JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
*J. Int. Math. Virtual Inst.*, Vol. **10**(2)(2020), 251-270
DOI: 10.7251/JIMVI2002251R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

# IMPLICATIVE SEMIGROUPS WITH APARTNESS, A REVUEW

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Abstract. This paper is a recapitulation of the analyzed ideas, newly introduced concepts and the obtained results about implicative semigroups with apartness.

## 1. INTRODUCTION

The notions of implicative semigroup and ordered filter were introduced by Chan and Shum [**6**]. Jun [**10, 11**], Jun, Meng and Xin [**12**] and Jun and Kim [**13**] discussed ordered filters and ideals of implicative semigroups.

In paper [**19**], in setting of Bishop's constructive mathematics [**1, 2**], following the ideas of Chan and Shum and other authors mentioned above, the author introduced the concept of implicative semigroups with (tight) apartness and gived some fundamental characterization of these semigroups. This work environment is recognized not only by the application of Intuitionistic logic (for example, [**27**]) instead of Classical logic but also by the principled-philosophical orientations of constructive mathematics (for example, [**3, 4, 5, 14**]). In this analysis, he used a set with an apartness relation and a co-order relation as a carrier on which the algebraic structure is constructed. In the articles [**20, 22, 23, 24, 25, 26**] he continues the analysis of implicative semigroups with apartness. While in the articles [**20, 25**] the concept of strongly extensional homomorphisms between implicative semigroups with apartnesss was introduced and analyzed, in the papers [**22, 23, 24, 26**] the focus was on the concepts of co-ideals ([**23, 24**]) and co-filters ([**23, 26**]) in such semigroups. Some of the introduced and analyzed concepts are

2010 *Mathematics Subject Classification.* 03F65, 20M12, 06F05, 06A06, 06A12.
*Key words and phrases.* Bishop's constructive mathematics, semigroup with apartness, co-ordered semigroups, implicative semigroups with apartness, co-ideals and co-filters, co-congruences on implicative semigroupwith apartness.

counterparts of classical concepts in the observed semigroups. Although some of the introduced concepts and processes with them are counterparts of classical concepts and processes in the observed semigroups in the classical case, the techniques used in analyzing their properties (in mentioned papers) differ significantly from those applied in their classical version. In order for the interested reader to gain at least some impression of these differences, we state one form of the First isomorphism theorem between implicative semigroups with apartness.

THEOREM 1.1 ([**25**], Theorem 9). *Let*

$$f : ((S, =, \neq), \cdot, \alpha, \otimes) \longrightarrow ((T, =, \neq), \cdot, \beta, \otimes)$$

*be a reverse isotone se-epimorphism between implicative semigroups. Then there exist a unique pair* $f_1 : S/(q_f^{\triangleleft}, q_f) \longrightarrow T$ *and* $f_2 : [S : q_f] \longrightarrow T$ *of embedding, injective and surjective se-homomorphisms such that*

$$f = f_1 \circ \pi = f_2 \circ \vartheta = f_2 \circ h \circ \pi.$$

Without going into detailed descriptions of the terms used in this introductory part, it is easy to see that using the homomorphism $f$, two algebraic structures $S/(q_f^{\triangleleft}, q_f)$ and $[S : q_f]$ can be constructed from the implicative semigroup $S$, both of which are isomorphic to semigroup $T$. Apart from the fact that the first quotient structure differs somewhat from the classical dual, the semigroup $[S : q_f]$ also appears, which has no the counterpart in the classical case.

This paper is a recapitulation of the analyzed ideas, newly introduced concepts and the obtained results.

## 2. PRELIMINARIES

In this section, we recall from [**7, 8, 9, 21**] some concepts and processes necessary in the sequel of this paper. An interested reader can look for more details in the papers [**19, 20, 22, 23, 24**]. This investigation is in Bishops constructive algebra in the sense of papers [**7, 8, 9, 21**] and books [**1, 2, 3, 4, 5, 14**] and Chapter 8: Algebra of [**27**].

**2.1. Set with apartness.** Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$"). A diversity relation "$\neq$" satisfying conditions

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z$$

is called apartness. In this paper, we assume that the apartness is tight, i.e. it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \Longrightarrow x = y).$$

A subset $X$ of $S$ is called a strongly extensional subset of $S$ if and only if

$$(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S).$$

Let $X$, $Y$ be subsets of $S$.

According with Bridge and Vita definition (see for instance [**5**]), we say that $X$ is set-set apartned from $Y$ (denoted $X \bowtie Y$) if and only if

$$(\forall x \in X)(\forall y \in Y)(x \neq y).$$

We set $x \lhd Y$ and $x \neq y$, instead of $\{x\} \bowtie Y$ and $\{x\} \bowtie \{y\}$ respectively. With $X^{\lhd} = \{x \in S : x \lhd X\}$ we denote the apartness complement of $X$.

We say that

- a function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$ is strongly extensional (an se-mapping, for short) if and only if

$$(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b);$$

- $f$ is an embedding if

$$(\forall a, b \in S)(a \neq b \implies f(a) \neq f(b))$$

holds.

Apartness is introduced in the direct product $(S, =_S, \neq_S) \times (T, =_T, \neq_T)$ as follows

$$(\forall x, y \in S)(\forall u, v \in T)((x, u) \neq (y, v) \iff (x \neq_S y \vee u \neq_T v)).$$

In writing relations of equality and apartness we will omit indices whenever it is possible and it do not lead to misunderstandings.

**2.2. Semigroup with apartness.** Let $(S, =, \neq)$ be a set with apartness. A total se-mapping $w : S \times S \longrightarrow S$ is internal binary operation on $S$ (shortly: the operation). By this is meant that the following statements

$$(\forall x, y, u.v \in S)((x, y) = (u, v) \implies w(x, y) = w(u, v));$$

$$(\forall x, y, u, v \in S)(w(x, y) \neq w(u, v) \implies (x \neq u, \vee y \neq v)).$$

are valid formulas. If the operation '$w$' on $S$ is associative, then the algebraic structure $(S, =, \neq), w)$ is a semigroup with apartness. It is common to write the operation '$w$' as multiplication, sometimes writing $xy$ instead of $x \cdot y$. For example, semigroups with apartness were the focus of the following articles [**7, 8, 9, 16, 17, 18**].

A relation $\alpha \subseteq S \times S$ S is a *co-order relation* on the semigroup with apartness $S$, if it fulfills the following properties

$(\forall x, y \in S)((x, y) \in \alpha \implies x \neq y)$ (consistency)

$(\forall x, y, z \in S)((x, z) \in \alpha \implies ((x, y) \in \alpha \vee (y, z) \in \alpha))$ (co-transitivity)

$(\forall x, y \in S)(x \neq y \implies ((x, y) \in \alpha \vee (y, z) \in \alpha))$ (linearity) and

compatibility with the product of $S$ in the following sense

$(\forall x, y, z \in S)(((xz, yz) \in \alpha \vee (zx, zy) \in \alpha) \implies (x, y) \in \alpha).$

In this case, we say that the semigroup $S$ is ordered by the co-order relation $\alpha$, or that it is a co-ordered semigroup with respect to $\alpha$. This author has dealt with these relations in semigroups with apartness in several of his publications (see, for example [**16, 18, 21**]).

A relation $q$ on $S$ is a *co-congruence* on $S$ if the following hold

$(\forall x, y \in S)((x, y) \in q \implies x \neq y)$ (consistency)

$(\forall x, y, z \in S)((x, z) \in \alpha \implies ((x, y) \in \alpha \vee (y, z) \in \alpha))$ (co-transitivity)

$(\forall x, y \in S)((x, y) \in q \implies (y, x) \in q)$ (symmetry) and compatibility with the operation in $S$ in the following sense

$(\forall x, y, z \in S)(((xz, yz) \in q \vee (zx, zy) \in q) \implies (x, y) \in q)$.

The relation of co-congruence on semigroups with apartness was studied by the author, too ([**16, 17**]).

