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f - DERIVATION AND d_a-DERIVATION OF ORDERED-SEMIRINGS

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Abstract. In this paper, we introduce the concept of f-derivation and the concept of d_a derivation of an ordered semiring. We study some of the properties of f and d_a derivations of ordered semirings. We prove that, if d is a f-derivation of an ordered integral semiring M then $kerd$ is a $m-k-$ideal of M.

1. Introduction

The notion of a semiring was introduced by Vandiver [**10**] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. Semiring is a generalization of ring but also of a generalization of distributive lattice. Semirings are structually similar to semigroups than to rings. Semiring theory has many applications in other branches of mathematics.

A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line, then $(I, \max, \min)$ is a semiring in which 0 is the additive identity and 1 is the mutiplicative identity. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. In structure, semiring lies between semigroup and ring. Many semirings have order structure in addition to their algebraic structure. Over the last few decades, several authors have investigated the relationship between the commutativity of ring R and the existence of certain specified derivations of R. The first result in this derivation is due to Posner [**5**] in 1957.

In the year 1990, Bresar and Vukman [**2**] established that a prime ring must be commutative if it admits a non-zero left derivation. Kim [**3**], [**4**] studied right

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derivation and generalized derivation of incline algebra. The notion of derivation of algebraic structures is useful for characterization of algebraic structures. The notion of derivation has also been generalized in various directions such as right derivation, left derivation, f-derivation, reverse derivation, orthogonal derivation, generalized right derivation, etc. M. K. Rao and Venkateswarlu [**6**], [**7**] introduced the notion of generalized right derivation of $\Gamma-$ incline and right derivation of ordered $\Gamma-$semiring. M. K. Rao [**8**], M.K.Sen [**9**] introduced and studied Γ-semiring and Γ-semigroup respectively.

In this paper,we introduce the concepts of d_a- derivation and f-derivation of ordered semirings. We study some of the properties of d_a, f-derivations of ordered semirings. We prove that if a derivation d_a is non-zero on an integral semiring M then it is non-zero on any non-zero ideal of M and we characterize k-ideal and $m-k$ ideal using derivations d_a, f of an ordered semiring.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. ([**1**]) A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Example 2.1. Let M be the set of all natural numbers. Then (M, max, min) is a semiring.

Definition 2.2. Let M be a semiring. If there exists $1 \in M$ such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in M$, is called an unity element of M then M is said to be semiring with unity.

Definition 2.3. An element a of a semiring S is called a regular element if there exists an element b of S such that $a = aba$.

Definition 2.4. A semiring S is called a regular semiring if every element of S is a regular element.

Definition 2.5. An element a of a semiring S is called a multiplicatively idempotent (an additively idempotent) element if $aa = a(a + a = a)$.

Definition 2.6. An element b of a semiring M is called an inverse element of a of M if $ab = ba = 1$.

Definition 2.7. A non-empty subset A of semiring M is called

(i) a subsemiring of M if A is an additive subsemigroup of M and $AA \subseteq A$.
(ii) a left (right) ideal of M if A is an additive subsemigroup of M and $MA \subseteq A$ $(AM \subseteq A)$.

(iii) an ideal if A is an additive subsemigroup of M, $MA \subseteq A$ and $AM \subseteq A$.
(iv) a $k-$ideal if A is a subsemiring of M, $AM \subseteq A$, $MA \subseteq A$ and $x \in M$, $x + y \in A$, $y \in A$ then $x \in A$.

Definition 2.8. A semiring M is called a division semiring if for each non-zero element of M has multiplication inverse.

Definition 2.9. A semiring M is called an ordered semiring if it admits a compatible relation $\leqslant$, i.e. $\leqslant$ is a partial ordering on M satisfies the following conditions. If $a \leqslant b$ and $c \leqslant d$ then

(i) $a + c \leqslant b + d, c + a \leqslant d + b$
(ii) $ac \leqslant bd$
(iii) $ca \leqslant db$, for all $a, b, c, d \in M$

Definition 2.10. An ordered semiring M is said to have zero element if there exists an element $0 \in M$ such that $0+x = x = x+0$ and $0x = x0 = 0$, for all $x \in M$.

An ordered semiring M is said to be commutative semiring if $xy = yx$, for all $x, y \in M$.

Definition 2.11. A non zero element a in an ordered semiring M is said to be zero divisor if there exists non zero element $b \in M$, such that $ab = ba = 0$.

Definition 2.12. An ordered semiring M with unity 1 and zero element 0 is called an integral ordered semiring if it has no zero divisors.

Definition 2.13. An ordered semiring M is said to be totally ordered semiring M if any two elements of M are comparable.

Definition 2.14. In an ordered semiring M

(i) the semigroup $(M, +)$ is said to be positively ordered , if $a \leqslant a + b$ and $b \leqslant a + b$, for all $a, b \in M$.
(ii) the semigroup $(M, +)$ is said to be negatively ordered, if $a + b \leqslant a$ and $a + b \leqslant b$, for all $a, b \in M$.
(iii) the semigroup $(M, \cdot)$ is said to be positively ordered, if $a \leqslant ab$ and $b \leqslant a\alpha b$, for all $\in \Gamma, a, b \in M$.
(iv) the semigroup $(M, \cdot)$ is said to be negatively ordered if $ab \leqslant a$ and $ab \leqslant b$ for all $a, b \in M$.

Definition 2.15. A non-empty subset A of an ordered semiring M is called a subsemiring M if $(A, +)$ is a subsemigroup of $(M, +)$ and $ab \in A$ for all $a, b \in A$.

Definition 2.16. Let M be an ordered semiring. A non-empty subset I of M is called a left (right) ideal of an ordered semiring M if I is closed under addition, $MI \subseteq I$ $(IM \subseteq I)$ and if for any $a \in M$, $b \in I$, $a \leqslant b \Rightarrow a \in I$. I is called an ideal of M if it is both a left ideal and a right ideal of M.

