JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
J. Int. Math. Virtual Inst., Vol. **10**(2)(2020), 215-234
DOI: 10.7251/JIMVI2002215M

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

A NOTE ON 0-ADJOINED SOFT SEMIGROUPS

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Abstract. In this paper, we construct a Galois connection between the complete lattice of (soft) substructures of a (soft) semigroup and the complete lattice of (soft) substructures of the 0 (1)-adjoined (soft) semigroup, which will be crucial in the Representation of Soft Substructures of a Soft Semigroup by their Crisp Cousins.

1. Introduction

The term semigroup first appeared in Mathematical literature (in French) by J. A. de Seguier in his book Elements de la Theorie des Groupes Abstraits (Elements of the Theory of Abstract Groups) in 1904 and the first paper about semigroups was a brief one by L. E. Dickson in 1905. But the theory really began from the paper Uber die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit (On finite groups without the rule of unique invertibility) written by A. Suschkewitsch in 1928. In current terminology, he showed that every finite semigroup contains a kernel (a simple ideal) and he completely determined the structure of finite simple semigroups. From that point on, the foundations of semigroup theory were further laid by D. Rees, J. A. Green, E. S. Lyapin, A. H. Clifford and G. Preston. The later two published a two-volume monographs on semigroup theory in 1961 and 1967.

From a historical point of view, it may be interesting to know that in 1956, the notion quasi-ideal of a semigroup was introduced by Steinfeld [**18**] as a generalization of the notions (left, right) ideal of a semigroup and interestingly, the notion of bi-ideal of a semigroup which further generalizes the notion of quasi-ideal of a semigroup was introduced by Good-Hughes [**10**] much earlier in 1952.

2010 *Mathematics Subject Classification.* 06B99, 20M10, 20M12, 06A15, 06D72.
Key words and phrases. Soft (sub) semigroup, Soft (left, right, quasi-, bi-) ideal, 0 (1)-adjoined (soft) semigroup.

On the other side, Molodtsov [**14**] introduced the notion of soft set as a mathematical tool for modelling uncertainties. Since its introduction, several mathematicians imposed various algebraic (sub) structures on them and studied some of their elementary properties. In 2010, Ali-Shabir-Shum [**2**] introduced the notions of soft semigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) over a semigroup and studied some of their properties.

In this paper, as in Grillet [**11**], we consider the empty set as a semigroup as there are *no* two elements in the empty set whose product is *not* in the empty set. Consequently, we modify the definitions of soft semigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) over a semigroup introduced by Ali-Shabir-Shum [**2**] by removing the conditions that the underlying soft set is *neither* empty *nor* null, so that by our definition, the empty soft set and the null soft set are trivially a soft (sub) semigroup, soft (left, right, quasi-, bi-) ideal. Next, we introduce the notions of 0 (1)-adjoined soft semigroups and studied some of their (algebraic) lattice theoretic properties. Further, we construct a Galois connection between the complete lattice of (soft) substructures of a (soft) semigroup and the complete lattice of (soft) substructures of the 0 (1)-adjoined (soft) semigroup.

In this paper, some proofs are left for three reasons, firstly in most cases they are simple or straight forward but a little involving, secondly we want to minimize the size of the document and lastly, in order to make the document more self contained, instead of proofs, we recall as many notions and results that are used in subsequent sections, as possible.

2. Preliminaries

In what follows we recall some basic definitions and elementary results in the theory of Lattices, Semigroups, Soft Sets and Soft Semigroups which are used in the main results:

We assume the following notions from Lattice Theory: (meet/join) complete poset, (meet/join) complete subposet, complete sublattice, (meet/join) complete homomorphism (isomorphism) of (meet/join) complete posets, one can refer to any standard text books on Lattice Theory for them. Observe that by a meet (join) complete poset we mean a poset in which every *non-empty* subset S has infimum (supremum), denoted by $\wedge S$ $(\vee S)$; by a complete poset or a complete lattice we mean a poset which is both a meet complete poset and a join complete poset; a subset of a meet (join) complete poset is a meet (join) complete subposet iff it is closed under infimum (supremum) for all its *non-empty* subsets; a subset of a complete lattice is a complete sublattice iff it is both a meet complete subposet and a join complete subposet; by a meet (join) complete homomorphism we mean any map between meet (join) complete posets which preserves infimums (supremums) for all *non-empty* subsets; by a complete homomorphism we mean any map between complete lattices which preserves infimums and supremums for all *non-empty* subsets and by a complete isomorphism we mean any complete homomorphism between complete lattices which is both one-one and onto.

LEMMA 2.1. *For any index set I, for any family of sets $(A_i)_{i\in I}$ and for any set B, the following are true:*

(1) $(\cup_{i\in I}A_i) \cap B = \cup_{i\in I}(A_i \cap B)$
(2) $(\cap_{i\in I}A_i) \cup B = \cap_{i\in I}(A_i \cup B)$
(3) $(\cup_{i\in I}A_i) \cup B = \cup_{i\in I}(A_i \cup B)$
(4) $(\cap_{i\in I}A_i) \cap B = \cap_{i\in I}(A_i \cap B)$.

LEMMA 2.2. *For any pair of sets A, B and for any function $f : A \to B$, the following are true:*

(1) $f = \phi$ *iff* $A = \phi$
(2) $B = \phi$ *implies* $f = \phi$ *but not conversely.*

DEFINITION 2.1. For any non-empty subset S of a meet (join) complete poset L with the largest (least) element 1_L (0_L), one can define
$\overline{\vee}S = \wedge\{\beta \in L/\alpha \wedge \beta = \alpha \text{ for all } \alpha \in S\}$ $(\overline{\wedge}S = \vee\{\beta \in L/\alpha \wedge \beta = \beta \text{ for all } \alpha \in S\})$
called the *meet (join) induced join (meet)* in L. Then L is a complete lattice with the $\overline{\vee}$ $(\overline{\wedge})$ called the *associated complete lattice* for the meet (join) complete poset L.

DEFINITION 2.2. A lattice L is said to be a *distributive lattice* iff for any $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Further, L is a *distributive complete lattice* iff it is both a distributive lattice and a complete lattice.

DEFINITION 2.3. A lattice L is said to be a *modular lattice* iff for any $x, y, z \in L$ such that $x \leqslant y$ implies $x \vee (y \wedge z) = y \wedge (x \vee z)$. Further, L is a *modular complete lattice* iff it is both a modular lattice and a complete lattice.

DEFINITION 2.4. For any pair of posets P, Q and for any pair of order preserving maps $f : P \to Q$, $g : Q \to P$, (g, f) is a Galois connection between P and Q iff for each $(a, b) \in P \times Q$, $fa \leqslant b$ iff $a \leqslant gb$ or equivalently, for any pair of posets P, Q and for any pair of maps $f : P \to Q$, $g : Q \to P$, (g, f) is a Galois connection iff $fg \leqslant I_Q$ and $I_P \leqslant gf$.

DEFINITION 2.5. A set S together with a binary operation which is associative is called a *semigroup.* Notice that the empty set is trivially a semigroup with the empty binary operation called the *empty semigroup.*

DEFINITION 2.6. For any pair of subsets A, B of a semigroup S, the set AB is defined by $AB = \{ab \in S/a \in A \text{ and } b \in B\}$ and it is a subset of S.