## 3. IMPLICATIVE SEMIGROUPS WITH APARTNESS

**3.1. Definition and Examples.** We recall some definitions and results. By a *negatively co-ordered* semigroup (briefly, n.a-o. semigroup) we mean a set $S$ with a co-order $\alpha$ and a binary operation '$\cdot$' (we will write $xy$ instead $x \cdot y$ ) such that for all $x, y, z \in S$, we have to have

(1) $(xy)z = x(yz)$,

(2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and

(3) $(xy, x) \lhd \alpha$ and $(xy, y) \lhd \alpha$.

In that case for co-order $\alpha$ we will say that it is a *negative co-order relation* on semigroup. The operation '$\cdot$' is extensional and strongly extensional function from $S \times S$ into $S$, i.e. it has to be

$$(x, y) = (x', y') \Longrightarrow xy = x'y'$$
$$(xy \neq x'y \ \vee \ yx \neq yx') \Longrightarrow x \neq x'$$

for any elements $x, x', y$ of $S$.

Let $\alpha$ be a relation on $S$. For an element $a$ of $S$ we put $a\alpha = \{x \in S : (a, x) \in \alpha\}$ and $\alpha a = \{x \in S : (x, a) \in \alpha\}$. In the following theorem we give some properties of negative co-order relation on semigroup.

Theorem 3.1 ([**19**], Theorem 3.1). *If $\alpha \subseteq S \times S$ is an anti-order relation on a semigroup $S$, then the following statements are equivalent:*

(i) *$\alpha$ is a negative co-order relation;*

(ii) *$\alpha b$ is a consistent subset of $S$ for any $b$ in $S$;*

(iii) *$(\forall a, b \in S)(\alpha a \cup \alpha b \subseteq \alpha(ab))$;*

(iv) *$a\alpha$ is an ideal of $S$ for any $a$ in $S$;*

(v) *$(\forall a, b \in S)((ab)\alpha \subseteq a\alpha \cap b\alpha)$.*

In order for the interested reader to gain insight into the techniques applied in the algebra of Bishop's constructive orientation, we present the proof of this theorem.

Proof. (i) $\Longrightarrow$ (ii). Let $x, y, a, b$ be arbitrary elements of $S$ and let $\alpha$ be negative co-order relation on semigroup $S$. If the product $xy$ lies in set $\alpha b$, i.e. if $(xy, b) \in \alpha$ holds, then we have $(xy, x) \in \alpha \vee (x, b) \in \alpha$ and $(xy, y) \in \alpha \vee (y, b) \in \alpha$. Since $\alpha$ is n.a-o. relation on semigroup $S$, the first cases in these disjunctions are impossible. So, we have $x \in \alpha b$ and $y \in \alpha b$. Therefore, the set $\alpha b$ is a consistent subset of $S$.

(ii) $\Longrightarrow$ (iii). If $x \in \alpha a \cup \alpha b$, i.e. if $(x, a) \in \alpha \ \vee \ (x, b) \in \alpha$, then $(x, ab) \in \alpha \ \vee \ (ab, a) \in \alpha \ \vee \ (x, ab) \in \alpha \ \vee \ (ab, b) \in \alpha$ holds by cotransitivity of $\alpha$. Since from

$ab \in \alpha a$ and $ab \in \alpha b$ implies $a \in \alpha a$ and $b \in \alpha b$ respectively, and since these is forbidden, in both cases we have $x \in \alpha(ab)$.

(iii) $\Longrightarrow$ (i). Let $(u, v)$ be an arbitrary element of $\alpha$. Then we have $(u, xy) \in \alpha$ or $(xy, x) \in \alpha$ or $(x, v) \in \alpha$. Thus, $u \neq xy \vee xy \in \alpha x \subseteq \alpha x \cup \alpha y \subseteq \alpha(xy) \vee x \neq v$. Since the second case is imposible, we have $(xy, x) \neq (u, v) \in \alpha$. So, we have $(xy, x) \lhd \alpha$. Proof for $(xy, y) \lhd \alpha$ we obtain analogously. Therefore, the relation $\alpha$ is negative co-order relation on $S$.

(i) $\Longrightarrow$ (iv). Suppose that $x \in a\alpha$ or $y \in a\alpha$, i.e. assume that $(a, x) \in \alpha$ or $(a, y) \in \alpha$. Thus, we have $(a, xy) \in \alpha \vee (xy, x) \in \alpha$ and $(a, xy) \in \alpha \vee (xy, y) \in \alpha$. So, in both cases we have $xy \in a\alpha$. Therefore, the set $a\alpha$ is an ideal of semigroup $S$ for any element $a \in S$.

(iv) $\Longrightarrow$ (v). Let $x$ be an arbitrary element of $(ab)\alpha$, i.e. assume that $(ab, x) \in \alpha$. Thus follows $(ab, a) \in \alpha \vee (a, x) \in \alpha$ and $(ab, b) \in \alpha \vee (b, x) \in \alpha$. Since $\alpha$ is a negative co-order relation on semigroup $S$, we have $x \in a\alpha$ and $x \in b\alpha$. So, finally we have $x \in a\alpha \cap b\alpha$.

(v) $\Longrightarrow$ (i). Suppose that the inclusion (v) holds for any two elements $a, b$ of semigroup $S$ and suppose that $(u, v)$ is an arbitrary element of $\alpha$. Because of cotransitivity of $\alpha$, we have $(u, xy) \in \alpha$ or $(xy, x) \in \alpha$ or $(x, v) \in \alpha$. In the second case we have $x \in (xy)\alpha \subseteq x\alpha \cap y\alpha \subseteq x\alpha$, which is impossible because $x \lhd x\alpha$. So, it has to be $(xy, x) \neq (u, v) \in \alpha$ and we have $(xy, x) \lhd \alpha$. Proof for $(xy, y) \lhd \alpha$ we obtained analogously. Therefore, the relation $\alpha$ is a negative anti-order relation on semigroup $S$. □

Let us note that for any $a$, $b$ in $S$ the following implication

$$(a, b) \in \alpha \Longrightarrow a\alpha \cup \alpha b = S$$

is valid.

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \longrightarrow S$ such that for any elements $x, y, z$ of $S$, the following is true

(4) $(z, x \otimes y) \in \alpha \Longleftrightarrow (zx, y) \in \alpha.$

Let us point out, as in the classical case, that in the definition of implicative semigroup we can take the stronger demand

(4') $(z, x \otimes y) \lhd \alpha \Longleftrightarrow (zx, y) \lhd \alpha$

instead of demand (4).

Theorem 3.2 ([**19**], Theorem 3.2). (4) *implies* (4').

In addition, let us recall that the internal binary operation must satisfy the following implications:

$$(a, b) = (u, v) \Longrightarrow a \otimes b = u \otimes v,$$
$$a \otimes b \neq u \otimes v \Longrightarrow (a, b) \neq (u, v).$$

The operation $\otimes$ is called *implication.* From now on, an implicative n.a-o. semigroup is simply called *implicative semigroup.*

EXAMPLE 3.1. Consider a set $S = \{1, a, b, c, d, 0\}$ with the operation tables

| $\cdot$ | 1 | a | b | c | d | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d | 0 |
| a | a | b | b | d | 0 | 0 |
| b | b | b | b | 0 | 0 | 0 |
| c | c | d | 0 | c | d | 0 |
| d | d | 0 | 0 | d | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

and

| $\otimes$ | 1 | a | b | c | d | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d | 0 |
| a | 1 | 1 | a | c | c | d |
| b | 1 | 1 | 1 | c | c | c |
| c | 1 | a | b | 1 | a | b |
| d | 1 | 1 | a | 1 | 1 | a |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |

under co-order relation $\alpha = \{(a, 0), (a, b), (a, c), (a, d), (b, 0), (b, c), (b, d), (c, 0), (c, a), (c, b), (c, d), (d, 0), (d, b), (1, 0), (1, a), (1, b), (1, c), (1, d)\}$. Then $S$ is an implicative semigroup with apartness.

