$-nI_A-continuous and g is contra n$\star$-continuous then $g \circ f$ is $\star$-nI_A-continuous.*
4. *If f is $\star$-nI_A-irresolute and g is contra $\star$-nI_A-irresolute then $g \circ f$ is contra $\star$-nI_A-irresolute.*
5. *If f is contra $\star$-nI_A-irresolute and g is $\star$-nI_A-irresolute then $g \circ f$ is contra $\star$-nI_A-irresolute.*
6. *If f is contra $\star$-nI_A-irresolute and g is contra n$\star$-continuous then $g \circ f$ is $\star$-nI_A-continuous.*

PROOF. (1) Let G be $\star$-nI_A-open-set in Q. Since g is contra $\star$-nI_A-irresolute, g^{-1}(G) is $\star$-nI_A-closed-set in P. Since f is contra $\star$-nI_A-irresolute, $(g \circ f)^{-1}$(G) = $f^{-1}(g^{-1}$(G)) is $\star$-nI_A-open-set in O. Therefore g ∘ f is $\star$-nI_A-irresolute.

The proofs of (2), (3), (4), (5) and (6) are similar to (1). □

THEOREM 3.7. *Let f: (O, $\mathcal{N}$, I) → (P, $\mathcal{N}'$, J) be a function and g: (O, $\mathcal{N}$, I) → ((O, $\mathcal{N}$, I) × (P, $\mathcal{N}'$, J)) the graph function of f, defined by g(o) = (o, f(o)) for every o ∈ O. If g is contra $\star$-nI_A-continuous, then f is contra $\star$-nI_A-continuous.*

Proof. Let G be an n-open set in (P, $\mathcal{N}'$, J). Then ((O, $\mathcal{N}$, I) × G) is an n-open set in ((O, $\mathcal{N}$, I) × (P, $\mathcal{N}'$, J)). It follows from Theorem 3.4, that f^{-1}(G) = g^{-1}((O, $\mathcal{N}$, I) × G) is $\star$-nI_A-closed-set in (O, $\mathcal{N}$, $\mathcal{I}$). Thus, f is contra $\star$-nI_A-continuous. □

Definition 3.5. A space (O, $\mathcal{N}$, I) is said to be a $\star$-nI_A-space if every $\star$-nI_A-open set is n-open in (O, $\mathcal{N}$, I).

Theorem 3.8. *A function f : (O, $\mathcal{N}$, I) → (P, $\mathcal{N}'$, J) is contra $\star$-nI_A-continuous and O is $\star$-nI_A-space, then f is contra n$\star$-continuous.*

Proof. Let G be n-closed set in P. Since f is contra $\star$-nI_A-continuous, f^{-1}(G) is $\star$-nI_A-open in O. Since O is a $\star$-nI_A-space, f^{-1}(G) is n-open in O. Therefore f is contra n$\star$-continuous. □

Definition 3.6. An nano ideal topological space (O, $\mathcal{N}$, I) is said to be $\star$-nI_A-connected if (O, $\mathcal{N}$, I) cannot be expressed as the union of two disjoint non empty $\star$-nI_A-open-set subsets of (O, $\mathcal{N}$, I).

Theorem 3.9. *A contra $\star$-nI_A-continuous image of a $\star$-nI_A-connected space is nano connected.*

Proof. Let f : (O, $\mathcal{N}$, I) → (P, $\mathcal{N}'$, J) be a contra $\star$-nI_A-continuous function of a $\star$-nI_A-connected space (O, $\mathcal{N}$, I) onto a nano topological space (P, $\mathcal{N}'$). If possible, let P be nano disconnected. Let G and S form a nano disconnection of P. Then G and S are nano clopen and P = G ∪ S where G ∩ S = ϕ. Since f is contra $\star$-nI_A-continuous, O = f^{-1}(P) = f^{-1}(G ∪ S) = f^{-1}(G) ∪ f^{-1}(S), where f^{-1}(G) and f^{-1}(S) are non empty $\star$-nI_A-open sets in O. Also f^{-1}(G) ∩ f^{-1}(S) = ϕ. Hence O is not $\star$-nI_A-connected. This is a contradiction. Therefore P is nano connected. □

Lemma 3.1. *For an nano ideal topological space (O, $\mathcal{N}$, $\mathcal{I}$), the following statements are equivalent.*

1. *O is $\star$-nI_A-connected.*
2. *The only subset of O which are both $\star$-nI_A-open-set and $\star$-nI_A-closed-set are the empty set ϕ and O.*

Proof. (1) ⇒ (2) Let G be a $\star$-nI_A-open-set and $\star$-nI_A-closed-set subset of O. Then O − G is both $\star$-nI_A-open-set and $\star$-nI_A-closed-set. Since O is $\star$-nI_A-connected, O can be expressed as union of two disjoint non empty $\star$-nI_A-open-set sets O and O − G, which implies O − G is empty.