Definition 2.17. A non-empty subset A of an ordered $\Gamma-$semiring M is called a $k-$ideal if A is an ideal and $x \in M$, $x + y \in A, y \in A$ then $x \in A$.

Definition 2.18. Let M and N be ordered semirings. A mapping $f : M \rightarrow N$ is called a homomorphism if

(i) $f(a + b) = f(a) + f(b)$
(ii) $f(ab) = f(a)f(b)$, for all $a, b \in M, \in \Gamma$.

Definition 2.19. Let M be an ordered semiring. A mapping $f : M \rightarrow M$ is called an endomorphism if

(i) f is an onto ,
(ii) $f(a + b) = f(a) + f(b)$,
(iii) $f(ab) = f(a)f(b)$, for all $a, b \in M$.

Definition 2.20. Let M be an ordered semiring. A mapping $d : M \rightarrow M$ is called a derivation if it satisfies
(i) $d(x + y) = d(x) + d(y)$
(ii) $d(xy) = d(x)y + xd(y)$ for all $x, y \in M$.

3. f-derivation of ordered semirings

In this section, we introduce the concept of f-derivation of ordered semirings and study some of their properties.

Definition 3.1. Let M be an ordered semiring and f be an endomorphism on M. A mapping $d : M \rightarrow M$ is called an f-derivation if it satisfies
(i) $d(x + y) = d(x) + d(y)$
(ii) $d(xy) = d(x)f(y) + xd(y)$ for all $x, y \in M$.

Theorem 3.1. *Let d be a f-derivation of an ordered semiring M. If $f(x) = x$ for all $x \in M$ then d is a derivation of M.*

Proof. Let $x, y \in M$. Then $d(xy) = d(x)f(y) + xd(y) = d(x)y + xd(y)$. Hence d is a derivation of M. Therefore f-derivation of M is a generalization of derivation d of M. □

Theorem 3.2. *Let d be a f-derivation of an ordered semiring M. If $f(0) = 0$ then $d(0) = 0$.*

Proof. Suppose d is a f-derivation of M. Then

$$d(0) = d(00) = d(0)f(0) + 0d(0) = d(0)0 + 0d(0) = 0 + 0 = 0.$$

□

Theorem 3.3. *Let f be an endomorphism on idempotent commutative ordered semiring M and $x \leqslant f(x)$ for all $x \in M$. Then f is a f-derivation of M.*

Proof. Let $x, y \in M$ .Then

$$\begin{aligned} f(xy) &= f(x)f(y) \\ &= f(x)f(y) + f(x)f(y) \\ &= f(x)f(y) + [x + f(x)]f(y) \\ &= f(x)f(y) + xf(y) + f(x)f(y) \\ &= f(x)f(y) + xf(y). \end{aligned}$$

Also, we have $f(x+y) = f(x) + f(y)$ since f is an endomorphism of M. Hence f is a derivation of M. □

THEOREM 3.4. *Let I be a non-zero ideal of an integral ordered semiring M in which multiplcative semigroup M is negatively ordered. If d is a non-zero f derivation on M , where f is a non-zero function on I then d is a non-zero f derivation on I.*

PROOF. Let d be a f derivation on I. Suppose that $x \in I$ such that $x \neq 0$, $d(x) = 0$ and $y \in M$. We have that $xy \leqslant x$ implies $d(xy) \leqslant d(x)$ and $d(xy) = 0$. Then $d(x)f(y) + xd(y) = 0$ and $xd(y) = 0$.. Thus $d(y) = 0$. since M is an integral ordered semiring. This contradicts that d is a non-zero f- derivation on M. Hence d is a non-zero f-derivation on I. □

THEOREM 3.5. *Let M be an idempotent ordered semiring and d be a f- derivation on M. If $d \circ d = d$ and $f \circ d = f$ then $d(xd(x)) = d(x)$ for all $x \in M$.*

PROOF. Let $x \in M$. Then $x = xx$. Thus

$$d(xd(x)) = d(x)f(d(x)) + xd(d(x)) = d(x)f(x) + xd(x) = d(xx) = d(x).$$

□

DEFINITION 3.2. An ordered semiring M is called a prime ordered semiring if $aMb = 0$ then $a = 0$ or $b = 0$.

DEFINITION 3.3. An ordered semiring M is called a $2-$ torsion free if $2a = 0 \Rightarrow a = 0$, for all $a \in M$.

THEOREM 3.6. *Let M be a prime ordered semiring and I be a non-zero ideal of M. If there exists f- derivation d on M and $d(I)x = 0$ then $x = 0$.*

PROOF. Suppose $d(I)x = 0$. Then $d(\gamma a)x = 0$ for all $\gamma \in I$ and $a \in M$. Thus $(d(\gamma)f(a) + \gamma d(a))x = 0$. Hence

$$\begin{aligned} d(\gamma)f(a)x + \gamma d(a)x &= 0 \\ \Rightarrow d(\gamma)f(a)x + \gamma d(a)x &= 0 \\ \Rightarrow 0 + \gamma d(a)x &= 0 \\ \Rightarrow \gamma d(a)x &= 0. \end{aligned}$$

Replacing a by ab, we have

$$\begin{aligned} \gamma d(ab)x &= 0 \\ \Rightarrow \gamma[d(a)f(b) + ad(b)]x &= 0 \\ \Rightarrow \gamma ad(b)x &= 0 \\ \Rightarrow d(b)x &= 0, \quad \textit{since } d \neq 0 \\ \Rightarrow x &= 0 \end{aligned}$$

Hence the theorem. □

THEOREM 3.7. *Let M be a 2-torsion free prime ordered semiring, d be an f-derivation on M such that $f \circ d = d \circ f$. If $d^2 = 0$ then $d = 0$.*