**3.2. Important Properties.** In any implicative semigroup $S$ there exists a special element of $S$, the biggest element in $(S, \alpha^{\lhd})$, which is almost neutral element in $(S, \cdot)$. Due to the specificity of the application of the technique in the proof of the previous statement, we state a complete theorem with proof.

THEOREM 3.3 ([**19**], Theorem 3.3). *In any inhabited implicative semigroup with apartness* $((S, =, \neq), \cdot, \alpha, \otimes)$ *holds*

$$(\forall x, y \in S)(x \otimes x = y \otimes y).$$

*and this element is the greatest element, written as* 1*, of* $(S, \alpha^{\lhd})$.

PROOF. By (3) we have $(tx, x) \lhd \alpha$ for any $t, x$ in $S$. By definition (4), it means $(t, x \otimes x) \lhd \alpha$. Thus, particularly for any elements $x$ and $y$ in $S$, we have $(y \otimes y, x \otimes x) \lhd \alpha$ and $(x \otimes x, y \otimes y) \lhd \alpha$. Therefore, we have

$$(x \otimes x, y \otimes y) \lhd \alpha \cup \alpha^{-1} = \neq$$

and $x \otimes x = y \otimes y$ since the apartness is tight. Finally, we conclude that the element $x \otimes x$, for any $x$ in $S$, is the greatest element in $(S, =, \neq, \alpha^{\lhd})$. The greatest element in $(S, =, \neq, \alpha^{\lhd})$ we denote by 1. So, for any element $t$ in $S$ we have $(t, 1) \lhd \alpha$. □

As a consequence of the previous theorem, we can prove that the following statement holds

COROLLARY 3.1 ([**19**], Corollary 3.1). *In semigroup* $(S, \cdot)$ *the following equation* $t = 1 \cdot t$ *holds.*

REMARK 3.1. From (3) we immediately conclude $(t \cdot 1, t) \lhd \alpha$ but we cannot conclude $t = t \cdot 1$. For that equation we also need $(t, t \cdot 1) \lhd \alpha$. In fact, Chan and Shum in [**6**] demonstrate in Example 1.2 that the top element does not have to be a multiplicative unit.

In the following theorem we describe the role of this special element.

THEOREM 3.4 ([**19**], Theorem 3.4). *If $S$ is an implicative semigroup, then for every $x, y \in S$ holds*

$$(x, y) \lhd \alpha \Longleftrightarrow 1 = x \otimes y$$
$$(x, y) \in \alpha \Longleftrightarrow 1 \neq x \otimes y.$$

As a consequence of above theorem we immediately get the following corollary.

COROLLARY 3.2 ([**19**], Corollary 3.2). *For any implicative semigroup $((S, =, \neq), \cdot, \alpha, \otimes)$, holds $1 = x \otimes 1$ for every element $x \in S$.*

As we saw, the greatest element 1 is a right annihilator for the semigroup operation $\otimes$. Some another fundamental properties of this operation we give in the following statements.

THEOREM 3.5 ([**19**], Theorem 3.5). *Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup with apartness. Then, for every $x, y, z \in S$, the following hold:*
(i) $(x, y \otimes (x \cdot y)) \lhd \alpha$;
(ii) $(x, x \otimes x^2) \lhd \alpha$;
(iii) $(x, y \otimes x) \lhd \alpha$;
(iv) *If $(z \otimes x, z \otimes y) \in \alpha$ or $(y \otimes z, x \otimes z) \in \alpha$ then $(x, y) \in \alpha$.*

As we saw from (iv) of above theorem, the relation $\alpha$ is not compatible with the operation $\otimes$ on both sides. There exists only left compatibility and so-called right 'anti-compatibility' between the co-order $\alpha$ and the operation $\otimes$.

THEOREM 3.6 ([**19**], Theorem 3.6). *Let $S$ be an implicative semigroup with apartness. Then for every $x, y, z \in S$, the following hold:*
(1) *If $(x, y) \lhd \alpha$ then $(y \otimes z, x \otimes z) \lhd \alpha$ and $(z \otimes x, z \otimes y) \lhd \alpha$,*
(2) $x \otimes (y \otimes z) = (x \cdot y) \otimes z$,

As a corollary of above theorem we conclude the following:

COROLLARY 3.3 ([**19**], Corollary 3.3). *For any implicative semigroup $(S, =, \neq, \cdot, \alpha, \otimes)$, holds $x = 1 \otimes x$ for every element $x \in S$.*

As we saw the biggest element of ordered semigroup $S$ is a left unity in $(S, \otimes)$.

**3.3. Important Substructures.** In this subsection we will begin with standard definition (Chan and Shum [[**6**], Definition 2.1]) of ordered filter. Let $S$ be an implicative semigroup and let $F$ be a nonempty subset of $S$. Then $F$ is called an *ordered filter* of $S$ if

(F1) $xy \in F$ for every $x, y \in F$, that is, $F$ is a subsemigroup of $S$, and

(F2) If $x \in F$ and $(x, y) \lhd \alpha$, then $y \in F$.

As we saw, condition (F1) supplies subset F with subsemigroup structure, until the condition (F2) supplies that F is an upper set. As it is usual in the Constructive mathematics, we can introduce a special (inhabited) proper subset of implicative semigroup $S$ claiming that subset $G$ of $S$ satisfies the following conditions:

(G1) $(\forall x, y \in S)(xy \in G \implies (x \in G \vee y \in G))$,

that is, $G$ is a cosubsemigroup of $S$ and

(G2) $(\forall x, y \in S)(y \in G \implies ((x, y) \in \alpha \vee x \in G))$.

This subset of $S$ we called *co-filter*. It is easy to check that co-filter is a strongly extensional subset of $S$. Moreover, strong compliment $G^{\lhd}$ of a co-filter $G$ is a filter in $S$. In fact, strong compliment $G^{\lhd}$ is obviously a subsemigroup of $S$. Assume that $x \lhd G$ and $(x, y) \lhd \alpha$ and let $u$ be an arbitrary element of $G$. Then, by strongly extensionality of $G$, we have $u \neq y$ or $y \in G$. In the second case we have $(x, y) \in \alpha \vee x \in G$. As both cases are impossible by hypothesis, we have $y \neq u \in G$. So, we have $y \in G^{\lhd}$.

The following two theorems give equivalent conditions of ordered co-filter.

Theorem 3.7 ([**19**], Theorem 3.7). *An inhabited proper subset $G$ of an implicative semigroup with apartness $S$ is an ordered co-filter of $S$ if and only if it satisfies the following conditions:*

(G3) $1 \lhd G$;

(G4) $(\forall x, y \in S)(y \in G \implies (x \otimes y \in G \vee x \in G))$.

Theorem 3.8 ([**19**], Theorem 3.8). *An inhabited proper subset $G$ of an implicative semigroup with apartness $S$ is an ordered co-filter of $S$ if and only if it satisfies the following condition:*

(G5) $(\forall x, y, z \in S)(z \in G \implies ((x, y \otimes z) \in \alpha \vee x \in G \vee y \in G))$.

Example 3.2. Consider a set $S = \{1, a, b, c, d, 0\}$ with the operation tables as in the above example. It is not so hard to check that set $G = \{c, d, 0\}$ is an ordered co-filter of implicative semigroup $S$.

Let $S$ be an implicative semigroup. For any $a \in S$, we define $G(a) = \{x \in S : (a, x) \in \alpha\}$, (i.e. $G(a)$ is the left class of relation $\alpha$ generated by the element $a$). Clearly that $1 \lhd G(a)$ and $a \lhd G(a)$. Generally speaking, $G(a)$ is not an ordered co-filter in $S$. Let $S$ be an implicative semigroup in Example 1. Then $G(c) = \{0, a, b, d\}$ is an ordered co-filter in $S$, but the set $G(a) = \{0, b, c, d\}$ is not an ordered co-filter in $S$ because, for example, holds $b \in G(a)$, $a \lhd G(a)$ and $a \otimes b = a \lhd G(a)$. Using Theorem 3.7 we can give one condition for the set $G(t)$ $(t \in S)$ to be an order anti-filter in $S$: The set $G(t)$ is an ordered anti-filter in an implicative semigroup $S$ if and only if the following condition

$$(t, y) \in \alpha \Longrightarrow (t, x \otimes y) \in \alpha \vee (t, x) \in \alpha$$

is true for all $x, y \in S$.