(2) ⇒ (1) Suppose O = G ∪ S where G and S are disjoint non empty $\star$-nI_A-open-set subsets of O. Then G is both $\star$-nI_A-open-set and $\star$-nI_A-closed-set. By assumption either G = ϕ or O which contradicts the assumption G and S are disjoint non empty $\star$-nI_A-open-set subsets of O. Therefore O is $\star$-nI_A-connected. □

4. Continuity and contra continuity via $\star$-nI_η-set

In this section, we introduce the notion of $\star$-nI_η-continuous, $\star$-nI_η-irresolute, contra $\star$-nI_η-continuous and contra $\star$-nI_η-irresolute functions. Moreover, we show some of their properties.

DEFINITION 4.1. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called $\star$-nI_η-continuous if $f^{-1}(V)$ is a $\star$-nI_η-open-set of $(O, \mathcal{N}, I)$ for every n-open set V of $(P, \mathcal{N}', J)$.

DEFINITION 4.2. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called $\star$-nI_η-irresolute if $f^{-1}(V)$ is a $\star$-nI_η-open-set of $(O, \mathcal{N}, I)$ for every $\star$-nI_η-open-set V of $(P, \mathcal{N}', J)$.

LEMMA 4.1. *For a function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$, the following statements hold:*

(1) *Every $\star$-nI_A continuous function is $\star$-nI_η continuous.*
(2) *Every $\star$-nI_A irresolute function is $\star$-nI_η irresolute.*

PROOF. (1) Let f be a $\star$-nI_A continuous function, then $f^{-1}(V)$ is $\star$-nI_A-open-set in O, by the remark 1.1, $f^{-1}(V)$ is $\star$-nI_η-open-set, since f is $\star$-nI_A continuous, V is n-open in P. Therefore, f is $\star$-nI_η continuous.
The proof of (2) is similar to (1). □

THEOREM 4.1. *Every $\star$-nI_η-continuous function is $\star$-nI_η-continuous.*

PROOF. Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a $n\star$-continuous function. Let V be a n-open set of $(P, \mathcal{N}', J)$. V is n-open-set. Since f is $n\star$-continuous, then $f^{-1}(V)$ is a n-open-set of $(O, \mathcal{N}, I)$. By the Remark 1.1, $f^{-1}(V)$ is $\star$-nI_η-open-set. Therefore f is $\star$-nI_η-continuous. □

THEOREM 4.2. *Every $\star$-nI_η-irresolute function is $\star$-nI_η-continuous.*

PROOF. Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a $\star$-nI_η-irresolute function. Let V be a n-open set of $(P, \mathcal{N}', J)$. V is $\star$-nI_η-open-set. Since f is $\star$-nI_η-irresolute, then $f^{-1}(V)$ is a $\star$-nI_η-open-set of $(O, \mathcal{N}, I)$. Therefore f is $\star$-nI_η-continuous. □

THEOREM 4.3. *Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \to (Q, \mathcal{N}_*, \mathcal{K})$ be any two functions. Then, the following statements hold:*

(1) *$g \circ f$ is $\star$-nI_η-irresolute if both f and g are $\star$-nI_η-irresolute.*
(2) *$g \circ f$ is $\star$-nI_η-continuous if g is $\star$-nI_η-continuous and f is $\star$-nI_η-irresolute.*

PROOF. Consider two $\star$-nI_η-irresolute functions, $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \to (Q, \mathcal{N}'_*, K))$ are $\star$-nI_η-irresolute functions. As g is consider to be a $\star$-nI_η-irresolute function, by Definition 4.2, for every $\star$-nI_η-open-set $q \subseteq (Q, \mathcal{N}'_*, K))$, $g^{-1}(q) = G$ which is $\star$-nI_η-open-set in $(P, \mathcal{N}', J)$. Since f is $\star$-nI_η-irresolute, $(g \circ f)^{-1}(q) = f^{-1}(g^{-1}(q)) = f^{-1}(G) = S$ and S is a$\star$-nI_η-open-set in $(O, \mathcal{N}, I)$. Hence $(g \circ f)$ is $\star$-nI_η-irresolute.
The proof of (2) is similar to (1). □

DEFINITION 4.3. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called contra $\star$-nI_η-continuous if $f^{-1}(V)$ is a $\star$-nI_η-closed-set of $(O, \mathcal{N}, I)$ for every n-open set V of $(P, \mathcal{N}', J)$.