DEFINITION 2.7. For any subset A of a semigroup S,

(1) A is a *subsemigroup* of S iff $A^2 \subseteq A$.

Notice that as in Grillet [**11**], the empty semigroup is trivially a subsemigroup of any semigroup.

(2) A is a *left (right) ideal* of S iff $SA \subseteq A$ $(AS \subseteq A)$
(3) A is an *ideal* of S iff $SA \cup AS \subseteq A$ iff it is both a left and a right ideal of S
(4) A is a *quasi-ideal* of S iff $SA \cap AS \subseteq A$
(5) A is a *bi-ideal* of S iff $AA \subseteq A$ and $ASA \subseteq A$.

LEMMA 2.3. *In any semigroup S, the following are true:*

(1) *The empty semigroup is trivially a (left, right, quasi-, bi-) ideal of S*

(2) *Arbitrary union of (left, right) ideals of S is a (left, right) ideal of S but arbitrary union of subsemigroups (quasi-ideals, bi-ideals) of S need not be a subsemigroup (quasi-ideal, bi-ideal) of S*

(3) *Arbitrary intersection of subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of S is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S*

(4) *The intersection of all subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of S containing a given subset is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S which is unique and smallest with respect to the containment of the given subset*

(5) *For any subset A of a semigroup S, whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the unique smallest subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) containing the given subset A defined as in (4) above is called the subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) generated by A and is denoted by $(A)_{s,\,S}$ $((A)_{*,\,S})$*

(6) *For any pair of subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) A, B of a semigroup S, A is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of B iff A is a subset of B*

(7) *Whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, for any subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) B of S and for any subset A of B, $(A)_{s,\,S}$ $((A)_{*,\,S})$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of B.*

DEFINITION 2.8. For any semigroup S, the semigroup $S \cup \{0\}$ such that S is a subsemigroup of $S \cup \{0\}$, where $00 = 0s = s0 = 0$ for all $s \in S$, is called the 0-*adjoined semigroup* and is denoted by S_0.

DEFINITION 2.9. For any semigroup S and for any subset B of the 0-adjoined semigroup S_0, $B - \{0\}$ is called the *0-contraction* of B in S. Notice that the 0-contraction of S_0 is S and ϕ is ϕ itself.

DEFINITION 2.10. For any semigroup S, the semigroup $S \cup \{1\}$ such that S is a subsemigroup of $S \cup \{1\}$, where $1s = s1 = s$ for all $s \in S$ and $11 = 1$, is called the 1-*adjoined semigroup* and is denoted by S_1.

DEFINITION 2.11. For any semigroup S and for any subset B of the 1-adjoined semigroup S_1, $B - \{1\}$ is called the *1-contraction* of B in S. Notice that the 1-contraction of S_1 is S and ϕ is ϕ itself.

Notation: For any semigroup S, whenever $* = s\,(l, r, i, q, b)$, $\mathcal{S}_s(S)$ $(\mathcal{S}_*(S))$ is the set of all subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of S and for any pair of subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) A, B of S, $A \leqslant_s B$ $(A \leqslant_* B)$ iff A is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of B.

THEOREM 2.1. *For any semigroup S, whenever $* = s,\, q,\, b,\, l,\, r,\, i$, the set $\mathcal{S}_*(S)$ is a complete lattice with*

(1) *the partial ordering defined by: for any $A,\, B \in \mathcal{S}_*(S)$, $A \leqslant B$ iff $A \leqslant_* B$;*

(2) *the largest and the least elements in* $\mathcal{S}_*(S)$ *are* S *and* ϕ *respectively;*
(3) *for any family* $(A_i)_{i\in I}$ *in* $\mathcal{S}_*(S)$, $\wedge_{i\in I}A_i = \cap_{i\in I}A_i$;
(4) *for any family* $(A_i)_{i\in I}$ *in* $\mathcal{S}_*(S)$,
(i) *for* $* = s,\, q,\, b,\, \vee_{i\in I}A_i = \overline{\vee}_{i\in I}A_i$,
where $\overline{\vee}$ *is the meet induced join in* $\mathcal{S}_*(S)$. *In fact,* $\overline{\vee}_{i\in I}A_i = (\cup_{i\in I}A_i)_{*,\,S}$;
(ii) *for* $* = l,\, r,\, i,\, \vee_{i\in I}A_i = \cup_{i\in I}A_i$.

Theorem 2.2. *For any semigroup* S*, whenever* $* = l,\, r,\, i$*, the set* $\mathcal{S}_*(S)$ *of all (left, right) ideals of* S *is a distributive complete lattice and so a modular complete lattice.*

The following Example shows that, whenever $* = s,\, q,\, b$, the set $\mathcal{S}_*(S)$ is *not* necessarily a modular lattice and hence *not* necessarily a distributive lattice.

Example 2.1. (1) Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot_S$	a	b	c
a	a	b	c
b	c	b	c
c	c	b	c

Then $\mathcal{S}_s(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, S\}$. Let $A = \{a\}$, $B = \{a, c\}$ and $C = \{b\}$. Clearly, $A \leqslant B$. Now $A \vee (B \wedge C) = (A \cup (B \cap C))_{s,S} = \{a\} \subset \{a, c\} = B \cap (A \cup C)_{s,\,S} = B \wedge (A \vee C)$. Therefore $\mathcal{S}_s(S)$ is *not* a modular lattice.

(2) Let $S = \{a, b, c, d, e\}$ be a semigroup with the following Cayley table:

$\cdot_S$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

Then $\mathcal{S}_q(S) = \{\phi,\ \{a\},\ \{a, b\},\ \{a, c\}, \{a, d\}, \{a, e\},\ \{a, b, c\},\ \{a, d, e\},\ \{a, b, d\}, \{a, c, e\},\ S\}$. Let $A = \{a, c\}$, $B = \{a, b, c\}$ and $C = \{a, d\}$. Clearly, $A \leqslant B$. Now $A \vee (B \wedge C) = (A \cup (B \cap C))_{q,S} = \{a, c\} \subset \{a, b, c\} = B \cap (A \cup C)_{q,\,S} = B \wedge (A \vee C)$. Therefore $\mathcal{S}_q(S)$ is *not* a modular lattice.

(3) Let S be the semigroup same as in (2) above. Then $\mathcal{S}_b(S) = \{\phi, \{a\}, \{a, b\}, \{a, c\},\ \{a, d\}, \{a, e\},\ \{a, b, c\}, \{a, d, e\}, \{a, b, d\}, \{a, c, e\}, S\}$. Let $A = \{a, c\}$, $B = \{a, b, c\}$ and $C = \{a, d\}$. Clearly, $A \leqslant B$. Now $A \vee (B \wedge C) = (A \cup (B \cap C))_{b,\,S} = \{a, c\} \subset \{a, b, c\} = B \cap (A \cup C)_{b,\,S} = B \wedge (A \vee C)$. Therefore $\mathcal{S}_b(S)$ is *not* a modular lattice.