**3.4. The concept of homomorphisms.** Let $S = (S, =, \neq, \cdot, \alpha, \otimes)$ and $T = (T, =, \neq, \cdot, \beta, \otimes)$ be two implicative semigroups and let $f : S \longrightarrow T$ be a strongly extensional mapping from $S$ into $T$. As the usual procedure in the construction of a mathematical system, for mapping $f$ we say that it is a homomorphism between implicative semigroups $S$ and $T$ if

$$(\forall x, y \in S)(f(x \otimes y) = f(x) \otimes f(y))$$

holds.

The homomorphisms between the implicative semigroups have been studied by Chan and Shum in [**6**]. In this section, our aim is to extend the results in Chan and Shum to implicative semigroups with apartness and strongly extensional homomorphisms. We first notice that the following implication holds

$$f(x) \otimes f(y) \neq f(x') \otimes f(y) \Longrightarrow f(x) \neq f(x').$$

In fact, for elements $x, x', y$ of $S$ we have the following sequence:

$$\begin{aligned} f(x) \otimes f(y) \neq f(x') \otimes f(y) &\Longleftrightarrow f(x \otimes y) \neq f(x' \otimes y) \\ &\Longleftrightarrow (f(x) \otimes f(y), f(x') \otimes f(y)) \in \beta \cup \beta^{-1} \\ &\Longrightarrow (f(x), f(x') \in \beta \cup \beta^{-1} \\ &\Longleftrightarrow f(x) \neq f(x'). \end{aligned}$$

The implication $f(y) \otimes f(x) \neq f(y) \otimes f(x') \Longrightarrow f(x) \neq f(x')$ follows analogously. Therefore, the homomorphism $f$ is compatible with the operation '$\otimes$'.

We now continue to study the implicative semigroups with extensional homomorphisms. It is noted that Theorem 2.2 [**6**] is a crucial result of this paper because we have to refer this theorem in proving the following theorem.

THEOREM 3.9 ([**20**], Theorem 3.1). *Let $f : S \longrightarrow T$ be an implicative homomorphism between implicative semigroups with apartness $S$ and $T$. Then the following hold:*

(1) $f(1) = 1$*;*
(2) $f(x) \neq 1 \Longrightarrow x \neq 1$ *for any* $x \in S$*;*
(3) $f$ *is a reverse isotone mapping, that is, if* $(f(x), f(y)) \in \beta$ *then* $(x, y) \in \alpha$*;*
(4) *If $f$ is surjective, then $f$ is a semigroup homomorphism, that is*

$$(\forall x, y \in S)(f(xy) = f(x)f(y));$$

(5) $G = f^{-1}(\{1\}^{\lhd})$ *is an ordered anti-filter in $S$ and valid* $G \subseteq \{1\}^{\lhd}$*;*
(6) $f$ *is an embedding homomorphism if and only if* $G = \{1\}^{\lhd}$.

PROOF. (1) By Theorem 3.3 in [**19**], we have

$$f(1) = f(1 \otimes 1) = f(1) \otimes f(1) = 1.$$

(2) This assertion immediately follows from (1) and from the fact that $f$ is a strongly extensional homomorphism. Indeed, from $f(x) \neq 1 = f(1)$ follows $x \neq 1$.

(3) Suppose that for $x, y \in S$ hold $(f(x), f(y)) \in \beta$. Then by Theorem 3.4 in [**19**] it follows that $f(x) \otimes f(y) \neq 1$. Since $f$ is a homomorphism of implicative semigroups, than we have $f(x \otimes y) \neq 1$. Hence, by (2) we have $x \otimes y \neq 1$. Thus, again by Theorem 3.4 in [**19**], it follows $(x, y) \in \alpha$. So, $f$ is a reverse isotone homomorphism.

(4) We first show that $(f(xy), f(x)f(y)) \lhd \beta$. As $f$ is onto, there exists an element $z \in S$ such that $f(z) = f(x)f(y)$. Since $f$ is a homomorphism, we have $f(xy) \otimes f(z) = f((xy) \otimes z) = f(x \otimes (y \otimes z)) = f(x) \otimes f(y \otimes z) = f(x) \otimes (f(y) \otimes f(z)) = (f(x)f(y)) \otimes f(z) = 1$ by Theorem 6.2, point (2), in [**19**]. Thus, by above mentioned Theorem 3.4, we got $(f(xy), f(x)f(y)) \lhd \beta$.

Conversely, by the fact $f(1) = 1$ we have the sequence of equivalent equations: the equation $(xy \otimes xy) = 1$ by assertion (2) of Theorem 3.6 ([**19**]) is equivalent to $f(x \otimes (y \otimes xy)) = 1$ and, since $f$ is a homomorphism, $f(x) \otimes f(y \otimes xy) = 1$, i.e. it is equivalent to the equation $f(x) \otimes (f(y) \otimes f(xy)) = 1$. Thus, again by using assertion (2) of Theorem 3.6 ([**19**]), we have the equivalent equation $(f(x)f(y)) \otimes f(xy) = 1$. The last equation means $(f(x)f(y), f(xy)) \lhd \beta$. Therefore, we have proved that $(f(x)f(y), f(xy)) \lhd \beta \cup \beta^{-1} = \neq$. Since the apartness is tight, we finally have $f(x)f(y) = f(xy)$.

(5) Let $u$ be an arbitrary element of $G$. Then $f(u) \in \{1\}^{\lhd}$, i.e. holds $f(u) \neq 1$. Thus, $u \neq 1$. So, we have $1 \lhd G$. Further on, let $x, y$ elements of S such that $y \in G$. Then, from $f(y) \neq 1$ follows $f(y) \neq f(x \otimes y) \vee f(x \otimes y) \neq 1$. Out of the first part, i.e. from $1 \otimes f(y) \neq f(x) \otimes f(y)$ by the comment before this theorem we conclude $1 \neq f(x)$. So, we have $x \in G$ or $x \otimes y \in G$. Therefore, the set $G$ is an ordered anti-filter in $S$. The last assertion is clear because $f$ is a strongly extensional mapping.

(6) Suppose that $f$ is an embedding function from $S$ in $T$. Then the implication $x \neq 1 \Longrightarrow f(x) \neq 1$ is true. So, holds $\{1\}^{\lhd} \subseteq G \subseteq \{1\}^{\lhd}$.

Let the equation $\{1\}^{\lhd} = f^{-1}(\{1\}^{\lhd})$ be true and let $x \neq y$ hold for elements $x, y \in S$. Then we have $(x, y) \in \alpha \vee (y, x) \in \alpha$, and thus $x \otimes y \neq 1 \vee y \otimes x \neq 1$. The last means $x \otimes y \in \{1\}^{\lhd} = f^{-1}(\{1\}^{\lhd})$ or $y \otimes x \in \{1\}^{\lhd} = f^{-1}(\{1\}^{\lhd})$. Hence, we have $f(x \otimes y) \neq 1 \vee f(y \otimes x) \neq 1$, i.e. we have $f(x) \otimes f(y) \neq 1$ or $f(y) \otimes f(x) \neq 1$. So, we have $(f(x), f(y)) \in \beta \cup \beta^{-1} = \neq$. This proves that $f$ is an embedding. □

Moreover, we can prove more then in assertion (5) of above theorem.

Theorem 3.10 ([**20**], Theorem 3.2). *Let $f$ be as in Theorem 3.9. If $G$ is an ordered co-filter in $T$, then the set $f^{-1}(G)$ is an ordered co-filter in $S$.*

At the end of this subsection, we repeat the two terms that will be used. The se-mapping $f : S \longrightarrow T$ between implicative semigroups is

- isotone if the following

$$(\forall x, y \in S)((x, y) \in \alpha \Longrightarrow (f(x), f(y)) \in \beta)$$

is valid;

-reverse isotone if the following

$$(\forall x, y \in S)((f(x), f(y)) \in \beta \Longrightarrow (x, y) \in \alpha)$$

holds.