Proof. Suppose $d^2 = 0$, $x, y \in M$. Then $d^2(xy) = 0$. Thus $d[d(x)f(y) + xd(y)] = 0$. From here it follows

$$\begin{aligned} &d^2(x)f(f(y)) + (d(x))d(f(y)) + d(x)f(d(y)) + xd(d(y)) = 0 \\ &\Rightarrow d(x)d(f(y)) + d(x)f(d(y)) = 0 \\ &\Rightarrow d(x)[d(f(y)) + f(d(y))] = 0 \\ &\Rightarrow d(x)[2d(f(y))] = 0. \end{aligned}$$

Therefore $d(x) = 0$ for all $x \in M$. Hence $d = 0$. □

Theorem 3.8. *Let d be a f- derivation on a prime ordered semiring M. If $a \in M$ such that $ad(x) = 0$ or $d(x)a = 0$, then $a = 0$ or $d = 0$.*

Proof. Let $x, y \in M$. Suppose that $ad(x) = 0$, for all $x \in M$. Then $ad(xy) = 0$. Thus $a[d(x)f(y) + xd(y)] = 0$ and

$$\begin{aligned} ad(x)f(y) + axd(y) &= 0 \\ \Rightarrow ad(x)f(y) + axd(y)] &= 0 \\ \Rightarrow axd(y) &= 0 \\ \Rightarrow a = 0 \text{ or } d &= 0. \end{aligned}$$

Similarly we can prove $d(x)a = 0$ then $a = 0$ *or* $d = 0$. □

Theorem 3.9. *Let d be a f- derivation of an ordered idempotent semiring M. If $d \circ d = d$ and $f \circ d = f$ then for each $x \in M$ $d(xd(x)) = d(x)$.*

Proof. Suppose d is a f- derivation of the ordered idempotent semiring M such that $d \circ d = d$, $f \circ d = f$ and $x \in M$. Then $xx = x$. Now

$$\begin{aligned} d(xd(x)) &= d(x)f(d(x)) + xd(d(x)) \\ &= d(x)f(x) + xd(x) \\ &= d(xx) \\ &= d(x). \end{aligned}$$

Therefore $d(xd(x)) = d(x)$. □

Theorem 3.10. *Let M be an ordered semiring in which $(M, +)$ is cancellative. Let d be a f- derivation of M, I be a subsemiring of M such that $f(I) = I$ and $d(xy) = d(x)d(y)$ for all x, y inI. Then $d(x)yf(x) = xyd(x) = d(x)yd(x)$ for all $x, y \in I$.*

Proof. Let $x, y \in I$. Then

$$\begin{aligned} d(xyx) &= d(x)f(yx) + xd(yx) \\ &= d(x)f(y)f(x) + xd(y)d(x) - - - - - (1) \end{aligned}$$

and

$$\begin{aligned} d(xyx) &= [d(x)f(y) + xd(y)]d(x) \\ &= d(x)f(y)d(x) + xd(y)d(x) - - - - - (2). \end{aligned}$$

From (1) and (2), we have $d(x)f(y)f(x) = d(x)f(y)d(x)$. Hence $d(x)zf(x) = d(x)zd(x)$ for all $x, z \in I$. Now

$$\begin{aligned} d(yxy) &= d(yx)f(y) + g(yx)d(y) \\ &= d(y)d(x)f(y) + yxd(y) - - - - - (3) \end{aligned}$$

and

$$\begin{aligned} d(yxy) &= d(y)d(xy) \\ &= d(y)[d(x)f(y) + xd(y)] \\ &= d(y)d(x)f(y) + d(y)xd(y) \\ &= d(y)d(x)f(y) + d(y)xd(y) - - - - - (4). \end{aligned}$$

From (3) and (4), we get $d(y)xd(y) = yxd(y),\ for\ all\ y \in I$. Therefore $d(x)yd(x) = xyd(x)$ for all $y \in I$. Hence $d(x)yd(x) = xyd(x) = d(x)yf(x)$. □

THEOREM 3.11. *Let M be a commutative ordered semiring and d_1, d_2 be f-derivations of M, $f \circ d_2 = f \circ d_1$ $d_1 \circ f = d_2 \circ f$, $f \circ f = f$. Define $d_1 d_2(x) = d_1(d_2(x))$ for all $x \in M$. If $d_1 d_2 = 0$ then $d_2 d_1$ is a f-derivation of M.*

PROOF. Suppose $d_1 d_2 = 0$,$x, y \in M$. Then $d_1 d_2(xy) = 0$ and $d_1[d_2(x)f(y) + xd_2(y)] = 0$. Thus $d_1(d_2(x)f(y)) + d_1(xd_2(y)) = 0$. Hence

$$d_1 d_2(x)f(f(y)) + d_2(x)d_1(f(y)) + (d_1(x))f(d_2(y)) + xd_1 d_2(y) = 0.$$

Therefore $d_2(x)d_1(f(y) + d_1(x)f(d_2(y)) = 0$ and

$$g(d_1(x))d_2(f(y)) + d_2(x)f(d_1(y)) = 0. - - - - - (1).$$

Now

$$\begin{aligned} d_2 d_1(xy) &= d_2[d_1(xy)] \\ &= d_2[d_1(x)f(y) + xd_1(y)] \\ &= d_2[d_1(x)f(y)] + d_2[xd_1(y)] \\ &= d_2 d_1(x) f \circ f(y) + d_1(x)d_2(f(y)) \\ &+ d_2(x)f(d_1(y)) + xd_2(d_1(y)) \\ &= d_2 d_1(x) f \circ f(y) + d_1(y) \quad from\ (1) \\ &= d_2 d_1(x)f(y) + xd_2 d_1(y). \end{aligned}$$

Hence $d_2 d_1$ is a f-derivation of M. □

THEOREM 3.12. *Let d be a f-derivation of idempotent ordered semiring M. If $d(1) = 1, x \leqslant d(x)$. Then the following hold for all $x, y \in M$.*

(i) $d(xy) \leqslant d(x)$

(ii) $d(xy) \leqslant d(y)$

(iii) *d is an isotone derivation.*

PROOF. Let $x, y \in M$. Then:

(1) $d(xy) = d(x)f(y) + xd(y) \leqslant d(x) + x \leqslant d(x) + d(x) = d(x)$.