Lemma 2.4. *In any semigroup* S*, the following are true:*
(1) *If* A *is a subsemigroup of* S *then* A *is also a subsemigroup of* S_0 *Further,* $\mathcal{S}_s(S)$ *is always a proper subset of* $\mathcal{S}_s(S_0)$

(2) *If A is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S then A_0 is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S_0*

(3) *If $\phi \neq B$ is a (left, right, quasi-, bi-) ideal of S_0 then $0 \in B$*

(4) *If B is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S_0 then $B - \{0\}$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S.*

Corollary 2.1. *For any subset A of a semigroup S, whenever $* = s, l, r, i, q, b$, $A \leqslant_* S$ iff $A \cup \{0\} \leqslant_* S_0$.*

Lemma 2.5. *In any semigroup S, the following are true:*

(1) *If A is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S then A is also a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S_1. In particular, S is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S_1. Further, whenever $* = s, l, r, i, q, b$, $\mathcal{S}_*(S)$ is always a proper subset of $\mathcal{S}_*(S_1)$*

(2) *If A is a subsemigroup of S then A_1 is a subsemigroup of S_1*

(3) *A_1 is a (left, right, quasi-, bi-) ideal of S_1 iff $A_1 = S_1$*

(4) *If B is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S_1 then $B - \{1\}$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S.*

Remark 2.1. Observe that, in view of the Lemma 2.4(2), while 0-adjoining is structure preserving, in view of the Lemma 2.5(1), 1-adjoining behaves like transitivity.

The following Example shows that if A is a (left, right, quasi-, bi-) ideal of S then A_1 need *not* be a (left, right, quasi-, bi-) ideal of S_1.

Example 2.2. Let $S = \{a, b\}$ be a semigroup with the following Cayley table. Then $S_1 = \{a, b, 1\}$ is also a semigroup with the following Cayley table:

$\cdot_S$	a	b
a	a	a
b	a	b

$\cdot_{S_1}$	a	b	1
a	a	a	a
b	a	b	b
1	a	b	1

Let $A = \{a\}$ be a (left, right, quasi-, bi-) ideal of S. Then $A_1 = A \cup \{1\} = \{a, 1\}$ is *not* a (left, right, quasi-, bi-) ideal of S_1.

Corollary 2.2. *In any semigroup S and for any subset A of S, the following are true:*

(1) *A is a subsemigroup of S iff $A \cup \{1\}$ is a subsemigroup of S_1*

(2) *Whenever $* = s, l, r, i, q, b$, $A \leqslant_* S$ iff $A \leqslant_* S_1$.*

Lemma 2.6. *For any semigroup S, for any subset A of S and for any subset B of S_0 (S_1) such that $A \subseteq B$, $(A)_{*,S} \subseteq (B)_{*,S_0}$ for $* = s, q, b$ and $(A)_{s,S} \subseteq (B)_{s,S_1}$.*

LEMMA 2.7. *For any semigroup S, for any subset A of S and for any subset B of S_0 (S_1), the following are true:*
(1) $(A)_{s,S} = (A)_{s,S_0}$ *and* $(A)_{s,S} \cup \{0\} = (A \cup \{0\})_{s,S_0}$
(2) $(A)_{*,S} \cup \{0\} = (A \cup \{0\})_{*,S_0}$ *for* $* = q, b$
(3) $(B)_{*,S_0} - \{0\} = (B - \{0\})_{*,S}$ *for* $* = s, q, b$
(4) $(A)_{s,S} = (A)_{s,S_1}$ *and* $(A)_{s,S} \cup \{1\} = (A \cup \{1\})_{s,S_1}$
(5) $(B)_{s,S_1} - \{1\} = (B - \{1\})_{s,S}$
(6) $(A)_{*,S} = (A)_{*,S_1}$ *for* $* = q, b$.

COROLLARY 2.3. *In any semigroup S, the following are true:*
(1) *For any family of subsemigroups (quasi-ideals, bi-ideals) $(A_i)_{i\in I}$ of S, for $* = s\,(q, b)$, $(\cup_{i\in I} A_i)_{*,S} \cup \{0\} = ((\cup_{i\in I} A_i) \cup \{0\})_{*,S_0}$*
(2) *For any family of subsemigroups (quasi-ideals, bi-ideals) $(B_i)_{i\in I}$ of S_0, for $* = s\,(q, b)$, $(\cup_{i\in I} B_i)_{*,S_0} - \{0\} = ((\cup_{i\in I} B_i) - \{0\})_{*,S}$*
(3) *For any family of subsemigroups $(A_i)_{i\in I}$ of S, $(\cup_{i\in I} A_i)_{s,S} \cup \{1\} = ((\cup_{i\in I} A_i) \cup \{1\})_{s,S_1}$*
(4) *For any family of subsemigroups $(B_i)_{i\in I}$ of S_1, $(\cup_{i\in I} B_i)_{s,S_1} - \{1\} = ((\cup_{i\in I} B_i) - \{1\})_{s,S}$.*

COROLLARY 2.4. *For any semigroup S, the following are true:*
(1) *The complete lattice $\mathcal{S}_s(S)$ of all subsemigroups of S is a complete sublattice of the complete lattice $\mathcal{S}_s(S_0)$ of all subsemigroups of S_0*
(2) *Whenever $* = s\,(q, b, l, r, i)$, the complete lattice $\mathcal{S}_s(S)$ ($\mathcal{S}_*(S)$) of all subsemigroups (quasi-ideals, bi-ideals, left ideals, right ideals, ideals) of S is a complete sublattice of the complete lattice $\mathcal{S}_s(S_1)$ ($\mathcal{S}_*(S_1)$) of all subsemigroups (quasi-ideals, bi-ideals, left ideals, right ideals, ideals) of S_1.*

In what follows we construct a Galois connection, which follows from the Theorem 2.3 (2.4) (5) and (6) below, between the complete lattice of all substructures of a given type for a semigroup and the complete lattice of all substructures of the same type for the 0 (1)-adjoined semigroup.

THEOREM 2.3. *For any semigroup S, whenever $* = s, q, b, l, r, i$, the maps $\varepsilon_* : \mathcal{S}_*(S) \to \mathcal{S}_*(S_0)$ defined by for any $A \in \mathcal{S}_*(S)$, $\varepsilon_* A = A \cup \{0\}$ and $\delta_* : \mathcal{S}_*(S_0) \to \mathcal{S}_*(S)$ defined by for any $C \in \mathcal{S}_*(S_0)$, $\delta_* C = C - \{0\}$, satisfy the following properties:*
(1) *The map ε_* is one-one;*
(2) *The map δ_* is onto;*
(3) *For any $A, B \in \mathcal{S}_*(S)$, $A \leqslant B$ implies $\varepsilon_* A \leqslant \varepsilon_* B$;*
(4) *For any $C, D \in \mathcal{S}_*(S_0)$, $C \leqslant D$ implies $\delta_* C \leqslant \delta_* D$;*
(5) *$\varepsilon_* \circ \delta_* \supseteq 1_{\mathcal{S}_*(S_0)}$, where $1_{\mathcal{S}_*(S_0)}$ is the identity map on $\mathcal{S}_*(S_0)$;*
(6) *$\delta_* \circ \varepsilon_* = 1_{\mathcal{S}_*(S)}$, where $1_{\mathcal{S}_*(S)}$ is the identity map on $\mathcal{S}_*(S)$.*
For any family $(A_i)_{i\in I}$ in $\mathcal{S}_*(S)$,
(7) *$\varepsilon_*(\cap_{i\in I} A_i) = \cap_{i\in I} \varepsilon_* A_i$;*
(8) (i) *for $* = s, q, b$, $\varepsilon_*(\triangledown_{i\in I} A_i) = \triangledown_{i\in I} \varepsilon_* A_i$;*
(ii) *for $* = l, r, i$, $\varepsilon_*(\cup_{i\in I} A_i) = \cup_{i\in I} \varepsilon_* A_i$.*
For any family $(C_i)_{i\in I}$ in $\mathcal{S}_*(S_0)$,
(9) *$\delta_*(\cap_{i\in I} C_i) = \cap_{i\in I} \delta_* C_i$;*