## 4. THE CONCEPT OF CO-IDEALS

In this section it is introduced and analyzed the concept of co-ideals of an implicative semigroup with apartness:

DEFINITION 4.1. ([**23**], Definition 3.1) A subset $K$ of $S$ is called *co-ideal* if the following holds:

(K1) $(\forall x, y \in S)(xy \in K \Longrightarrow y \in K)$ and

(K2) $(\forall x, y, z \in S)(x \otimes z \in K \Longrightarrow ((xy, z) \in \alpha \vee y \in K))$.

We say for co-ideal $K$ of $S$ that it is *proper* co-ideal if $K \subset S$ is valid.

The condition (K2) is equivalent to

(K2') $(\forall x, y, z \in S)(x \otimes z \in K \Longrightarrow ((x, y \otimes z) \in \alpha \vee y \in K))$

according to (4).

It is easy to check that the following holds:

PROPOSITION 4.1 ([**23**]). *If $K$ is a co-ideal in an implicative semigroup with apartness $S$, then*

(K3) $(\forall y, z \in S)(z \in K \Longrightarrow ((y, z) \in \alpha \vee y \in K))$.

COROLLARY 4.1 ([**23**]). *A co-ideal $K$ of an implicative semigroup with apartness $S$ is a strongly extensional subset of $S$.*

COROLLARY 4.2 ([**23**]). *For a proper co-ideal $K$ in an implicative semigroup with apartness $S$, the following holds*

(K4) $1 \lhd K$.

More information about the properties of the set $K^{\lhd}$ is given by the following theorem

THEOREM 4.1 ([**23**]). *The strong compliment $K^{\lhd}$ of a co-ideal $K$ in a semigroup $S$ satisfies the following two conditions:*

(i) *$K^{\lhd}$ is a right ideal in $(S, \cdot)$; and*

(ii) $(\forall x, y \in S)((y \in K^{\lhd} \wedge (y, x) \in \alpha^{\lhd}) \Longrightarrow x \in K^{\lhd})$.

PROPOSITION 4.2 ([**23**]). *If $S$ is a commutative semigroup, then (K3) implies (K2).*

THEOREM 4.2 ([**23**]). *The family $\mathfrak{K}(S)$ of all co-ideals in an implicative semigroup with apartness $S$ forms a complete lattice.*

COROLLARY 4.3 ([**23**]). *Let $B$ be a subset of an implicative semigroup with apartness $S$. Then there exists the maximal co-ideal included in $B$.*

COROLLARY 4.4 ([**23**]). *Let $a$ be an arbitrary element in an implicative semigroup with apartness $S$. Then there exists the maximal co-ideal $M_a$ in $S$ such that $a \lhd M_a$.*

In the article [**13**], Definition 3.1, the concept of ideals in implicit semigroups is defined as a subset $J$ in $S$ that satisfies the following two conditions

(J1) $(\forall x, y \in S)(y \in J \implies x \otimes y \in J)$ and
(J2) $(\forall x, yz \in S)(x \in J \wedge y \in J \implies (x \otimes (y \otimes z)) \otimes z \in J)$.

Let us show that our determination of co-ideal $K$ in an implicative semigroup $S$ with apartness by Definition 4.1 is correct, i.e. let us show that the concepts of ideals and co-ideals are associated. In order to achieve this, we will show that $K^{\lhd}$ is an ideal in the sense of the foregoing description.

THEOREM 4.3 ([**24**], Theorem 4). *Let $K$ be a co-ideal of an implicative semigroup $S$ with apartness. Then $K^{\lhd}$ satisfies the conditions (J1) and (J2).*

PROOF. Let $x, y, u \in S$ be arbitrary elements such that $y \in K^{\lhd}$ and $u \in K$. Then $u \neq x \otimes y \vee x \otimes y \in K$ by strongly extensionality of $K$. From option $x \otimes y \in K$ follows $y \in K$ by (K4). It is impossible in accordance with hypothesis $y \lhd K$. So, have to be $x \otimes y \neq u \in K$. Therefore, $x \otimes y \in K^{\lhd}$. So, we have proved that set $K^{\lhd}$ satisfies condition (J1).

Let $x, y, z, u \in S$ be arbitrary elements such that $x \in K^{\lhd}$, $y \in K^{\lhd}$ and $u \in K$. Then

$$u \neq (x \otimes (y \otimes z)) \otimes z \vee (x \otimes (y \otimes z)) \otimes z \in K$$

by strongly extensionality of $K$ in $S$. From $(x \otimes (y \otimes z)) \otimes z \in K$ follows $x \cdot y \in K$ by Proposition 5 in [**24**]. Thus $y \in K$ by (K1). This contradicts the hypothesis $y \lhd K$. So, it has to be $(x \otimes (y \otimes z)) \otimes z \neq u \in K$. Finally, we have $(x \otimes (y \otimes z)) \otimes z \in K^{\lhd}$. We have proved that the set $K^{\lhd}$ satisfies the condition (J2). □ □

## 5. THE CONCEPT OF CO-CONGRUENCES

In this section we introduce the concept of co-congruences on implicative semigroups with apartness and show some of its basic properties.

Let $(S, =, \neq)$ be a set with apartness. A relation $q$ on $S$ is a co-equality relation on $S$ if the following hold

(a) $(\forall x, y \in S)((x, y) \in q \implies x \neq y)$;

(b) $(\forall x, y \in S)((x, y) \in q \implies (y, x) \in q)$ and

(c) $(\forall x, y, z \in S)((x, z) \in q \implies ((x, y) \in q \vee (y, z) \in q))$ .

DEFINITION 5.1. ([**25**]) Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup with apartness. A co-equality relation $q$ on $S$ is a *co-congruence* on $S$ if the following hold

(d1) $(\forall x, y, y \in S)((xu, yu) \in q \implies (x, y) \in q)$,
(d2) $(\forall x, y, y \in S)((x \otimes u, y \otimes u) \in q \implies (x, y) \in q)$,
(e1) $(\forall x, y, v \in S)((vx, vy) \in q \implies (x, y) \in q)$ and
(e2) $(\forall x, y, v \in S)((v \otimes x, v \otimes y) \in q \implies (x, y) \in q)$.

NOTE 5.1. In the language of classical algebra, a co-equality relation $q$ on an implicative semigroup $S$ is compatible with the internal operations if those operations are cancellative with respect to $q$.

LEMMA 5.1 ([**25**]). *The condition* (d1)$\wedge$(e1) *is equivalent to the condition*
(f) $(\forall x, y, u, v \in S)((xu, yv) \in q \implies ((x, y) \in q \vee (u, v) \in q))$.

LEMMA 5.2 ([**25**]). *The condition* (d2)$\wedge$(e2) *is equivalent to the condition* (g) $(\forall x, y, u, v \in S)((x \otimes u, y \otimes v) \in q \Longrightarrow ((x, y) \in q \vee (u, v) \in q))$.

PROPOSITION 5.1 ([**25**]). *Let $q$ be a co-congruence on an implicative semigroup with apartness $S$. Then the class $xq = \{t \in S : (x, t) \in \alpha\}$, generated by the element $x \in S$, is a strongly extensional subset of $S$.*

PROPOSITION 5.2 ([**25**]). *Let $q$ be a co-congruence on an implicative semigroup with apartness $S$. Then the subset $C_1 = \{x \in S : (x, 1) \in q\}$ is a co-filter of $S$.*

## 6. QUOTIENT STRUCTURES

A relation $\sigma \subseteq S \times S$ is a co-quasiorder on $S$ if it is consistent and co-transitive. In the following, we assume that $\sigma$ is compatible with the operations in $S$ and the following $\sigma \subseteq \alpha$ holds. In [**16**], Lemma 1, the author proved that the relation $q = \sigma \cup \sigma^{-1}$ is a co-congruence on $S$.