(ii) Proof of (ii) is similar to proof of (i).

(iii) Let $x \leqslant y$. Then $x + y = y$. Thus $d(x) + d(y) = d(y)$. Therefore $d(x) \leqslant d(y)$. □

THEOREM 3.13. *Let M be an ordered semiring with unity in which $(M, +)$ is positively ordered, and d be a f-derivation. If $d(1) = 1$ then $x \leqslant d(x)$ for all $x \in M$.*

PROOF. Let $x \in M$. Then $x1 = x$.. Therefore

$$d(x) = d(x1) = d(x)f(1) + xd(1) \geqslant xd(1) = xd(1).$$

Suppose $d(1) = 1$. Then $x1 \leqslant d(x)$. Thus $x \leqslant d(x)$. □

THEOREM 3.14. *Let M be an idempotent ordered semiring in which multiplicative semigroup M is negatively ordered and d be a f-derivation such that $f(x) \leqslant x$ for all $x \in M$. Then $d(x) \leqslant x$.*

PROOF. Let $x \in M$. Then $x = xx$. Thus

$$d(x) = d(xx) = d(x)f(x) + xd(x) \leqslant f(x) + x \leqslant x + x = x.$$

Therefore $d(x) \leqslant x$. □

THEOREM 3.15. *Let M be an idempotent ordered semiring with unity in which $(M, +)$ is positively ordered and multiplicative semigroup M is negatively ordered and d be a f-derivation of M, such that $f(x) \leqslant x$ for all $x \in M$. Then $d(1) = 1$ if and only if $d(x) = x$.*

PROOF. Suppose $d(1) = 1$. By Theorem 3.13, we have $x \leqslant d(x)$ and by Theorem 3.14, we have $d(x) \leqslant x$. Therefore $d(x) = x$.

Converse is obvious. □

COROLLARY 3.1. *Let M be an idempotent ordered semiring in which multiplicativ semigroup M is negatively ordered, semigroup $(M, +)$ is positively ordered, and $d(1) = 1$. Then d is a f-derivation such that $f(x) \leqslant x$ for all $x \in M$ if and only if $d(x) = x$.*

PROOF. Suppose d is a f- derivation of the ordered semiring M such that $f(x) \leqslant x$ for all $x \in M$. By Theorem 3.15, we have $d(x) = x$.

Conversely, suppose that $d(x) = x$, for $x \in M$. Then there exists $\in \Gamma$ such that $xx = x$. Thus $d(x) = d(xx)$ and $x = xf(x) + xx$ and $x \leqslant f(x) + x$. Hence $x \leqslant f(x)$. Now, we have $x \geqslant xf(x)$ and $xx \geqslant xf(x)$. So, $x \geqslant f(x)$. Therefore $f(x) = x$. □

THEOREM 3.16. *Let d be a f- derivation of an ordered semiring M in which multiplicative semigroup M is negatively ordered and semigroup $(M, +)$ is positively ordered. Then $kerd$ is a $k-$ideal of M.*

PROOF. Let $x, y \in kerd$. Then $d(x) = d(y) = 0$. and $d(x + y) = d(x) + d(y) = 0 + 0 = 0$. Thus $d(xy) = d(x)f(y) + xd(y) = 0f(y) + x0 = 0 + 0 = 0$. Therefore xy, $x + y \in M$. Hence $kerd$ is a subsemiring of M.

Suppose $x \in kerd$ and $y \in M$. Then $d(x) = 0$. We have $xy \leqslant x$. Thus $d(xy) \leqslant d(x)$ and $d(xy) = 0$. So, $xy \in kerd$.

Suppose $x \leqslant y$ and $y \in kerd$. Then $x + y \leqslant y + y$ and $x + y \leqslant y \leqslant x + y$. Thus $x + y = y$ and hence $d(x + y) = d(y)$. So, $d(x) + d(y) = 0$ and therefore $d(x) + 0 = 0$. Finally, we have $d(x) = 0$. Hence $x \in kerd$.

Suppose $x + y \in kerd$, $x \in kerd$. Then $d(x + y) = 0$ and $d(x) = 0$. Thus $d(y) = 0$. This means $y \in kerd$. Hence $kerd$ is a $k-$ideal. □

DEFINITION 3.4. An ideal I of an ordered semiring M is said to be $m-k-$ideal if $xy \in I$, $x \in I$, $1 \neq y \in M$ then $y \in I$.

THEOREM 3.17. *Let d be a f-derivation of an ordered integral semiring M in which multiplicative semigroup M is negatively ordered and semigroup $(M,+)$ is positively ordered. Then $kerd$ is a $m-k$ ideal of M.*

PROOF. By Theorem 3.16, $kerd$ is an ideal of M. Suppose $xy \in kerd$, $x \in kerd$, $y \in M$. Then $d(xy) = d(x)f(y) + xd(y)$ and $0 = 0f(y) + xd(y)$. Thus $0 = xd(y)$. Therefore $d(y) = 0$, since M is an f ordered integral semiring. So, $y \in kerd$. Hence $kerd$ is a $m-k-$ideal of the ordered integral semiring M. □

THEOREM 3.18. *Let d be a f-derivation of an idempotent commutative ordered semiring M in which multiplicaative semigroup M is negatively ordered. If $f(x) \leqslant x$ for all $x \in M$ then $d(x) \leqslant x$ for all $x \in M$.*

PROOF. Suppose $f(x) \leqslant x$, for all $x \in M$. Then $f(x) + x = x$. Let $x \in M$. Then $x = xx$. Thus

$$d(x) = d(xx) = d(x)f(x) + xd(x) = d(x)[f(x) + x] = d(x)x \leqslant x.$$

□

THEOREM 3.19. *Let d be a f- derivation of a commutative idempotent ordered semiring M in which multiplicative semigroup M is cancellative negatively ordered, semigroup $(M,+)$ is a band. Define a set $\{x \in M : f(x) \leqslant x \wedge d(x) = x\}$ and it is denoted by $Fix_d(M)$. Then $Fix_d(M)$ is a $m-k-$ideal of M.*

PROOF. Let $x, y \in Fix_d(M)$. Then $f(x) \leqslant x, d(x) = x, f(y) \leqslant y, d(y) = y$. Therefore $f(xy) = f(x)f(y) \leqslant xy$ and $f(x+y) = f(x) + f(y) \leqslant x + y$. from here it follows $d(x+y) = x+y$. Therefore $xy \in Fix_d(M)$ and $x+y \in Fix_d(M)$.