Definition 4.4. A function $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ is called contra $\star$-nI_η-irresolute if $f^{-1}(V)$ is a $\star$-nI_η-closed-set of $(O, \mathcal{N}, I)$ for every $\star$-nI_η-open-set V of $(P, \mathcal{N}', J)$.

Lemma 4.2. *For a function $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$, the following statements hold:*

(1) *Every contra $\star$-nI_A continuous function is contra $\star$-nI_η continuous.*
(2) *Every contra $\star$-nI_A irresolute function is contra $\star$-nI_η irresolute.*

Proof. (1) Let f be a contra $\star$-nI_A continuous function, then $f^{-1}(V)$ is $\star$-nI_A-closed-set in O, by the remaar 1.1, $f^{-1}(V)$ is $\star$-nI_η-closed-set, since f is contra $\star$-nI_A continuous, V is n-open in P. Therefore, f is contra $\star$-nI_η continuous.

The proof of (2) is similar to (1). □

Theorem 4.4. *Let f: $(O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ be a function. Then the following conditions are equivalent*

(1) *f is contra $\star$-nI_η-continuous.*
(2) *The inverse image of each n-open set in P is $\star$-nI_η-closed-set in O.*
(3) *The inverse image of each n-closed set in P is $\star$-nI_η-open-set in O.*
(4) *For each point o in O and each n-closed set G in P with f(o) $\in$ G, there is a$\star$-nI_η-open-set U in O containing o such that f(U) $\subset$ G.*

Proof. (1) $\Rightarrow$ (2). Let G be n-open in P. Then P $-$ G is n-closed in P. By definition of contra $\star$-nI_η-continuous, f^{-1}(P $-$ G) is $\star$-nI_A-open-set in O. But f^{-1}(P $-$ G) = O - f^{-1}(G). This implies f^{-1}(G) is $\star$-nI_η-closed-set in O.

(2) $\Rightarrow$ (3) Let G be any n-closed set in P. Then P$-$G is n-open set in P. By the assumption of (2), f^{-1}(P $-$ G) is $\star$-nI_η-closed-set in O. But f^{-1}(P $-$ G) = O - f^{-1}(G). This implies f^{-1}(G) is $\star$-nI_η-open-set in O.

(3) $\Rightarrow$ (4). Let o $\in$ O and G be any n-closed set in P with f(o) $\in$ G. By (3), f^{-1}(G) is $\star$-nI_η-open-set in O. Set U= f^{-1}(G). Then there is a $\star$-nI_η-open-set U in O containing o such that f(U) $\subset$ G.

(4) $\Rightarrow$ (1). Let o $\in$ O and G be any n-closed set in P with f(o) $\in$ G. Then P $-$ G is n-open in P with f(o) $\in$ G. By (4), there is a $\star$-nI_η-open-set U in O containing o such that f(U) $\subset$ G. This implies U= f^{-1}(G) . Therefore, O $-$ U = O $-$ f^{-1}(G) = f^{-1}(P$-$G) which is $\star$-nI_η-closed-set in O. □

Theorem 4.5. *Let f: $(O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ be a function. Then the following conditions are equivalent*

(1) *f is contra $\star$-nI_η-irresolute.*
(2) *The inverse image of each $\star$-nI_η-open-set in P is $\star$-nI_η-closed-set in O.*
(3) *The inverse image of each $\star$-nI_η-closed-set in P is $\star$-nI_η-open-set in O.*
(4) *For each point o in O and each $\star$-nI_η-closed-set set G in P with f(o) $\in$ G, there is a $\star$-nI_η-open-set U in O containing o such that f(U) $\subset$ G.*

Proof. The proof is similar to the Theorem 4.4 □

Theorem 4.6. *Let $f : (O, \mathcal{N}, I) \rightarrow (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \rightarrow (Q, \mathcal{N}'_*, K)$. Then the following properties hold:*

(1) *If f is contra $\star$-nI_η-irresolute and g is contra $\star$-nI_η-irresolute then $g \circ f$ is $\star$-nI_η-irresolute.*
(2) *If f is contra $\star$-nI_η-irresolute and g is contra $\star$-nI_η-continuous then $g \circ f$ is $\star$-nI_η-continuous.*
(3) *If f is contra $\star$-nI_η-continuous and g is contra $n\star$-continuous then $g \circ f$ is $\star$-nI_η-continuous.*
(4) *If f is $\star$-nI_η-irresolute and g is contra $\star$-nI_η-irresolute then $g \circ f$ is contra $\star$-nI_η-irresolute.*
(5) *If f is contra $\star$-nI_η-irresolute and g is $\star$-nI_η-irresolute then $g \circ f$ is contra $\star$-nI_η-irresolute.*
(6) *If f is contra $\star$-nI_η-irresolute and g is contra $n\star$-continuous then $g \circ f$ is $\star$-nI_η-continuous.*