It is known ([**21**], Proposition 1.1) that the strong complement $q^{\lhd}$ of $q$ is a congruence on $S$ associate with the co-congruence $q$ in the following sense $q^{\lhd} \circ q \subseteq q$ and $q \circ q^{\lhd} \subseteq q$. So, the factor-set $S/(q^{\lhd}, q) = \{[x] : x \in S\}$ can be constructed ([**15**], Theorem 2) with

$$[x] =_1 [y] \iff (x.y) \lhd q, \ [x] \neq_1 [y] \iff (x, y) \in q.$$

If we define the operations '$\cdot_1$' and '$\otimes_1$' and the co-order $\theta$ on $S/(q^{\lhd}, q)$ as follows

$$[x] \cdot_1 [y] =_1 [x \cdot y], \ \ [x] \otimes_1 [y] =_1 [x \otimes y], \ \ ([x], [y]) \in \theta \iff (x, y) \in \sigma,$$

the following theorem can be proved.

THEOREM 6.1 ([**25**], Theorem 4). *Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup with apartness such that the co-quasiorder $\sigma \subseteq \alpha$ satisfies condition (4). If $q = \sigma \cup \sigma^{-1}$, then the system $((S/(q^{\lhd}, q), =_1, \neq_1), \cdot_1, \theta, \otimes_1)$ is an implicative semigroup with apartnes and there is a unique reverse isotone se-epimorphism $\pi : S \longrightarrow S/(q^{\lhd}, q)$.*

In order for the interested reader to gain insight into the techniques applied in the algebra of Bishop's constructive orientation, we present the proof of this theorem.

PROOF. It can be directly verified that '$\cdot_1$' is a well-defined operation in quotient structure w$S/(q^{\lhd}, q)$ that satisfies the conditions (1) and (2).

Let $x, y, z, u, v \in S$ be elements such that $[x] =_1 [y]$ and $(u, v) \in q$. Then $(x, y) \lhd q$. From $(u, v) \in q$ it follows $(u, xz) \in q \vee (xz, yz) \in q \vee (yz, v) \in q$ by co-transitivity of $q$. Because from the second option it follows $(x, y) \in q$ by (d1), we have to have $u \neq xz \vee v \neq yz$ by consistency of $q$. Thus $(xz, yz) \neq (u, v) \in q$. Hence $[x] \cdot_1 [z] =_1 [xz] \neq_1 [yz] =_1 [y] \cdot_1 [z]$.

Let $x, y, z, u, v \in S$ be elements such that $[x] \cdot_1 [z] \neq_1 [y] \cdot_1 [z]$. Then $[xz] \neq_1 [yz]$ and $(xz, yz) \in q$. Thus $(x, y) \in q$ by (d1). Hence $[x] \neq_1 [y]$.

We have shown that the multiplication on the right is well defined, i.e. that it is an extensive and strictly extensive total function on $S/(q^{\lhd}, q)$. Analogously, it can be shown that the multiplication on the left is also well defined.

Let us show that the multiplication in $S/(q^{\lhd}, q)$ satisfies condition (3). Let $x, y, u, v \in S$ be such $([u], [v]) \in \theta$. Then $(u, v) \in \sigma$. Thus

$$(u, xy) \in \sigma \subseteq q \vee (xy, x) \in \sigma \subseteq \alpha \vee (x, v) \in \sigma \subseteq q.$$

hence $[u] \neq_1 [x] \cdot_1 [y]$ or $[x] \neq_1 [v]$ because $(xy, x) \in \alpha$ is impossible by (3). So, $([x] \cdot_1 [y], [x]) \neq_1 ([u], [v]) \in \theta$. therefore, $([x] \cdot_1 [y], [x]) \lhd \theta$. The proof of the second part of (3) is analogous to the one of the first part.

Let us check the condition (4). Let $x, y, z \in S$ be elements such that $([z], [x] \otimes_1 [y]) \in \theta$. Then $(z, x \otimes y) \in \sigma$. Thus $(zx, y) \in \sigma$. This means $([z] \cdot_1 [y], [y]) \in \theta$.

Let us define $\pi(x) = [x]$ for any $x \in S$. It is easy to prove that $\pi$ is a se-monomorphism. The reverse isotonicity remains to be verified. Suppose $([x], [y]) \in \theta$. Then $(x, y) \in \sigma \subseteq \alpha$ by definition. This means that $\pi$ is a reverse isotone se-epimorphism. The uniqueness of this mapping is obvious. □

On the other hand, we can construct ([**15**], Theorem 3) the family $[S : q] = \{xq : x \in S\}$, where

$$xq =_2 yq \iff (x, y) \lhd q, \ \ xq \neq_2 yq \iff (x, y) \in q.$$

If we define the operations '$\cdot_2$' and '$\otimes_2$' and the co-order $\Theta$ on $[S : q]$ as follows

$$xq \cdot yq =_2 (x \cdot y)q, \ \ xq \otimes_2 yq =_2 (x \otimes y)q, \ \ (xq, yq) \in \Theta \iff (x, y) \in \sigma$$

it can be verified.

Theorem 6.2 ([**25**], Theorem 5). *Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup with apartness such that the co-quasiorder $\sigma \subseteq \alpha$ satisfies condition (4). Let $q = \sigma \cup \sigma^{-1}$. Then $(([S : q], =_2, \neq_2), \cdot_2, \Theta, \otimes_2)$ is an implicative semigroup with apartnes and there exists a unique reverse isotone se-epimorphism $\vartheta : S \longrightarrow [S : q]$.*

Proof. (i) Let us show that the operation '$\cdot_2$' is well defined.

We first show that '$\cdot_2$' is an extensive function with respect to the equality '$=_2$' in set $[S : q]$. Let $x, y, u, v, s, t \in S$ be such that $xq =_2 uq$, $qy =_2 vq$ and $(s, t) \in q$. Then $(x, u) \lhd q$, $(y, v) \lhd q$. From $(s, t) \in q$ it follows

$$(s, xy) \in q \vee (xy, xv) \in q \vee (xv, uv) \in q \vee (uv, t) \in q$$

by co-transitivity of $q$. Thus,

$$s \neq xy \vee (y, v) \in q \vee (x, u) \in q \vee uv \neq t$$

by consistency of $q$ and since the second and third options are impossible because $(x, u) \lhd q$ and $(y, v) \lhd q$ hold hypothesis. This shows that $(xy, uv) \neq (s, t) \lhd q$. So $xyq =_2 uvq$. This means $xq \cdot_2 yq = uq \cdot_2 vq$ according to the definition of the operation '$\cdot_2$'.

To show that '$\cdot_2$' is a strictly extensive function, we take $x, u, y, v \in S$ such that $xyq \neq_2 uvq$, ie. such that $(xy, uv) \in q$. From $(xy, uv) \in q$ immediately follows $(x, y) \in q \vee (y, v) \in q$ according to Lemma 1 in [**25**]. Therefore, we have $xq \neq_2 uq$ or $yq \neq_2 = vq$ which shows that '$\cdot_2$' is an extensive function.

(ii) Let $x, y, z, y, v \in S$ be arbitrary elements such that $(u, v) \in q$. Then $u, x(yz)) \in q \vee (x(yz), (xy)z) \in q \vee ((xy)z, v) \in q$ by co-transitivity of $q$. Thus, we have $u \neq x(yz) \vee (xy)z \neq v$ by consistency of $q$ since the second option is impossible by (1). This gives $(x(yz),(xy)z) \neq (u,v) \in q$ which means that $x(yz)q =_2 (xy)zq$ is valid. This it shown that the multiplication '$\cdot_2$' in the set $[S:q]$ satisfies the condition (1).