Suppose $x \leqslant y$ and $y \in M$. Then by Corollary 3.17, $d(y) = y$. Now from $x \leqslant y$ it follows $x+y \leqslant y+y = y \leqslant x+y$ and $x+y = y$. Thus $d(x+y) = d(y)$ implies that $d(x) + d(y) = d(y)$ and $d(x) + y = y = x + y$. So, $d(x) = x$. Hence $x \in Fix_d(M)$.

Suppose $x + y \in Fix_d(M)$ and $y \in Fix_d(M)$. Then $d(x + y) = x + y$ and $d(y) = y$. Thus $d(x) + d(y) = x + y$. So, $d(x) + y = x + y$ and $d(x) = x$. Hence $Fix_d(M)$ is a $k-$ideal of M.

Suppose $xy \in Fix_d(M)$ and $x \in Fix_d(M)$. Then $f(xy) \leqslant xy$ and $d(xy) = xy$. Thus $f(x) \leqslant x$ and $d(x) = x$. Hence $f(x)f(y)+xy = xy$ and $f(x)+x = x$. Now, we have $f(x)f(y)+(f(x)+x)y = (f(x)+x)y$ and $f(x)f(y)+f(x)y+xy = f(x)y+xy$. From here, it follows $f(x)f(y) + f(x)y = f(x)y$ and $f(x)(f(y) + y) = f(x)y$. So, $f(y) + y = y$. Therefore $f(y) \leqslant y$.

First, from $d(xy) = xy$ and $d(xy) = d(x)f(y)+xd(y)$ it follows $xy = d(x)f(y)+xd(y)$ and $xy = d(x)f(y) + xd(y) \leqslant xy + xd(y)$. Thus $xy \leqslant xy + xd(y)$ and $y \leqslant y+d(y)$. So, we have $y \leqslant d(y) \leqslant y.$. This means $y = d(y)$. Hence $y \in Fix_d(M)$. Thus $Fix_d(M)$ is a $m-k-$ideal of M. □

4. Derivation d_a of ordered semirings

In this section, we introduce the notion of derivation of the form d_a of ordered semirings. We study some of the properties of derivation d_a of ordered semirings.

Let M be an ordered semiring. Then for any $a \in M$ we define a mapping $d : M \to M$ by $d(x) = xa$, for all $x \in M$.This function d is denoted by d_a.

DEFINITION 4.1. Let M be an ordered semiring and d_a be a function. Then the function d_a is said to be derivation of M if

(i) $d_a(x + y) = d_a(x) + d_a(y)$ and

(ii) $d_a(xy) = d_a(x)y + xd_a(y)$, for all $x, y \in M$.

If d_a be a derivation of an ordered semiring M and f(x)=x for all $x \in M$ then derivation $d_a$is a f derivation. Hence f derivation is a generalization of d_a derivation.

EXAMPLE 4.1. Let $M = \{0, a, b, 1\}$. If we define the the additive and multiplicative operations on M by

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

.	0	0	b	1
0	0	a	0	0
a	0	a	b	1
b	0	b	b	1
1	0	1	1	1

and $x \leqslant y$ if and only if $x + y = y$, for all $x, y \in M$ then M is an ordered semiring. Let $a \in M$ Define $d_{a,} = xa$, for all $x \in M$. Obviously $d_{a,}$ is a derivation of M

EXAMPLE 4.2. Let $M = [0, 1]$. Define the binary operations + on M by $a + b = \max\{a, b\}$ and binary operation by $ab = \min\{a, b\}$, for all $a, b \in M$, and $a \leqslant b$ if and only if $a + b = b$, for all $a, b \in M$. Then M is an ordered semiring. Let $a \in M$. Define $d_a(x) = xa$, for all $x \in M$. Obviously d_a is a derivation of M.

LEMMA 4.1. *Let M be an ordered commutative semiring in which semigroup $(M, +)$ is a band. Then d_a is a derivation of M.*

PROOF. Let M be an ordered commutative semiring in which semigroup $(M, +)$ is a band and $x, y \in M$. Then $d_a(xy) = (xy)a$ and

$$\begin{aligned} d(x)y + xd_a(y) &= (xa)y + xya \\ &= y(xa) + xya \\ &= (yx)a + xya \\ &= xya + xya \\ &= xya. \end{aligned}$$

Hence d_a is a derivation of M. □

LEMMA 4.2. *Let M be an ordered commutative semiring in which semigroup semigroup $(M, +)$ is a band with unity element* 1. *Then there exists a derivation d_1, such that $d_1(x) = x$.*

PROOF. Let $x \in M$. Then $x1 = x$. By Lemma 4.1, d_1 is a derivation and $d_1(x) = x1 = x$. □

THEOREM 4.1. *Let M be an ordered semiring in which semigroup $(M, +)$ is is a band and positively ordered,multiplicative semigroup M is negatively ordered and d_a be a derivation.Then*
(i) $d_a(xy) \leqslant d_a(x) + d_a(y)$
(ii) $d_a(x) \leqslant x$
(iii) *if $x \leqslant y$ then $d_a(xy) \leqslant y$.*