PROOF. (1) Let G be $\star$-nI_η-open-set in Q. Since g is contra $\star$-nI_η-irresolute, g^{-1}(G) is $\star$-nI_η-closed-set in P. Since f is contra $\star$-nI_η-irresolute, $(g \circ f)^{-1}$(G) = $f^{-1}(g^{-1}$(G)) is $\star$-nI_η-open-set in O. Therefore g $\circ$ f is $\star$-nI_η-irresolute.
The proofs of (2), (3), (4), (5) and (6) are similar to (1). □

THEOREM 4.7. *Let f: (O, $\mathcal{N}$, I) $\to$ (P, $\mathcal{N}'$, J) be a function and g: (O, $\mathcal{N}$, I) $\to$ ((O, $\mathcal{N}$, I) $\times$ (P, $\mathcal{N}'$, J)) the graph function of f, defined by g(o) = (o, f(o)) for every $o \in O$. If g is contra $\star$-nI_η-continuous, then f is contra $\star$-nI_η-continuous.*

PROOF. Let G be an n-open set in (P, $\mathcal{N}'$, J). Then ((O, $\mathcal{N}$, I) $\times$ G) is an n-open set in ((O, $\mathcal{N}$, I) $\times$ (P, $\mathcal{N}'$, J)). It follows from Theorem 4.4, that f^{-1}(G) = g^{-1}((O, $\mathcal{N}$, I) $\times$ G) is $\star$-nI_η-closed-set in (O, $\mathcal{N}$, $\mathcal{I}$). Thus, f is contra $\star$-nI_η-continuous. □

DEFINITION 4.5. A space (O, $\mathcal{N}$, I) is said to be a $\star$-nI_η-space if every $\star$-nI_η-open set is n-open in (O, $\mathcal{N}$, I).

THEOREM 4.8. *A function f : (O, $\mathcal{N}$, I) $\to$ (P, $\mathcal{N}'$, J) is contra $\star$-nI_η-continuous and O is $\star$-nI_η-space, then f is contra $n\star$-continuous.*

PROOF. Let G be n-closed set in P. Since f is contra $\star$-nI_η-continuous, f^{-1}(G) is $\star$-nI_η-open in O. Since O is a $\star$-nI_η-space, f^{-1}(G) is n-open in O. Therefore f is contra n$\star$-continuous. □

DEFINITION 4.6. An nano ideal topological space (O, $\mathcal{N}$, I) is said to be $\star$-nI_η-connected if (O, $\mathcal{N}$, I) cannot be expressed as the union of two disjoint non empty $\star$-nI_η-open-set subsets of (O, $\mathcal{N}$, I).

THEOREM 4.9. *A contra $\star$-nI_η-continuous image of a $\star$-nI_η-connected space is nano connected.*

PROOF. Let f : (O, $\mathcal{N}$, I) $\to$ (P, $\mathcal{N}'$, J) be a contra $\star$-nI_η-continuous function of a $\star$-nI_η-connected space (O, $\mathcal{N}$, I) onto a nano topological space (P, $\mathcal{N}'$). If possible, let P be nano disconnected. Let G and S form a nano disconnection of P. Then G and S are nano clopen and P = G $\cup$ S where G $\cap$ S = ϕ. Since f is contra $\star$-nI_η-continuous, O = f^{-1}(P) = f^{-1}(G $\cup$ S) = f^{-1}(G) $\cup$ f^{-1}(S), where

f^{-1}(G) and f^{-1}(S) are non empty $\star$-nI_η-open sets in O. Also $f^{-1}(G) \cap f^{-1}(S) = \phi$. Hence O is not $\star$-nI_η-connected. This is a contradiction. Therefore P is nano connected. □

Lemma 4.3. *For an nano ideal topological space (O, $\mathcal{N}$, $\mathcal{I}$), the following statements are equivalent.*

(1) *O is $\star$-nI_η-connected.*
(2) *The only subset of O which are both $\star$-nI_η-open-set and $\star$-nI_η-closed-set are the empty set ϕ and O.*

Proof. (1) $\Rightarrow$ (2) Let G be a $\star$-nI_η-open-set and $\star$-nI_η-closed-set subset of O. Then O − G is both $\star$-nI_η-open-set and $\star$-nI_η-closed-set. Since O is $\star$-nI_ηconnected, O can be expressed as union of two disjoint non empty $\star$-nI_η-open-set sets O and O − G, which implies O − G is empty.