Let $x,y,z \in S$ be such that $(xzq,yzq) \in \Theta$. Them $(xz,yz) \in \sigma$ by definition of $\Theta$. Thus $(x,y) \in \sigma$ by compatibility of the multiplication in $S$ with the co-quasiorder $\sigma$. So, we have $(xq,yq) \in \Theta$. The proof for the implication $(zxq,zyq) \in \Theta \implies (zq,yq) \in \Theta$ can be shown analogously to the previous one. Thus, multiplication in [S: q] satisfies the condition (2).

To show that the multiplication in semigroup $[S:q]$ satisfies condition (3), we take the elements $x,y,u,v \in S$ such that $(uq,vq) \in \Theta$. Then, from $(u,v) \in \sigma$ it follows $(u,xy) \in \sigma \vee (xy,x) \in \sigma \subseteq \alpha \vee (x,v) \in \sigma$ by co-transitivity of $\sigma$. Since the second option is impossible by (3), we have $u \neq xy$ or $x \neq v$ by consistency of $\sigma$. This means $(xy,x) \neq (u,v) \in \sigma$. So, $(xyq,xq) \neq_2 (uq,vq) \in \Theta$. Therefore, we have $(xyq,xq) \lhd \Theta$. The proof that for the elements $x,y \in S$ holds $(xyq,yq) \lhd \Theta$ can be shown analogously to the previous one.

(iii) Let us show that the operation a is well defined and that the elements semigroup $[S:q]$ satisfy the condition (4).

Let $x,y,u,v,s,t \in S$ be such that $xq =_2 uq$, $yq =_2 vq$ and $(s,y) \in q$. Then $(x,u) \lhd q$ and $(y,v) \lhd q$. From $(s,t) \in q$ it follows

$$(s,x\otimes y) \in q \vee (x\otimes y,u\otimes y) \in q \vee (u\otimes y,u\otimes v) \in q \vee (u\otimes v,t) \in q$$

by co-transitivity of $q$. The second and third possibilities would lead to $(x,y) \in q$ and $y,v) \in q$ which is in contradiction with the hypothesis. So, have to have $s \neq x\otimes y$ or $u\otimes v \neq t$. This means $(x\otimes y,u\otimes v) \neq (s,y) \in q$. Therefore, $(x\otimes y)q =_2 (u\otimes v)q$.

Let us choose the elements $x,y,u,v \in S$ such that $(x\otimes y)q \neq_2 (y\otimes v)q$. Then $(x\otimes y,u\otimes v) \in q$. Hence immediately follows $(x,u) \in q$ or $(y,v) \in q$ according to Lemma **??**. This means $xq \neq_2 uq$ or $yq \neq_2 vq$.

Let $x,y,z \in S$ be such $(zq,(x\otimes y)q) \in \Theta$. Then $(z,x\otimes y) \in \sigma$ by definition of $\Theta$. Since $(z,x\otimes y) \in \sigma \iff (zx,y) \in \sigma$ according to (4), we have $((zx)q,yq) \in \Theta$. Obviously, the reverse is also true. Thus, in the semigroup $[S:q]$, the condition (4) is a valid formula.

(iv) By direct checking it can be determined that the function: $\vartheta : S \longrightarrow [S:q]$, defined by $(\forall x \in S)(\vartheta(x) = xq)$, is a unique se-surjective function. Further on, for $x,y \in S$ we have $\vartheta(x\otimes y) =_2 (x\otimes y)q =_2 xq \otimes_2 yq =_2 \vartheta(x) \otimes_2 \vartheta(y)$. Let $x,y \in S$ be such that $(\vartheta(x),\vartheta(y)) \in \Theta$. Then $(xq,yq) \in \Theta$. Thus $(x,y) \in \sigma \subseteq \alpha$ by definition of $\Theta$. This means that $\vartheta$ is a reverse isotone se-epimorphism. □

It should be emphasized here that the semigroup [S: q] has no counterpart in the classical theory of implicit semigroups. However, there is a strong link between the semigroup $S/(q^{\lhd},q)$ and the semigroup $[S:q]$.

Theorem 6.3 ([**25**], Theorem 6). *Let $((S,=,\neq),\cdot,\alpha,\otimes)$ be an implicative semigroup with apartness such that the co-quasiorder $\sigma \subseteq \alpha$ satisfies condition (4). Let $q = \sigma \cup \sigma^{-1}$. Then there exists a unique reverse isotone se-epimorphism*

*$\pi : S \longrightarrow S/(q^{\triangleleft}, q)$, defined by $\pi(x) = [x]$, and a unique reverse isotone se-epimorphism $\vartheta : S \longrightarrow [S : q]$, defined by $\vartheta(x) = xq$, and a unique strongly extensional, embedding, injective and surjective homomorphism $h : S/(q^{\triangleleft}, q) \longrightarrow [S : q]$, defined by $h([x]) = xq$, such that $\vartheta = h \circ \pi$ and $\pi = h^{-1} \circ \vartheta$.*

## 7. THE ISOMORPHISM THEOREM

We will start this section with two important technical lemmas.

LEMMA 7.1 ([**25**], Lemma 7). *Let $f : S \longrightarrow T$ be a reverse isotone se-epimorphism from implicative semigroups $((S, =, \neq), \cdot, \alpha, \otimes)$ onto implicative semigroup $((T, =, \neq), \cdot, \beta, \otimes)$. Then the relation $f^{-1}(\beta) = \{(\in S \times S : (f(x), f(y)) \in \beta\}$ is a co-quasiorder on $S$ such that $f^{-1}(\beta) \subseteq \alpha$ and $f^{-1}(\beta)$ satisfies the condition (4).*

LEMMA 7.2 ([**25**], Lemma 8). *Let $f : S \longrightarrow T$ be a reverse isotone se-epimorphism from implicative semigroups $((S, =, \neq), \cdot, \alpha, \otimes)$ onto implicative semigroup $((T, =, \neq), \cdot, \beta, \otimes)$. Then the relation $q_f = \{(x, y) \in S \times S : f(x) \neq f(y)\}$ is a co-congruence on $S$ such that $q_f = f^{-1}(\beta) \cup (f^{-1}(\beta))^{-1}$.*

The following theorem can be viewed as the First Isomorphism Theorem for implicative semigroups with apartness.

THEOREM 7.1 ([**25**], Theorem 9). *Let $f : S \longrightarrow T$ be a reverse isotone se-epimorphism from implicative semigroups with apartness $((S, =, \neq), \cdot, \alpha, \otimes)$ onto implicative semigroup with apartness $((T, =, \neq), \cdot, \beta, \otimes)$. Then there exist a unique pair $f_1 : S/(q_f^{\triangleleft}, q_f) \longrightarrow T$ and $f_2 : [S : q_f] \longrightarrow T$ of embedding, injective and surjective se-homomorphisms such that*

$$f = f_1 \circ \pi = f_2 \circ \vartheta = f_2 \circ h \circ \pi.$$

PROOF. The existence and uniqueness of the se-epimorphisms $\pi$ and $\vartheta$ is proved in Theorem 6.1. and Theorem 6.2 respectively.

In what follows, let $x, y \in S$ be such that $[x] =_1 [y]$. Then

$$[x] =_1 [y] \iff xq^{\triangleleft} =_1 yq^{\triangleleft} \iff (x, y) \triangleleft q \iff h([x]) = xq =_2 yq = h([y]).$$

This shows that $h$ is an injective function.

Take $x, y \in S$ such that $h([x]) \neq_2 h([y])$. Then the following holds

$$h([x]) \neq_2 h([y]) \iff xq \neq_2 yq \iff (x, y) \in q \iff [x] \neq_1 [y].$$

This proves that $h$ is a strongly extensive and embedding function.