(10) (i) *for* $* = s, q, b$, $\delta_*(\bigtriangledown_{i\in I} C_i) = \bigtriangledown_{i\in I}\delta_* C_i$*;*

(ii) *for* $* = l, r, i$, $\delta_*(\cup_{i\in I} C_i) = \cup_{i\in I}\delta_* C_i$.

(11) *The map* ε_* *is a complete monomorphism;*

(12) *The map* δ_* *is a complete epimorphism.*

Proof. (1): It is straightforward.

(2): It follows from the definition of δ_* and the Lemma 2.4(2).

(3): It follows from the definition of ε_* and the Lemmas 2.4(2), 2.3(6).

(4): It follows from the definition of δ_* and the Lemmas 2.4(4) and 2.3(6).

(5): (a) For $* = s$

Case I: Let $C \in \mathcal{S}_s(S) \subseteq \mathcal{S}_s(S_0)$. Then

$$(\varepsilon_s \circ \delta_s)(C) = \varepsilon_s(\delta_s(C)) = \varepsilon_s(C) = C \cup \{0\} \supset C.$$

Case II: Let $D \in \mathcal{S}_s(S_0) - \mathcal{S}_s(S)$. Then

$$(\varepsilon_s \circ \delta_s)(D) = \varepsilon_s(\delta_s(D)) = \varepsilon_s(D - \{0\}) = (D - \{0\}) \cup \{0\} = D.$$

From the above two Cases, we have

$$(\varepsilon_s \circ \delta_s)(X) \supseteq X \text{ for all } X \in \mathcal{S}_s(S_0) \text{ or } \varepsilon_s \circ \delta_s \supseteq 1_{\mathcal{S}_s(S_0)},$$

where $1_{\mathcal{S}_s(S_0)}$ is the identity map on $\mathcal{S}_s(S_0)$.

(b) For $* = q$

Case I: Let $\phi \in \mathcal{S}_q(S_0)$. Then

$$(\varepsilon_q \circ \delta_q)(\phi) = \varepsilon_q(\delta_q(\phi)) = \varepsilon_q(\phi) = \phi \cup \{0\} = \{0\} \supset \phi.$$

Case II: Let $\phi \neq C \in \mathcal{S}_q(S_0)$. Then

$$(\varepsilon_q \circ \delta_q)(C) = \varepsilon_q(\delta_q(C)) = \varepsilon_q(C - \{0\}) = (C - \{0\}) \cup \{0\} = C.$$

From the above two Cases, we have

$$(\varepsilon_q \circ \delta_q)(X) \supseteq X \text{ for all } X \in \mathcal{S}_q(S_0) \text{ or } \varepsilon_q \circ \delta_q \supseteq 1_{\mathcal{S}_q(S_0)},$$

where $1_{\mathcal{S}_q(S_0)}$ is the identity map on $\mathcal{S}_q(S_0)$.

For $* = b, l, r, i$, the proofs follow in a similar way as in (b) above.

(6): It follows from the definitions of ε_* and δ_*.

(7): It follows from the definition of ε_* and the Lemma 2.1(2).

(8): It follows from the definition of ε_*, Lemma 2.1(3) and the Corollary 2.3(1).

(9): It follows from the definition of δ_* and the Lemma 2.1(4).

(10): It follows from the definition of δ_*, Lemma 2.1(1) and the Corollary 2.3(2).

(11): It follows from (1), (7) and (8) above.

(12): It follows from (2), (9) and (10) above. □

The following Example shows that (1) ε_* is *not* onto and (2) δ_* is *not* one-one in the above Theorem for $* = s, q, b, l, r, i$.

EXAMPLE 2.3. Let $S = \{a, b\}$ be the semigroup with the following Cayley table. Then $S_0 = \{a, b, 0\}$ is the semigroup with the following Cayley table:

$\cdot_S$	a	b
a	a	a
b	a	b

$\cdot_{S_0}$	a	b	0
a	a	a	0
b	a	b	0
0	0	0	0

(1) For $* = s, q, b, l, r, i$, let $B = \phi \in \mathcal{S}_*(S_0)$. Then there is *no* $A \in \mathcal{S}_s(S)$ such that $\varepsilon_* A = A \cup \{0\} = B$ or ε_* is *not* onto.

(2) For $* = s, q, b, l, r, i$, let $B = \phi$ and $D = \{0\} \in \mathcal{S}_*(S_0)$. Then $\delta_* B = \phi = \delta_* D$ but $B \neq D$ or δ_* is *not* one-one.

THEOREM 2.4. *For any semigroup S, the maps $\varepsilon : \mathcal{S}_s(S) \to \mathcal{S}_s(S_1)$ defined by for any $A \in \mathcal{S}_s(S)$, $\varepsilon A = A \cup \{1\}$ and $\delta : \mathcal{S}_s(S_1) \to \mathcal{S}_s(S)$ defined by for any $C \in \mathcal{S}_s(S_1)$, $\delta C = C - \{1\}$, satisfy the following properties:*
(1) *The map ε is one-one;*
(2) *The map δ is onto;*
(3) *For any $A, B \in \mathcal{S}_s(S)$, $A \leqslant B$ implies $\varepsilon A \leqslant \varepsilon B$;*
(4) *For any $C, D \in \mathcal{S}_s(S_1)$, $C \leqslant D$ implies $\delta C \leqslant \delta D$;*
(5) $\varepsilon \circ \delta \supseteq 1_{\mathcal{S}_s(S_1)}$, *where $1_{\mathcal{S}_s(S_1)}$ is the identity map on $\mathcal{S}_s(S_1)$;*
(6) $\delta \circ \varepsilon = 1_{\mathcal{S}_s(S)}$, *where $1_{\mathcal{S}_s(S)}$ is the identity map on $\mathcal{S}_s(S)$.*
For any family $(A_i)_{i \in I}$ in $\mathcal{S}_s(S)$,
(7) $\varepsilon(\cap_{i \in I} A_i) = \cap_{i \in I} \varepsilon A_i$;
(8) $\varepsilon(\triangledown_{i \in I} A_i) = \triangledown_{i \in I} \varepsilon A_i$.
For any family $(C_i)_{i \in I}$ in $\mathcal{S}_s(S_1)$,
(9) $\delta(\cap_{i \in I} C_i) = \cap_{i \in I} \delta C_i$;
(10) $\delta(\triangledown_{i \in I} C_i) = \triangledown_{i \in I} \delta C_i$.
(11) *The map ε is a complete monomorphism;*
(12) *The map δ is a complete epimorphism.*