(2) $\Rightarrow$ (1) Suppose O = G ∪ S where G and S are disjoint non empty $\star$-nI_η-open-set subsets of O. Then G is both $\star$-nI_η-open-set and $\star$-nI_η-closed-set. By assumption either G = ϕ or O which contradicts the assumption G and S are disjoint non empty $\star$-nI_η-open-set subsets of O. Therefore O is $\star$-nI_η-connected. □

5. Continuity and contra continuity via $\star$-nI_C-set

In this section, we introduce the notion of $\star$-nI_C-continuous, $\star$-nI_C-irresolute, contra $\star$-nI_C-continuous and contra $\star$-nI_C-irresolute functions. Furthermore, we show some of their properties.

Definition 5.1. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called $\star$-nI_C-continuous if $f^{-1}(V)$ is a $\star$-nI_C-open-set of $(O, \mathcal{N}, I)$ for every n-open set V of $(P, \mathcal{N}', J)$.

Definition 5.2. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called $\star$-nI_C-irresolute if $f^{-1}(V)$ is a $\star$-nI_C-open-set of $(O, \mathcal{N}, I)$ for every $\star$-nI_C-open-set V of $(P, \mathcal{N}', J)$.

Lemma 5.1. *For a function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$, the following statements hold:*

(1) *If f is $nI^\star_C$ continuous, then f is $\star$-nI_C-continuous.*
(2) *If f is $\star$-nI_η continuous, then f is $\star$-nI_C-continuous.*
(3) *If f is $nI^\star_C$ irresolute, then f is $\star$-nI_C-irresolute.*
(4) *If f is $\star$-nI_η irresolute, then f is $\star$-nI_C-irresolute.*
(5) *If f is nI_A continuous, then f is $\star$-nI_C-continuous.*
(6) *If f is nI_A irresolute, then f is $\star$-nI_C-irresolute.*

Proof. (1) Let f be a $nI^\star_C$ continuous, then $f^{-1}(V)$ is $nI^\star_C$-open-set in O, but By the Remark 1.1, $f^{-1}(V)$ is $\star$-nI_C-open-set in O, since f is $nI^\star_C$ continuous, then V is n-open in P. Therefore, f is $\star$-nI_C-continuous.

(2) Let f be a $\star$-nI_η continuous, then $f^{-1}(V)$ is $\star$-nI_η-open-set in O, but By the Remark 1.1, $f^{-1}(V)$ is $\star$-nI_C-open-set in O, since f is $\star$-nI_η continuous, then V is n-open in P. Therefore, f is $\star$-nI_C-continuous.

The proof of (3) is similar to part (1). The proof of (4) is similar part (2). The proof of (5) is followed by part (2) and Lemma 4.1 and the proof of (5) is followed by part (4) and Lemma 4.1. □

THEOREM 5.1. *Every* $\star$-nI_C*-continuous function is* $\star$-nI_C*-continuous.*

PROOF. Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a $n\star$-continuous function. Let V be a n-open set of $(P, \mathcal{N}', J)$. V is n-open-set. Since f is $n\star$-continuous, then $f^{-1}(V)$ is a n-open-set of $(O, \mathcal{N}, I)$. By the Remark 1.1, $f^{-1}(V)$ is $\star$-nI_C-open-set. Therefore f is $\star$-nI_C-continuous. □

THEOREM 5.2. *Every* $\star$-nI_C*-irresolute function is* $\star$-nI_C*-continuous.*

PROOF. Let $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a $\star$-nI_C-irresolute function. Let V be a n-open set of $(P, \mathcal{N}', J)$. V is $\star$-nI_C-open-set. Since f is $\star$-nI_C-irresolute, then $f^{-1}(V)$ is a $\star$-nI_C-open-set of $(O, \mathcal{N}, I)$. Therefore f is $\star$-nI_C-continuous. □

THEOREM 5.3. *Let* $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ *and* $g : (P, \mathcal{N}', J) \to (Q, \mathcal{N}_*, \mathcal{K})$ *be any two functions. Then, the following statements hold:*

(1) $g \circ f$ *is* $\star$-nI_C*-irresolute if both* f *and* g *are* $\star$-nI_C*-irresolute.*
(2) $g \circ f$ *is* $\star$-nI_C*-continuous if* g *is* $\star$-nI_C*-continuous and* f *is* $\star$-nI_C*-irresolute.*