PROOF. Let $x, y \in M$. Then:

$$d_a(xy) = (xy)a = (x + x)ya = xya + xya \leqslant xa + ya = d_a(x) + d_a(y).$$

(ii) $d_a(x) = xa \leqslant x$.
(iii) Suppose $x \leqslant y$. Then $x + y \leqslant y + y$ and $x + y \leqslant y \leqslant x + y$. This $x + y = y$. Now

$$d_a(xy) \leqslant d_a(x) + d_a(y) \leqslant x + y = y.$$

This completes the proof. □

THEOREM 4.2. *Let d_a be a derivation of an ordered semiring M. Then $d_a(0) = 0$.*

PROOF. By Definition 4.1, $d_a(x) = xa$, for all $x \in M$. Then $d_a(0) = 0a = 0$. Therefore $d_a(0) = 0$. □

THEOREM 4.3. *Let d_a be a derivation of an idempotent ordered semiring M in which multiplicative semigroup M is negatively ordered, semigroup $(M, +)$ is a band. Then $d_a(x) \leqslant x$, for all $x \in M$.*

PROOF. Let d_a be a derivation of an idempotent ordered semiring M in which semigroup M is negatively ordered. Suppose $x \in M$. Then there exists such that $xx = x$. Then

$$d_a(x) = d_a(xx) = d_a(x)x + xd_a(x) \leqslant x + x.$$

Therefore $d_a(x) \leqslant x$. This completes the proof. □

THEOREM 4.4. *Let M be an ordered semiring in which multiplicaative semigroup M is negatively ordered. Then $d_a(xy) \leqslant d_a(x + y)$, for all $x, y \in M$.*

PROOF. Let M be an ordered semiring in which multiplicaative semigroup M is negatively ordered. Suppose $x, y \in M$. Then $d_a(x)y \leqslant d_a(x)$ and $xd_a(y) \leqslant d_a(y)$. Therefore

$$d_a(xy) = d_a(x)y + xd_a(y) \leqslant d_a(x) + d_a(y) = d_a(x + y).$$

This completes the proof. □

THEOREM 4.5. *Let M be an idempotent ordered semiring in which multiplicative semigroup M is negatively ordered and semigroup $(M, +)$ is a band. Then the following hold for all $x, y \in M$:*
(i) $d_a(xy) \leqslant d_a(x) + d_a(y)$

(ii) *If* $x \leqslant y$ *then* $d_a(xy) \leqslant y$
(iii) $d_a(x) \leqslant x$.

PROOF. (i)

$$d_a(xy) = d_a(x)y + xd_a(y) \leqslant d_a(x) + d_a(y).$$

(ii) Suppose $x \leqslant y$. Then $xd_a(y) \leqslant yd_a(y) \leqslant y$ and $d_a(x)y \leqslant y$. Thus

$$d_a(xy) = d_a(x)y + xd_a(y) \leqslant y + y = y.$$

(iii) Let $x \in M$. Then $xx = x$. Then

$$d_a(x) = d_a(xx) = d_a(x)x + xd_a(x) \leqslant x + x = x.$$

This completes the proof. □

THEOREM 4.6. *Let* M *be an idempotent ordered semiring with unity* 1 *in which semigroup* $(M, +)$ *is a band and positively ordered, multiplicative semigroup* M *is negatively ordered and* d_a *be a derivation of* M*. Then the following hold for all* $x \in M$*:*
(i) $xd_a(1) \leqslant d_a(x)$.
(ii) *If* $d_a(1) = 1$ *then* $d_a(x) = x$*, for all* $x \in M$.

PROOF. (i) Let $x \in M$. Then $x1 = x$. Then $d_a(x1) = d_a(x)$ and $d_a(x)1 + xd_a(1) = d_a(x)$. Thus $xd_a(1) \leqslant d_a(x)$.

(ii) Suppose $d_a(1) = 1$. We have $xd_a(1) \leqslant d_a(x)$. From here, ir follows $x1 \leqslant d_a(x)$. Therefore $x \leqslant d_a(x)$. By Theorem 4.5, holds $d_a(x) \leqslant x$. Hence $d_a(x) = x$, for all $x \in M$. □

THEOREM 4.7. *Let* M *be an ordered semiring with unity* 1 *in which semigroup* $(M, +)$ *is is a band and positively ordered, multiplicative-semigroup* M *is negatively ordered and* d_a *be a derivation of* M*. If* $x \in M$ *then*
(i) $xd_a(1) \leqslant d_a(x)$
(ii) *If* $d_a(1) = 1$ *then* $x \leqslant d_a(x)$.

PROOF. (i) Let M be an ordered -semiring with unity 1, d_a be a derivation of M and $x \in M$. Then $x1 = x$. Thus

$$d_a(x) = d_a(x1) = d_a(x)1 + xd_a(1).$$

So, it follows $xd_a(1) \leqslant d_a(x)1 + xd_a(1) = d_a(x)$.

(ii) Suppose $d_a(1) = 1$ and $xd_a(1) \leqslant d_a(x)$. Then $x1 \leqslant d_a(x)$ and $x \leqslant d_a(x)$. This completes the proof. □

THEOREM 4.8. *Let* M *be an idempotent ordered semiring in which multiplicative semigroup* M *is negatively ordered and semigroup* $(M, +)$ *is a band. If* $d^2_{a,}(x) = d_a(d_a(x)) = d_a(x)$ *then* $d_a(xd_a(x)) \leqslant d_a(x)$*, for all* $x \in M$.