PROOF. (1): It is straightforward.
(2): It follows from the definition of δ and the Lemma 2.5(2).
(3): It follows from the definition of ε and the Lemmas 2.5(2), 2.3(6).
(4): It follows from the definition of δ and the Lemmas 2.5(4), 2.3(6).
(5): It follows from the definitions of δ, ε and the Lemma 2.5(1).
(6): It follows from the definitions of ε and δ.
(7): It follows from the definition of ε and the Lemma 2.1(2).
(8): It follows from the definition of ε, Lemma 2.1(3) and the Corollary 2.3(3).
(9): It follows from the definition of δ and the Lemma 2.1(4).
(10): It follows from the definition of δ, Lemma 2.1(1) and the Corollary 2.3(4).
(11): It follows from (1), (7) and (8) above.
(12): It follows from (2), (9) and (10) above. □

The following Example shows that in the above Theorem (1) ε is *not* onto and (2) δ is *not* one-one.

EXAMPLE 2.4. Let S and S_1 be the semigroups same as in the Example 2.2.

(1) Let $B = \{a\} \in \mathcal{S}_s(S_1)$. Then there is *no* $A \in \mathcal{S}_s(S)$ such that $\varepsilon A = A \cup \{1\} = B$ or ε is *not* onto.

(2) Let $B = \{a\}$ and $D = \{a, 1\} \in \mathcal{S}_s(S_1)$. Then $\delta B = \{a\} = \delta D$ but $B \neq D$ or δ is *not* one-one.

REMARK 2.2. Observe that from the Example 2.2, it is clear that adjoining of 1 does *not* preserve (left, right, quasi-, bi-) ideals. Consequently, the above Theorem has *no* analogues in the case of (left, right, quasi-, bi-) ideals.

DEFINITION 2.12. ([**14**]) Let U be a universal set, $P(U)$ be the power set of U and E be a set of parameters. A pair (F, E) is called a *soft set* over U iff $F : E \to P(U)$ is a mapping defined by for each $e \in E$, $F(e)$ is a subset of U.

Notice that a collective presentation of all the notions algebras, soft sets, fuzzy soft sets, f-soft algebras, f-fuzzy soft algebras in the single paper, Murthy-Maheswari [**15**] raised some serious notational conflicts and to fix the same we deviated from the above notation for a soft set and adapted the following notation for convenience as follows:

Let U be a universal set. A typical *soft set* over U is an ordered pair $\mathsf{E} = (\sigma_E, E)$, where E is a set of *parameters*, called the *underlying parameter set* for E, $P(U)$ is the power set of U and $\sigma_E : E \to P(U)$ is a map, called the *underlying set valued map* for E.

DEFINITION 2.13. ([**4**]) The *empty soft set* over U is a soft set with the empty parameter set, denoted by $\Phi = (\sigma_\phi, \phi)$. Clearly, it is unique.

DEFINITION 2.14. ([**4**]) A soft set E over U is said to be a *null soft set* iff $\sigma_E e = \phi$ for all $e \in E$.

DEFINITION 2.15. ([**17**]) For any pair of soft sets A, B over U, A is a *soft subset* of B, denoted by $\mathsf{A} \subseteq \mathsf{B}$, iff (i) $A \subseteq B$ (ii) $\sigma_A a \subseteq \sigma_B a$ for all $a \in A$.

DEFINITION 2.16. For any family of soft subsets $(\mathsf{A}_i)_{i \in I}$ of E,

(1) ([**8**]) the *soft union* of $(\mathsf{A}_i)_{i \in I}$, denoted by $\cup_{i \in I} \mathsf{A}_i$, is defined by the soft set A, where

(i) $A = \cup_{i \in I} A_i$

(ii) $\sigma_A a = \cup_{i \in I_a} \sigma_{A_i} a$ for all $a \in A$, where $I_a = \{i \in I / a \in A_i\}$

(2) the *soft intersection* of $(\mathsf{A}_i)_{i \in I}$, denoted by $\cap_{i \in I} \mathsf{A}_i$, is defined by the soft set A, where

(i) $A = \cap_{i \in I} A_i$

(ii) $\sigma_A a = \cap_{i \in I} \sigma_{A_i} a$ for all $a \in A$.

DEFINITION 2.17. ([**2**]) A soft set (F, A) over a semigroup S which is *neither* empty *nor* null is said to be a soft semigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) over S iff $F(a)$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S for all $a \in A$ whenever $F(a) \neq \phi$.

Notice that the definitions of soft semigroup, soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) used in this paper are different from the above ones in two ways. Firstly, the substructure notions defined above are over/of a *crisp* semigroup and the substructure notions defined below are a slight generalizations of the above, namely, those of a soft semigroup and secondly, as empty set is trivially a (sub) semigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) for us, in our definitions below, we do *not* need the two pre-conditions that a soft set (F, A) be neither empty nor null, as in the above definitions.

3. Soft Substructures of a Soft Semigroup

In what follows we introduce the notions of soft (sub) semigroup, soft (left, right, quasi-, bi-) ideal of a soft semigroup and make a (algebraic) lattice theoretic study of (sub) collections of them. Notice that throughout this section U is a semigroup unless otherwise explicitly stated.

Definition 3.1. A soft set E over a semigroup U is said to be a *soft semigroup* over U iff $\sigma_E e$ is a subsemigroup of U for all $e \in E$. Consequently, for us the empty soft set Φ and the null soft set Φ_E over U are trivially soft semigroups over U.

Definition 3.2. For any soft subset A of a soft semigroup E over U, A is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E iff $\sigma_A a$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of $\sigma_E a$ for all $a \in A$. Notice that the empty soft subset Φ and a null soft subset Φ_A of E are trivially soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of E.

Lemma 3.1. *For any soft semigroup E over U, the following are true:*

(1) *For any pair of soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) A, B of E, A is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of B iff A is a soft subset of B*

(2) *Arbitrary union of soft (left, right) ideals of E is always a soft (left, right) ideal of E but arbitrary union of soft subsemigroups (quasi-ideals, bi-ideals) of E need not be a soft subsemigroup (quasi-ideal, bi-ideal) of E*

(3) *Arbitrary intersection of soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of E is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E*

(4) *The intersection of all soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) containing a given soft subset is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) which is unique and smallest with respect to the containment of the given soft subset.*

Proof. (1) follows from the Definition 3.2 and the Lemma 2.3(6).

(2) follows from the definition 2.16(1) and the Lemma 2.3(2).

(3) and (4) follows from the definition 2.16(2) and the Lemma 2.3 (3) and (4). □

Definition 3.3. For any soft subset A of a soft semigroup E over U, whenever $* = s\,(l,\ r,\ i,\ q,\ b)$, the unique smallest soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E containing A defined as in the Lemma 3.1(4) is called the *soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) generated by* A and is denoted by $(\mathsf{A})_{s,\,\mathsf{E}}$ $((\mathsf{A})_{*,\,\mathsf{E}})$.