PROOF. Consider two $\star$-nI_C-irresolute functions, $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ and $g : (P, \mathcal{N}', J) \to (Q, \mathcal{N}'_*, K))$ are $\star$-nI_C-irresolute functions. As g is consider to be a $\star$-nI_C-irresolute function, by Definition 5.2, for every $\star$-nI_C-open-set $q \subseteq (Q, \mathcal{N}'_*, K))$, $g^{-1}(q) = G$ which is $\star$-nI_C-open-set in $(P, \mathcal{N}', J)$. Since f is $\star$-nI_C-irresolute, $(g \circ f)^{-1}(q) = f^{-1}(g^{-1}(q)) = f^{-1}(G) = S$ and S is a $\star$-nI_C-open-set in $(O, \mathcal{N}, I)$. Hence $(g \circ f)$ is $\star$-nI_C-irresolute.
The proof of (2) is similar to (1) □

DEFINITION 5.3. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called contra $\star$-nI_C-continuous if $f^{-1}(V)$ is a $\star$-nI_C-closed-set of $(O, \mathcal{N}, I)$ for every n-open set V of $(P, \mathcal{N}', J)$.

DEFINITION 5.4. A function $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ is called contra $\star$-nI_C-irresolute if $f^{-1}(V)$ is a $\star$-nI_C-closed-set of $(O, \mathcal{N}, I)$ for every $\star$-nI_C-open-set V of $(P, \mathcal{N}', J)$.

LEMMA 5.2. *For a function* $f : (O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$*, the following statements hold:*

(1) *If* f *is contra* $nI^\star_C$ *continuous, then* f *is contra* $\star$-nI_C*-continuous.*
(2) *If* f *is contra* $\star$-nI_η *continuous, then* f *is contra* $\star$-nI_C*-continuous.*
(3) *If* f *is contra* $nI^\star_C$ *irresolute, then* f *is contra* $\star$-nI_C*-irresolute.*
(4) *If* f *is contra* $\star$-nI_η *irresolute, then* f *is contra* $\star$-nI_C*-irresolute.*
(5) *If* f *is contra* nI_A *continuous, then* f *is contra* $\star$-nI_C*-continuous.*
(6) *If* f *is contra* nI_A *irresolute, then* f *is contra* $\star$-nI_C*-irresolute.*

PROOF. (1) Let f be a contra $nI^\star_C$ continuous, then $f^{-1}(V)$ is $nI^\star_C$-closed-set in O, but By the Remark 1.1, $f^{-1}(V)$ is $\star$-nI_C-closed-set in O, since f is contra $nI^\star_C$ continuous, then V is n-open in P. Therefore, f is contra $\star$-nI_C-continuous.

(2) Let f be a contra $\star$-nI_η continuous, then $f^{-1}(V)$ is $\star$-nI_η-closed-set in O, but By the Remark 1.1, $f^{-1}(V)$ is $\star$-nI_C-closed-set in O, since f is contra $\star$-nI_η continuous, then V is n-open in P. Therefore, f is contra $\star$-nI_C-continuous.

The proof of (3) is similar to part (1). The proof of (4) is similar part (2). The proof of (5) is followed by part (2) and Lemma 4.2 and the proof of (5) is followed by part (4) and Lemma 4.2. □

THEOREM 5.4. *Let f: $(O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a function. Then the following conditions are equivalent*

(1) *f is contra $\star$-nI_C-continuous.*
(2) *The inverse image of each n-open set in P is $\star$-nI_C-closed-set in O.*
(3) *The inverse image of each n-closed set in P is $\star$-nI_C-open-set in O.*
(4) *For each point o in O and each n-closed set G in P with f(o) $\in$ G, there is a $\star$-nI_C-open-set U in O containing o such that f(U) $\subset$ G.*

PROOF. (1) $\Rightarrow$ (2). Let G be n-open in P. Then P $-$ G is n-closed in P. By definition of contra $\star$-nI_C-continuous, f^{-1}(P $-$ G) is $\star$-nI_C-open-set in O. But f^{-1}(P $-$ G) = O - f^{-1}(G). This implies f^{-1}(G) is $\star$-nI_C-closed-set in O.

(2) $\Rightarrow$ (3) Let G be any n-closed set in P. Then P$-$G is n-open set in P. By the assumption of (2), f^{-1}(P $-$ G) is $\star$-nI_C-closed-set in O. But f^{-1}(P $-$ G) = O - f^{-1}(G). This implies f^{-1}(G) is $\star$-nI_C-open-set in O.