PROOF. Let M be an idempotent ordered semiring and $d^2_{a,}(x) = d_a(d_a(x)) = d_a(x)$, for all $x \in M$. Then

$$\begin{aligned} d_a(xd_a(x)) &= d_a(x)d_a(x) + xd(d_a(x)) = d_a(x) + xd_a(x) \\ &\leqslant d_a(x) + d_a(x) = d_a(x). \end{aligned}$$

Therefore $d_a(xd_a(x)) \leqslant d_a(x)$. This completes the proof. □

THEOREM 4.9. *Let d_a be a derivation of an ordered integral-semiring M with unity and $a \in M$. If $ad_a(x) = 0$ for all $x \in M$, then either $a = 0$ or $d_a = 0$.*

PROOF. Suppose $ad_a(x) = 0$ for all $x \in M$, Let $y \in M$. Replace x by xy, then $ad_a(xy) = 0$ and $a[d_a(x)y + xd_a(y)] = 0$. Thus $axd_a(y) = 0$ and $a1d_a(y) = 0$. Hence $ad_a(y) = 0$. Therefore $a = 0$ or $d_a(y) = 0$ since M has no zero divisors. □

DEFINITION 4.2. An ideal I of an ordered semiring M is said to be $m - k$ ideal if $xy \in I$, $x \in I$, $1 \neq y \in M$ then $y \in I$.

DEFINITION 4.3. Let d_a be a derivation of an ordered semiring M. Derivation d_a is called an isotone derivation if $x \leqslant y$ then $d_a(x) \leqslant d_a(y)$ for all $x, y \in M$.

THEOREM 4.10. *Let d_a be an isotone derivation of an ordered semiring M. Define $ker\ d_a = \{x \in M/d_a(x) = 0\}$. Then $ker\ d_a$ is a $k-$ideal of an ordered semiring M.*

PROOF. Let $x, y \in ker\ d_a$. Then $xa = ya = 0$. Thus $d_a(x + y) = (x + y)a = 0$. Therefore $x + y \in ker\ d_a$. Now, we have

$$d_a(xy) = d_a(x)y + xd_a(y) = (xa)y + x(ya) = 0y + x0 = 0.$$

Therefore $xy \in ker\ d_a$.

Suppose $y \in ker\ d_a, x \in M$ and $x \leqslant y$. Then $d_a(x) \leqslant d_a(y)$. It follows $xa \leqslant ya = 0$ and $xa = 0$.. So, $x \in ker\ d_a$. Hence $ker\ d_a$ is an ideal.

Suppose $x+y \in ker\ d_a$ and $y \in ker\ d_a$. Then $d_a(x+y) = 0$ and $d_a(x)+d_a(y) = 0$. Thus $d_a(x) = 0$. Hence $x \in ker\ d_a$. This completes the proof. □

THEOREM 4.11. *Let d_a be an isotone derivation of an integral ordered semiring M. Then $ker\ d_a$ is a $m - k$ ideal of M.*

PROOF. By Theorem 4.10, $ker\ d_a$ is an ideal of an ordered semiring M. Let $0 \neq y \in ker\ d_a$, $x \in M$ and $xy \in ker\ d_a$. Then $d_a(xy) = 0$ and $d_a(x)y+xd_a(y) = 0$. Thus $d_a(x)y = 0$. So, $d_a(x) = 0$, since M is an integral ordered semiring. Therefore $ker\ d_a$ is a $m - k$ ideal of M. □

THEOREM 4.12. *Let d_a be a derivation of multiplicative cancellative commutative ordered semiring M where $(M, +)$ is positively ordered and band, semigroup M is negatively ordered and $d_a(1) = 1$. Define a set $Fixd_a(M) = \{x \in M : d_a(x) = x\}$. Then $Fixd_a(M)$ is a $m - k$ ideal.*

PROOF. Obviously $Fixd_a(M) = \{x \in M : d_a(x) = x\}$ is an ideal of M. Suppose $xy \in Fixd_a(M)$, $x \in Fixd_a(M)$. Then $d_a(xy) = xy$. Thus

$$\begin{aligned}
&d_a(x)y + xd_a(y) = xy \\
&\Rightarrow xy + xd_a(y) = xy \\
&\Rightarrow x[y + d_a(y)] = xy \\
&\Rightarrow y + d_a(y) = y \\
&\Rightarrow d_a(y) \leqslant y + d_a(y) = y.
\end{aligned}$$

By Theorem 4.7, we have $y \leqslant d_a(y)$. Hence $d_a(y) = y$ and $y \in Fixd_a(M)$. So, $Fixd_a(M)$ is a $m-k-$ideal of M. □

THEOREM 4.13. *Let M be an ordered commutative semiring and d_a, d_b be derivations of M. If $d_a\ d_b = 0$ then $d_b\ d_a$ is a derivation of M.*

PROOF. Let $x, y \in M$. Then

$$\begin{aligned} 0 &= d_a\ d_b\ (xy) = d_a[\ d_b\ (x)y + x\ d_b\ (y)] \\ &= d_a(\ d_b\ (x))y + \ d_b\ (x)d_a(y) + d_a(x)\ d_b\ (y) + xd_a(\ d_b\ (y)) \end{aligned}$$

and

$$0 = \ d_b\ (x)d_a(y) + d_a(x)\ d_b\ (y).$$

Thus

$$d_b\ d_a(x+y) = \ d_b\ [d_a(x) + d_a(y)] = \ d_b\ (d_a(x)) + \ d_b\ (d_a(y)).$$

and

$$\begin{aligned} d_b\ d_a(xy) &= \ d_b\ [d_a(x)y + xd_a(y)] \\ &= \ d_b\ d_a(x)y + d_a(x)\ d_b\ (y) + \ d_b\ (x)d_a(y) + x\ d_b\ (d_a(y)) \\ &= \ d_b\ (d_a(x))y + x\ d_b\ (d_a(y)). \end{aligned}$$

Hence $d_b\ d_a$ is a derivation of M. □

THEOREM 4.14. *Let d_a be a derivation of an ordered integral semiringM with unity and $b \in M$. If $bd_a(x) = 0$, for all $x \in M$, then either $b = 0$ or d_a is zero.*

PROOF. Suppose $bd_a(x) = 0$ for all $x \in M$. Let $y \in M$. Replace x by xy we have $bd_a(xy) = 0$. Then $b[d_a(x)y + xd_a(y)] = 0$ and $bxd_a(y) = 0$. Since $b \in M$ and $b1 = b$ we have $bd_a(y) = 0$. Thus $b = 0$ or $d_a(y) = 0$. hence $b = 0$ or $d_a = 0$. □

THEOREM 4.15. *Let M be an ordered semiring in which $(M, +)$ is positively ordered and band, multiplicative semigroup M is negatively ordered and d_a be a derivation of M. Then the following hold:*
(i) $d_a(xy) \leqslant d_a(x)$
(ii) $d_a(xy) \leqslant d_a(y)$
(iii) $x \leqslant y$ *then* $d_a(x) \leqslant d_a(y)$ *for all* $x, y \in M$.