As the following equations

$$h([x] \cdot_1 [y]) =_2 h([xy]) =_2 (xy)q =_2 xq \cdot_2 yq =_2 h([x]) \cdot_2 h([y])$$

and

$$h([x] \otimes_1 [y]) =_2 h([xy]) =_2 (xy)q =_2 xq \otimes_2 yq =_2 h([x]) \otimes_2 h([y])$$

are obviously true, we conclude that $h$ is a se-homomorphism. Finally, $h$ is a unique embedding, injective and surjective se-homomorphism such that $\vartheta = h \circ \pi$ is valid. □

## 8. CONCEPT OF IMPLICATIVE CO-FILTERS

DEFINITION 8.1. ([**26**], Definition 1) Let $((S,=,\neq),\cdot,\alpha,\otimes)$ be an implicative semigroup. An inhabited subset $G$ of $S$ is called an implicative co-filter of $S$ if it satisfies (G3) and

(GI) $(\forall x,y,z \in S)(x \otimes z \in G \implies (x \otimes (y \otimes z) \in G \vee x \otimes y \in G))$.

EXAMPLE 8.1. Let $S = \{1,2,3,4,5,0\}$ and operations '$\cdot$' and '$\otimes$' defined on $S$ as follows:

| $\cdot$ | 1 | 2 | 3 | 4 | 5 | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 2 | 3 | 3 | 5 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 | 0 | 0 |
| 4 | 4 | 5 | 0 | 4 | 5 | 0 |
| 5 | 5 | 0 | 0 | 5 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

and

| $\otimes$ | 1 | 2 | 3 | 4 | 5 | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 1 | 1 | 2 | 4 | 4 | 5 |
| 3 | 1 | 1 | 1 | 4 | 4 | 4 |
| 4 | 1 | 2 | 3 | 1 | 2 | 3 |
| 5 | 1 | 1 | 2 | 1 | 1 | 2 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |

Then $S = ((S,=,\neq),\cdot,\nleqslant,\otimes)$ is an implicative semigroup where the co-order relation '$\nleqslant$' is defined as follows $\nleqslant= \{(1,2),(1,3),(1,4),(1,5),(1,0),(2,3),(2,4),(2,5),(2,0), (3,4), (3,5),(3,0),(4,2),(4,3),(4,5),(4,0),(5,3),(5,0)\}$. Then the subsets $\{4,5,0\}$, $\{2,3,5,0\}$ and $\{2,3,4,5,0\}$ are ordered co-filters of $S$. The subset $\{4,5,0\}$ is an implicative co-filters of $S$ but the ordered co-filter $\{2,3,4,5,0\}$ is not an implicative co-filter of $S$.

PROPOSITION 8.1 ([**26**]). *Every implicative co-filter is a co-filter.*

PROOF. Let $G$ be an implicative co-filter of an implicative semigroup $S$. Let $y,z \in S$ be such that $y \in G$. Then $1 \otimes y \in G$ by Corollary 3.3. in [**19**]. Thus $1\otimes y \in G \implies (1\otimes(y\otimes z) \vee 1\otimes y \in G)$ by (GI) . Hence $y \in G \implies (y\otimes z \in G \vee z \in G)$ in accordance with Corollary 3.3. in [**19**]. We have proved that $G$ satisfies the conditions of Theorem 3.7 in [**19**]. So, $G$ is a co-filter in $S$. □

In what follows, the following proposition will be useful to us.

PROPOSITION 8.2 ([**26**]). *Let $S$ be an implicative semigroup with apartness. If $G$ is an implicative co-filter of $S$, then the following holds:*

(G5) $(\forall x,y \in S)(x \otimes y \in G \implies x \otimes (x \otimes y) \in G)$

Let $G$ be an ordered co-filter of an implicative semigroup with apartness $S$ and let $a \in S$. Define $G_a := \{x \in S : a \otimes x \in G\}$. Note that $G_1 = G$ according Corollary 3.3 in [**19**].

By using the set $G_a$ $(a \in G)$, we can design a condition for an ordered co-filter to be an implicative co-filter.

THEOREM 8.1 ([**26**]). *Let $G$ be an ordered co-filter of an implicative semigroup with apartness $S$. Then $G$ is an implicative co-filter of $S$ if and only if for any $a \in S$, the set $G_a$ is an ordered co-filter of $S$.*

In what follows, we will deal with commutative implicative semigroups with apartness: An implicative semigroup with apartness $((S,=,\neq),\cdot,\alpha,\otimes)$ is commutative if it satisfies $(\forall x,y \in S)(x \cdot y = y \cdot x)$.

Lemma 8.1. *If $S$ is a commutative implicative semigroup with apartness, then the following holds*
(5) $(\forall x, y, z \in S)(x \otimes (y \otimes z) = y \otimes (x \otimes z))$.

Lemma 8.2. *If $S$ is a commutative implicative semigroup with apartness, then the following holds*
(6) $(\forall x, y, z \in S)((y \otimes z, (x \otimes y) \otimes (x \otimes z)) \lhd \alpha)$.

Theorem 8.2. *Let $S$ be a commutative implicative semigroup with apartness. If an inhabited subset $G$ of $S$ satisfies the condition (G2), (G3) and the condition*
(G6) $(\forall x, y, z \in S)(y \otimes z \in G \implies (x \otimes (y \otimes (y \otimes z)) \in G \vee x \in G))$,
*then $G$ is an implicative co-filter of $S$.*

Theorem 8.3 ([**26**]). *Let $S$ be a commutative implicative semigroup with apartness. If an ordered co-filter $G$ of $S$ satisfies the condition*
(G7) $(\forall x, y, z \in S)((x \otimes y) \otimes (x \otimes z) \in G \implies x \otimes (y \otimes z) \in G)$,
*then $G$ is an implicative co-filter of $S$.*

Let us prove that for an ordered co-filter $G$ of an implicative semigroup with apartness $S$ condition (G5) is sufficient for it to be an implicit co-filter of $S$.

Theorem 8.4 ([**26**]). *Let $S$ be a commutative implicative semigroup with apartness. If an ordered co-filter $G$ of $S$ satisfies the condition (G5), then it is an implicative co-filter of $S$.*

Proof. Suppose that an ordered co-filter $G$ of $S$ satisfies the condition (G5). To prove that $G$ is an implicative co-filter of $S$, it suffices to prove that it satisfies the condition (G7).

Let $x, y, z \in S$ be arbitrary elements such that $(x \otimes y) \otimes (x \otimes z) \in G$. Then $x \otimes ((x \otimes y) \otimes z) \in G$ by (5). Thus $x \otimes (x \otimes (x \otimes y) \otimes z) \in G$ by (G5). This, according to (5), can be transformed into $x \otimes ((x \otimes y) \otimes (x \otimes z)) \in G$. Hence it follows

$$(x \otimes (y \otimes z), x \otimes ((x \otimes y) \otimes (x \otimes z)) \in \alpha \;\vee\; x \otimes (y \otimes z) \in G$$

in accordance with (G2) since $G$ is an ordered co-filter of $S$. The first option gives us

$$(y \otimes z, (x \otimes y) \otimes (x \otimes z)) \in \alpha$$

due to the strongly extensionality of the operation $x$, which is in contrast to (6). The obtained contradiction confirms the possibility of $x \otimes (y \otimes z) \in G$. □

## 9. FINAL OBSERVATION

Bishops constructive mathematics includes the following two basements:
(1) The Intuitionistic logic and
(2) The principled-philosophical orientations of constructivism.

Intuitionistic logic does not accept the TND principle (tertium non datur i.e. the logical princess exclusion of the third) as an axiom. In addition, Intuitionistic logic does not accept the validity of the double negation principle. This makes it possible to have a difference relation in sets which is not a negation of equality relation.

Therefore, we accept that in Bishops constructive mathematics we consider set $S$ as a relational system $(S, =, \neq)$. These orientation allow us design a co-substructure in $S$ as a dual of the observed sub-structure.

In Bishops constructive algebra we always encounter with at least the following three problems:

(a) How to choose a predicate (or more predicates) between several classically equivalent ones by which an algebraic concept is determined?

(b) Since every predicate has at least one of its duales, how to construct a dual of given an algebraic structure?

(c) What are the specifics of this approach to looking at a given algebraic structure and what are the particularities of the case that cannot be found in classical algebra?

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Received by editors 09.01.2020; Revised version 22.06.2020; Available online 06.07.2020.

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