(3) $\Rightarrow$ (4). Let o $\in$ O and G be any n-closed set in P with f(o) $\in$ G. By (3), f^{-1}(G) is $\star$-nI_C-open-set in O. Set U= f^{-1}(G). Then there is a $\star$-nI_C-open-set U in O containing o such that f(U) $\subset$ G.

(4) $\Rightarrow$ (1). Let o $\in$ O and G be any n-closed set in P with f(o) $\in$ G. Then P $-$ G is n-open in P with f(o) $\in$ G. By (4), there is a $\star$-nI_C-open-set U in O containing o such that f(U) $\subset$ G. This implies U= f^{-1}(G) . Therefore, O $-$ U = O $-$ f^{-1}(G) = f^{-1}(P$-$G) which is $\star$-nI_C-closed-set in O. □

THEOREM 5.5. *Let f: $(O, \mathcal{N}, I) \to (P, \mathcal{N}', J)$ be a function. Then the following conditions are equivalent*

(1) *f is contra $\star$-nI_C-irresolute.*
(2) *The inverse image of each $\star$-nI_C-open-set in P is $\star$-nI_C-closed-set in O.*
(3) *The inverse image of each $\star$-nI_C-closed-set in P is $\star$-nI_C-open-set in O.*
(4) *For each point o in O and each $\star$-nI_C-closed-set set G in P with f(o) $\in$ G, there is a $\star$-nI_C-open-set U in O containing o such that f(U) $\subset$ G.*

PROOF. The proof is similar to the Theorem 5.4 □

THEOREM 5.6. *Let $f : (O,\mathcal{N}, I) \to (P,\mathcal{N}', J)$ and $g : (P,\mathcal{N}', J) \to (Q,\mathcal{N}'_*, K)$. Then the following properties hold:*

(1) *If f is contra $\star$-nI_C-irresolute and g is contra $\star$-nI_C-irresolute then $g \circ f$ is $\star$-nI_C-irresolute.*
(2) *If f is contra $\star$-nI_C-irresolute and g is contra $\star$-nI_C-continuous then $g \circ f$ is $\star$-nI_C-continuous.*
(3) *If f is contra $\star$-nI_C-continuous and g is contra n$\star$-continuous then $g \circ f$ is $\star$-nI_C-continuous.*

(4) *If f is $\star$-nI_C-irresolute and g is contra $\star$-nI_C-irresolute then $g \circ f$ is contra $\star$-nI_C-irresolute.*
(5) *If f is contra $\star$-nI_C-irresolute and g is $\star$-nI_C-irresolute then $g \circ f$ is contra $\star$-nI_C-irresolute.*
(6) *If f is contra $\star$-nI_C-irresolute and g is contra n$\star$-continuous then $g \circ f$ is $\star$-nI_C-continuous.*

Proof. (1) Let G be $\star$-nI_C-open-set in Q. Since g is contra $\star$-nI_C-irresolute, g^{-1}(G) is $\star$-nI_C-closed-set in P. Since f is contra $\star$-nI_C-irresolute, $(g \circ f)^{-1}$(G) = $f^{-1}(g^{-1}$(G)) is $\star$-nI_C-open-set in O. Therefore g $\circ$ f is $\star$-nI_C-irresolute.
The proofs of (2), (3), (4), (5) and (6) are similar to (1). □

Theorem 5.7. *Let f: (O, $\mathcal{N}$, I) $\to$ (P, $\mathcal{N}'$, J) be a function and g: (O, $\mathcal{N}$, I) $\to$ ((O, $\mathcal{N}$, I) $\times$ (P, $\mathcal{N}'$, J)) the graph function of f, defined by g(o) = (o, f(o)) for every $o \in O$. If g is contra $\star$-nI_C-continuous, then f is contra $\star$-nI_C-continuous.*

Proof. Let G be an n-open set in (P, $\mathcal{N}'$, J). Then ((O, $\mathcal{N}$, I) $\times$ G) is an n-open set in ((O, $\mathcal{N}$, I) $\times$ (P, $\mathcal{N}'$, J)). It follows from Theorem 5.4, that f^{-1}(G) = g^{-1}((O, $\mathcal{N}$, I) $\times$ G) is $\star$-nI_C-closed-set in (O, $\mathcal{N}$, $\mathcal{I}$). Thus, f is contra $\star$-nI_C-continuous. □

Definition 5.5. A space (O, $\mathcal{N}$, I) is said to be a $\star$-nI_C-space if every $\star$-nI_C-open set is n-open in (O, $\mathcal{N}$, I).