LEMMA 3.2. *For any soft subset A of a soft semigroup E over U, whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) generated by A, $(A)_{s,\,E}$ $((A)_{*,\,E})$, is given by C, where $C = A$ and $\sigma_C e = (\sigma_A e)_{s,\,\sigma_E e}$ $((\sigma_A e)_{*,\,\sigma_E e})$ for all $e \in C$.*

PROOF. It is straightforward. □

Notation: For any soft semigroup E over U, whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, $\mathcal{S}_s(\mathsf{E})$ $(\mathcal{S}_*(\mathsf{E}))$ is the set of all soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of E and for any pair of soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) A, B of E, $\mathsf{A} \leqslant_s \mathsf{B}$ $(\mathsf{A} \leqslant_* \mathsf{B})$ iff A is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of B.

THEOREM 3.1. *For any soft semigroup E over U, whenever $* = s,\, q,\, b,\, l,\, r,\, i$, the set $\mathcal{S}_*(E)$ is a complete lattice with*

(1) *the partial ordering defined by: for any A, $B \in \mathcal{S}_*(E)$, $A \leqslant B$ iff $A \leqslant_* B$;*
(2) *the largest and the least elements in $\mathcal{S}_*(E)$ are E and Φ respectively;*
(3) *for any family $(A_i)_{i\in I}$ in $\mathcal{S}_*(E)$, $\wedge_{i\in I} A_i = \cap_{i\in I} A_i$;*
(4) *for any family $(A_i)_{i\in I}$ in $\mathcal{S}_*(E)$, however,*
 (i) *for $* = s,\, q,\, b$, $\vee_{i\in I} A_i = \overline{\vee}_{i\in I} A_i$, where $\overline{\vee}$ is the meet induced join in $\mathcal{S}_*(E)$ and $\overline{\vee}_{i\in I} A_i = A$, where $A = \cup_{i\in I} A_i$ and $\sigma_A e = (\cup_{i\in I_e} \sigma_{A_i} e)_{*,\,\sigma_E e}$ for all $e \in A$, where $I_e = \{i \in I / e \in A_i\}$, and*
 (ii) *for $* = l,\, r,\, i$, $\vee_{i\in I} A_i = \cup_{i\in I} A_i$.*

PROOF. It is straightforward. □

THEOREM 3.2. *For any soft semigroup E over U, whenever $* = (l,\, r)\, i$, the set $(\mathcal{S}_*(E))$ $\mathcal{S}_i(E)$ of all soft (left, right) ideals of E is a distributive complete lattice and so a modular complete lattice.*

PROOF. It follows from the Definitions 2.16 and 2.2 and the Theorem 3.1. □

The Examples to show that, whenever $* = s\,(q,\, b)$, the set $\mathcal{S}_s(\mathsf{E})$ $(\mathcal{S}_*(\mathsf{E}))$ of all soft subsemigroups (quasi-ideals, bi-ideals) of E is *neither* a distributive *nor* a modular lattice, follow from that of Example 2.1.

4. $0\,(1)$-Adjoined Soft Semigroups

In this section, we introduce the notions of $0\,(1)$-Adjoined Soft Semigroups and studied some of their properties. Further, we construct a Galois connection between the complete lattice of soft substructures of a soft semigroup and the complete lattice of soft substructures of the 0 (1)-adjoined soft semigroup.

DEFINITION 4.1. For any soft semigroup E over U, the soft semigroup $\mathsf{E}_0 = (\sigma_{E_0}, E_0)$ over U_0 (cf.Definition 2.8), where $E_0 = E$ and $\sigma_{E_0} e = \sigma_E e \cup \{0\}$ for all $e \in E_0$, such that E is a soft subsemigroup of E_0 is called the 0-*adjoined soft semigroup.*

Notice that
(1) $\Phi_0 = \Phi$
(2) if $\mathsf{E} = \Phi_E$ then $\mathsf{E}_0 = \mathsf{F}$, where $F = E$ and $\sigma_F e = \{0\}$ for all $e \in F$
(3) if E is the whole soft semigroup over U then E_0 is the whole soft semigroup over U_0.

Definition 4.2. For any soft semigroup E over U and for any soft subset B of the 0-adjoined soft semigroup E_0, the soft subset $\mathsf{B} - \{0\} = \mathsf{C}$ of E, where $C = B$ and $\sigma_C e = \sigma_B e - \{0\}$ for all $e \in C$, is called the *0-contraction* of B in E.
Notice that
(1) the 0-contraction of Φ is Φ itself and
(2) the 0-contraction of E_0 is E.

Definition 4.3. For any soft semigroup E over U, the soft semigroup $\mathsf{E}_1 = (\sigma_{E_1}, E_1)$ over U_1 (cf.Definition 2.10), where $E_1 = E$ and $\sigma_{E_1} e = \sigma_E e \cup \{1\}$ for all $e \in E_1$, such that E is a soft subsemigroup of E_1 is called the 1-*adjoined soft semigroup.*
Notice that
(1) $\Phi_1 = \Phi$
(2) if $\mathsf{E} = \Phi_E$ then $\mathsf{E}_1 = \mathsf{F}$, where $F = E$ and $\sigma_F e = \{1\}$ for all $e \in F$
(3) if E is the whole soft semigroup over U then E_1 is the whole soft semigroup over U_1.

Definition 4.4. For any soft semigroup E over U and for any soft subset B of the 1-adjoined soft semigroup E_1, the soft subset $\mathsf{B} - \{1\} = \mathsf{C}$, where $C = B$ and $\sigma_C e = \sigma_B e - \{1\}$ for all $e \in C$, is called the *1-contraction* of B in E.
Notice that
(1) the 1-contraction of Φ is Φ itself and
(2) the 1-contraction of E_1 is E.

Lemma 4.1. *For any soft semigroup E over U, the following are true:*
(1) *If A is a soft subsemigroup of E then A is also a soft subsemigroup of E_0. Further, $\mathcal{S}_s(\mathsf{E})$ is a proper subset of $\mathcal{S}_s(\mathsf{E}_0)$*
(2) *If A is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E then A_0 is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E_0*
(3) *If B is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E_0 then $\mathsf{B} - \{0\} = \mathsf{C}$ is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E. Further, for any $\mathsf{B} \in \mathcal{S}_s(\mathsf{E}) \subseteq \mathcal{S}_s(\mathsf{E}_0)$, $\mathsf{B} - \{0\} = \mathsf{B}$.*

Proof. It follows from the Lemma 2.4. □

Lemma 4.2. *For any soft semigroup E over U, the following are true:*
(1) *If A is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E then A is also a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E_1. In particular, E is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E_1. Further, whenever $* = s, l, r, i, q, b$, $\mathcal{S}_*(\mathsf{E})$ is a proper subset of $\mathcal{S}_*(\mathsf{E}_1)$*

(2) *If A is a soft subsemigroup of E then A_1 is a soft subsemigroup of E_1*

(3) *If B is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E_1 then $B - \{1\} = C$ is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E. Further, whenever $* = s, l, r, i, q, b$, for any $B \in \mathcal{S}_*(E) \subseteq \mathcal{S}_*(E_1)$, $B - \{1\} = B$.*

Proof. It follows from the Lemma 2.5. □

Remark 4.1. Observe that, in view of the Lemma 4.1(2), while 0-adjoining is structure preserving, in view of the Lemma 4.2(1), 1-adjoining behaves like transitivity.