PROOF. (i) Let $x, y \in M$,. Then $d_a(xy) = (xy)a \leqslant xa = d_a(x)$.
(ii) Similarly we can prove $d_a(xy) \leqslant d_a(y)$.
(iii) Suppose $x \leqslant y$. Then it follows $x + y \leqslant y + y$ and $x + y \leqslant y \leqslant x + y$. Thus $x + y = y$. From here, it follows $d_a(x + y) = d_a(y)$ and $d_a(x) + d_a(y) = d_a(y)$. Therefore $d_a(x) \leqslant d_a(y)$. □

THEOREM 4.16. *Let d_a be a derivation of an integral ordered semiring M in which semigroup $(M, +)$ is a band. Define $d_a^2(x) = d_a(d_a(x))$, for all $x \in M$. If $d_a^2 = 0$, then $d_a = 0$.*

PROOF. Let $x, y \in M$ Then $d_a^2(xy) = 0$ and

$$\begin{aligned}
&d_a[d_a(xy)] = 0 \\
&\Rightarrow d_a[d_a(x)y + xd_a(y)] = 0 \\
&\Rightarrow d_a^2(x)y + d_a(x)d_a(y) + d_a(x)d_a(y) + xd_a^2(y) = 0 \\
&\Rightarrow d_a(x)d_a(y) + d_a(x)d_a(y) = 0 \\
&\Rightarrow d_a(x)d_a(y) = 0 \\
&\Rightarrow d_a(x) = 0 \ or \ d_a(y) = 0, \text{ for all } x, y \in M..
\end{aligned}$$

Therefore in both the cases we have $d_a = 0$. □

THEOREM 4.17. *Let I be a non-zero ideal of an integral ordered semiring M in which semigroup M is negatively ordered. If d_a is a non-zero derivation of M then d_a is non-zero on I.*

PROOF. Suppose d_a is a non-zero derivation of M and $d_a(x) = 0$ for all $x \in I$. Let $y \in M, \in \Gamma$ and $x \in I$. Then $xy \leqslant x$. Therefore $xy \in I$. From here it follows $d_a(xy) = 0$ and $d_a(x)y + xd_a(y) = 0$. So, $xd_a(y) = 0$. Since M has no zero divisors, we have $x = 0$ or $d_a(y) = 0$, for all $y \in M$. Since I is a non-zero ideal, we get $d_a(y) = 0$ for all $y \in M$ which is a contradiction to $d_a \neq 0$ on M. Hence d_a is non-zero derivation on I. □

THEOREM 4.18. *Let d_a be a non-zero derivation of an integral ordered semiring M. If I is a non-zero ideal of M and $t \in M$ such that $td_a(I) = 0$ then $t = 0$.*

PROOF. By Theorem 4.17, there exists $x \in I$ such that $d_a(x) \neq 0$. Suppose $td_a(I) = 0$. Then $td_a(xx) = 0$. Now, we have

$$\begin{aligned}
&t[d_a(x)x + xd_a(x)] = 0 \\
&\Rightarrow td_a(x)x + t(xd_a(x)) = 0 \\
&\Rightarrow t(xd_a(x)) = 0.
\end{aligned}$$

Therefore $t = 0$. □

5. Conclusion

In this paper, we introduced the concept of f-derivation and derivation d_a of an ordered semiring We characterized $m-k-$ideal using f and d_a -derivations of an ordered semiring M. We studied some of the properties of the f and d_a derivations of ordered semirings.

References

[1] P. J. Allen. A fundamental theorem of homomorphism for semirings. *Proc. Am. Math. Soc.*, **21**(2)(1969), 412–416.

[2] M. Bresar and J. Vukman. On the left derivation and related mappings. *Proc. Am. Math. Soc.*, **110**(1)(1990), 7–16.

[3] K. H. Kim. On right derivation of incline algebras. *Journal of The Chung Cheong Mathematical Society* **26**(4)(2013), 683–690.

[4] K. H. Kim. On genralized right derivation of incline algebras. *Gulf J. Math.*, **3**(1)(2015), 36–46.
[5] E. C. Posner. Derivations in prime rings. *Proc. Amer. Math. Soc.*, **8** (1957), 1093–1100.
[6] M. M. K. Rao and B. Venkateswarlu. On generalized right derivations of $\Gamma-$incline. *J. Int. Math. Virtual Inst.*, **6**(2016), 31–47.
[7] M. M. K. Rao and B. Venkateswarlu. Right derivation of ordered $\Gamma-$semiring. *Discuss. Math., Gen. Algebra Appl.*, **36**(2)(2016), 209–221.
[8] M. M. K. Rao. $\Gamma-$semirings - I. *Southeast Asian Bull. of Math.*, **19**(1)(1995), 49–54.
[9] M. K. Sen. On $\Gamma-$semigroup. *Proc. of Inter. Conf. of Alg. and its Appl.*, Decker Publicaiton, New York, (1981), 301–308.
[10] H. S. Vandiver. Note on a simple type of algebra in which cancellation law of addition does not hold. *Bull. Amer. Math. Soc.*, (N.S.), **40** (1934), 914–920.

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