Theorem 5.8. *A function f : (O, $\mathcal{N}$, I) $\to$ (P, $\mathcal{N}'$, J) is contra $\star$-nI_C-continuous and O is $\star$-nI_C-space, then f is contra n$\star$-continuous.*

Proof. Let G be n-closed set in P. Since f is contra $\star$-nI_C-continuous, f^{-1}(G) is $\star$-nI_C-open in O. Since O is a $\star$-nI_C-space, f^{-1}(G) is n-open in O. Therefore f is contra n$\star$-continuous. □

Definition 5.6. An nano ideal topological space (O, $\mathcal{N}$, I) is said to be $\star$-nI_C-connected if (O, $\mathcal{N}$, I) cannot be expressed as the union of two disjoint non empty $\star$-nI_C-open-set subsets of (O, $\mathcal{N}$, I).

Theorem 5.9. *A contra $\star$-nI_C-continuous image of a $\star$-nI_C-connected space is nano connected.*

Proof. Let f : (O, $\mathcal{N}$, I) $\to$ (P, $\mathcal{N}'$, J) be a contra $\star$-nI_C-continuous function of a $\star$-nI_C-connected space (O, $\mathcal{N}$, I) onto a nano topological space (P, $\mathcal{N}'$). If possible, let P be nano disconnected. Let G and S form a nano disconnection of P. Then G and S are nano clopen and P = G $\cup$ S where G $\cap$ S = ϕ. Since f is contra $\star$-nI_C-continuous, O = f^{-1}(P) = f^{-1}(G $\cup$ S) = f^{-1}(G) $\cup$ f^{-1}(S), where f^{-1}(G) and f^{-1}(S) are non empty $\star$-nI_C-open sets in O. Also f^{-1}(G) $\cap$ f^{-1}(S) = ϕ. Hence O is not $\star$-nI_C-connected. This is a contradiction. Therefore P is nano connected. □

Lemma 5.3. *For an nano ideal topological space (O, $\mathcal{N}$, $\mathcal{I}$), the following statements are equivalent.*

(1) *O is $\star$-nI_C-connected.*

(2) *The only subset of O which are both* $\star$-nI_C*-open-set and* $\star$-nI_C*-closed-set are the empty set* ϕ *and O.*

Proof. (1) $\Rightarrow$ (2) Let G be a $\star$-nI_C-open-set and $\star$-nI_C-closed-set subset of O. Then O $-$ G is both $\star$-nI_C-open-set and $\star$-nI_C-closed-set. Since O is $\star$-nI_C-connected, O can be expressed as union of two disjoint non empty $\star$-nI_C-open-set sets O and O $-$ G, which implies O $-$ G is empty.

(2) $\Rightarrow$ (1) Suppose O $=$ G $\cup$ S where G and S are disjoint non empty $\star$-nI_C-open-set subsets of O. Then G is both $\star$-nI_C-open-set and $\star$-nI_C-closed-set. By assumption either G $= \phi$ or O which contradicts the assumption G and S are disjoint non empty $\star$-nI_C-open-set subsets of O. Therefore O is $\star$-nI_C-connected. □

References

[1] J. Dontchev. Contra-continuous functions and strongly S-closed spaces. *Int. J. Math. Math. Sci.*, **19**(2)(1996), 303–310.

[2] O. Nethaji, R. Asokan and I. Rajasekaran. New generalized classes of ideal nanotopologicalspaces. *Bull. Int. Math. Virtual Inst.*, **9**(3)(2019), 543–552.

[3] R. Premkumar. and M. Rameshpali. New sets in ideal nano topological spaces. *Bull. Int. Math. Virtual Inst.*, **10**(1)(2020), 19–27.

[4] M. Parimala, S. Jafari and S. Murali. Nano ideal generalized closed sets in nano ideal topological spaces. *Annales Univ. Sci. Budapest.*, **60**(2017), 3–11.

[5] M. Parimala, T. Noiri and S. Jafari. New types of nano topological spaces via nano ideals. Abailable at https://www.researchgate.net/publication/315892279 (2016).

[6] I. Rajasekaran and O. Nethaji. Unified approach of several sets in ideal nanotopological spaces. *Asia Mathematika*, **3**(1)(2019), 70–78.

[7] R. Vaidyanathaswamy. *Set topology.* Chelsea Publishing Company, New York, 1946.

Received by editors 29.05.2020; Revised version 17.07.2020; Available online 27.07.2020.

Corporción Universitaria Latinoamericana, Barranquilla, Colombia
E-mail address: carlosgranadosortiz@outlook.es