The Example to show that if A is a soft (left, right, quasi-, bi-) ideal of E then A_1 need *not* be a soft (left, right, quasi-, bi-) ideal of E_1, follow from that of Example 2.2.

Corollary 4.1. *For any soft semigroup E over U, the following are true:*

(1) *The complete lattice $\mathcal{S}_s(E)$ of all soft subsemigroups of E is a complete sublattice of the complete lattice $\mathcal{S}_s(E_0)$ of all soft subsemigroups of E_0*

(2) *Whenever $* = s\,(q, b, l, r, i)$, the complete lattice $\mathcal{S}_s(E)$ $(\mathcal{S}_*(E))$ of all soft subsemigroups (quasi-ideals, bi-ideals, left ideals, right ideals, ideals) of E is a complete sublattice of the complete lattice $\mathcal{S}_s(E_1)$ $(\mathcal{S}_*(E_1))$ of all soft subsemigroups (quasi-ideals, bi-ideals, left ideals, right ideals, ideals) of E_1.*

Proof. It is straightforward. □

In what follows we construct a Galois connection, which follows from the Theorem 4.1 (4.2) (5) and (6) below, between the complete lattice of all soft substructures of a given type for a soft semigroup and the complete lattice of all soft substructures of the same type for the 0 (1)-adjoined soft semigroup.

Theorem 4.1. *For any soft semigroup E over U, whenever $* = s, q, b, l, r, i$, the maps $\varepsilon_* : \mathcal{S}_*(E) \to \mathcal{S}_*(E_0)$ defined by for any $A \in \mathcal{S}_*(E)$, $\varepsilon_* A = A_0$, where $A_0 = A$ and $\sigma_{A_0} e = \sigma_A e \cup \{0\}$ for all $e \in A_0$, and $\delta_* : \mathcal{S}_*(E_0) \to \mathcal{S}_*(E)$ defined by for any $C \in \mathcal{S}_*(E_0)$, $\delta_* C = C - \{0\} = F$, where $F = C$ and $\sigma_F e = \sigma_C e - \{0\}$ for all $e \in F$, satisfy the following properties:*

(1) *The map ε_* is one-one;*

(2) *The map δ_* is onto;*

(3) *For any $A, B \in \mathcal{S}_*(E)$, $A \leqslant B$ implies $\varepsilon_* A \leqslant \varepsilon_* B$;*

(4) *For any $C, D \in \mathcal{S}_*(E_0)$, $C \leqslant D$ implies $\delta_* C \leqslant \delta_* D$;*

(5) *$\varepsilon_* \circ \delta_* \supseteq 1_{\mathcal{S}_*(E_0)}$, where $1_{\mathcal{S}_*(E_0)}$ is the identity map on $\mathcal{S}_*(E_0)$;*

(6) *$\delta_* \circ \varepsilon_* = 1_{\mathcal{S}_*(E)}$, where $1_{\mathcal{S}_*(E)}$ is the identity map on $\mathcal{S}_*(E)$.*

For any family $(A_i)_{i \in I}$ in $\mathcal{S}_*(E)$,

(7) *$\varepsilon_*(\cap_{i \in I} A_i) = \cap_{i \in I} \varepsilon_* A_i$;*

(8) (i) *for $* = s, q, b$, $\varepsilon_*(\triangledown_{i \in I} A_i) = \triangledown_{i \in I} \varepsilon_* A_i$;*

(ii) *for $* = l, r, i$, $\varepsilon_*(\cup_{i \in I} A_i) = \cup_{i \in I} \varepsilon_* A_i$.*

For any family $(C_i)_{i \in I}$ in $\mathcal{S}_*(E_0)$,

(9) *$\delta_*(\cap_{i \in I} C_i) = \cap_{i \in I} \delta_* C_i$;*

(10) (i) *for* $* = s$, q, b, $\delta_*(\nabla_{i\in I}\mathsf{C}_i) = \nabla_{i\in I}\delta_*\mathsf{C}_i$;

(ii) *for* $* = l$, r, i, $\delta_*(\cup_{i\in I}\mathsf{C}_i) = \cup_{i\in I}\delta_*\mathsf{C}_i$.

(11) *The map* ε_* *is a complete monomorphism;*

(l2) *The map* δ_* *is a complete epimorphism.*

Proof. In (a) below, we prove the Theorem for $* = s$. Then in (b) below, the proof is outlined for $* = q$, b. Finally, in (c) below, for $* = l$, r, i, the proof is outlined.

(a): (1): It is straightforward.

(2): It follows from the definition of ε_s and the Lemma 4.1 (1) and (3).

(3) and (4): Follows from the definitions of ε_s and δ_s and the Lemmas 3.1(1) and 2.15.

(5): Case I: Let $\mathsf{C} \in \mathcal{S}_s(\mathsf{E}) \subseteq \mathcal{S}_s(\mathsf{E}_0)$. Then

$$(\varepsilon_s \circ \delta_s)(\mathsf{C}) = \varepsilon_s(\delta_s(\mathsf{C})) = \varepsilon_s(\mathsf{C} - \{0\}) = \varepsilon_s(\mathsf{C}) = \mathsf{C}_0 \supset \mathsf{C}.$$

Case II: Let $\mathsf{C} \in \mathcal{S}_s(\mathsf{E}_0) - \mathcal{S}_s(\mathsf{E})$. Then $(\varepsilon_s \circ \delta_s)(\mathsf{C}) = \varepsilon_s(\delta_s(\mathsf{C}))$. Let $\delta_s\mathsf{C} = \mathsf{C} - \{0\} = \mathsf{F}$. Then $F = C$ and $\sigma_F e = \sigma_C e - \{0\}$ for all $e \in F$. Let $\varepsilon_s\mathsf{F} = \mathsf{F}_0$. Then $F_0 = F$ and $\sigma_{F_0} e = \sigma_F e \cup \{0\}$ for all $e \in F_0$.

We show that $\mathsf{F}_0 = \mathsf{C}$ or (i) $F_0 = C$ (ii) $\sigma_{F_0} e = \sigma_C e$ for all $e \in F_0$.

(i): $F_0 = F = C$.

(ii): Let $e \in F_0 = C$ be fixed.

Now $\sigma_{F_0} e = \sigma_F e \cup \{0\} = (\sigma_C e - \{0\}) \cup \{0\} = \sigma_C e$.

Now (i) and (ii) imply $\mathsf{F}_0 = \mathsf{C}$ or $(\varepsilon_s \circ \delta_s)(\mathsf{C}) = \varepsilon_s(\delta_s(\mathsf{C})) = \mathsf{C}$.

From the above two Cases, we get that

$$(\varepsilon_s \circ \delta_s)(\mathsf{X}) \supseteq \mathsf{X} \text{ for all } \mathsf{X} \in \mathcal{S}_s(\mathsf{E}_0) \text{ or } \varepsilon_s \circ \delta_s \supseteq 1_{\mathcal{S}_s(\mathsf{E}_0)},$$

where $1_{\mathcal{S}_s(\mathsf{E}_0)}$ is the identity map on $\mathcal{S}_s(\mathsf{E}_0)$.

(6): Case I: $\Phi \in \mathcal{S}_s(\mathsf{E})$, $(\delta_s \circ \varepsilon_s)(\Phi) = \delta_s(\varepsilon_s(\Phi)) = \delta_s(\Phi) = \Phi$.

Case II: If $\Phi \neq \mathsf{A} \in \mathcal{S}_s(\mathsf{E})$ then $(\delta_s \circ \varepsilon_s)(\mathsf{A}) = \delta_s(\varepsilon_s(\mathsf{A}))$.

Let $\varepsilon_s\mathsf{A} = \mathsf{A}_0$. Then $A_0 = A$ and $\sigma_{A_0} e = \sigma_A e \cup \{0\}$ for all $e \in A_0$.

Let $\delta_s\mathsf{A}_0 = \mathsf{A}_0 - \{0\} = \mathsf{F}$. Then $F = A_0$ and $\sigma_F e = \sigma_{A_0} e - \{0\}$ for all $e \in F$.

We show that $\mathsf{F} = \mathsf{A}$ or (i) $F = A$ (ii) $\sigma_F e = \sigma_A e$ for all $e \in F$.

(i): $F = A_0 = A$.

(ii): Let $e \in F = A$ be fixed.

Now $\sigma_F e = \sigma_{A_0} e - \{0\} = (\sigma_A e \cup \{0\}) - \{0\} = \sigma_A e$.

Now (i) and (ii) imply $\mathsf{F} = \mathsf{A}$ or $(\delta_s \circ \varepsilon_s)(\mathsf{A}) = \delta_s(\varepsilon_s(\mathsf{A})) = \mathsf{A}$.

From the above two Cases, we get that

$$(\delta_s \circ \varepsilon_s)(\mathsf{A}) = \mathsf{A} \text{ for all } \mathsf{A} \in \mathcal{S}_s(\mathsf{E}) \text{ or } \delta_s \circ \varepsilon_s = 1_{\mathcal{S}_s(\mathsf{E})},$$

where $1_{\mathcal{S}_s(\mathsf{E})}$ is the identity map on $\mathcal{S}_s(\mathsf{E})$.

(7): Let $\cap_{i\in I}\mathsf{A}_i = \mathsf{A}$. Then $A = \cap_{i\in I} A_i$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i} e$ for all $e \in A$. Let $\varepsilon_s\mathsf{A} = \mathsf{A}_0$. Then $A_0 = A$ and $\sigma_{A_0} e = \sigma_A e \cup \{0\}$ for all $e \in A_0$. Let $\varepsilon_s\mathsf{A}_i = \mathsf{B}_{i0}$.

Then $B_{i0} = A_i$ and $\sigma_{B_{i0}} e = \sigma_{A_i} e \cup \{0\}$ for all $e \in B_{i0}$. Let $\cap_{i\in I}\mathsf{B}_{i0} = \mathsf{B}_0$. Then $B_0 = \cap_{i\in I} B_{i0}$ and $\sigma_{B_0} e = \cap_{i\in I}\sigma_{B_{i0}} e$ for all $e \in B_0$.

We show that $\mathsf{A}_0 = \mathsf{B}_0$ or (i) $A_0 = B_0$ (ii) $\sigma_{A_0} e = \sigma_{B_0} e$ for all $e \in A_0$.

(i): $A_0 = A = \cap_{i\in I} A_i = \cap_{i\in I} B_{i0} = B_0$.

(ii): Let $e \in A_0 = B_0$ be fixed.

Now $\sigma_{A_0} e = \sigma_A e \cup \{0\} = (\cap_{i\in I}\sigma_{A_i} e) \cup \{0\} = \cap_{i\in I}(\sigma_{A_i} e \cup \{0\}) = \cap_{i\in I}\sigma_{B_{i0}} e = \sigma_{B_0} e$.

Now (i) and (ii) imply $\mathsf{A}_0 = \mathsf{B}_0$ or $\varepsilon_s(\cap_{i\in I}\mathsf{A}_i) = \cap_{i\in I}\varepsilon_s\mathsf{A}_i$.

(8): Let $\overline{\vee}_{i\in I}\mathsf{A}_i = \mathsf{A}$. Then $A = \cup_{i\in I} A_i$ and $\sigma_A e = (\cup_{i\in I_e}\sigma_{A_i} e)_{s,\,\sigma_E e}$ for all $e \in A$, where $I_e = \{i \in I / e \in A_i\}$. Let $\varepsilon_s\mathsf{A} = \mathsf{A}_0$. Then $A_0 = A$ and $\sigma_{A_0} e = \sigma_A e \cup \{0\}$ for all $e \in A_0$. Let $\varepsilon_s\mathsf{A}_i = \mathsf{B}_{i0}$. Then $B_{i0} = A_i$ and $\sigma_{B_{i0}} e = \sigma_{A_i} e \cup \{0\}$ for all $e \in B_{i0}$. Let $\overline{\vee}_{i\in I}\mathsf{B}_{i0} = \mathsf{B}_0$. Then $B_0 = \cup_{i\in I} B_{i0}$ and $\sigma_{B_0} e = (\cup_{i\in I_e}\sigma_{B_{i0}} e)_{s,\,\sigma_{E_0} e}$ for all $e \in B_0$, where $I_e = \{i \in I / e \in B_{i0}\}$.

We show that $\mathsf{A}_0 = \mathsf{B}_0$ or (i) $A_0 = B_0$ (ii) $\sigma_{A_0} e = \sigma_{B_0} e$ for all $e \in A_0$.

(i): $A_0 = A = \cup_{i\in I} A_i = \cup_{i\in I} B_{i0} = B_0$.

(ii): Let $e \in A_0 = B_0$ be fixed.

Now $\sigma_{B_0} e = (\cup_{i\in I_e}\sigma_{B_{i0}} e)_{s,\,\sigma_{E_0} e} = (\cup_{i\in I_e}(\sigma_{A_i} e \cup \{0\}))_{s,\,\sigma_{E_0} e} = ((\cup_{i\in I_e}\sigma_{A_i} e) \cup \{0\})_{s,\,\sigma_{E_0} e} = (\cup_{i\in I_e}\sigma_{A_i} e)_{s,\,\sigma_E e} \cup \{0\} = \sigma_A e \cup \{0\} = \sigma_{A_0} e$.

Now (i) and (ii) imply $\mathsf{A}_0 = \mathsf{B}_0$ or $\varepsilon_s(\overline{\vee}_{i\in I}\mathsf{A}_i) = \overline{\vee}_{i\in I}\varepsilon_s\mathsf{A}_i$.

(9): Let $\cap_{i\in I}\mathsf{C}_i = \mathsf{C}$. Then $C = \cap_{i\in I} C_i$ and $\sigma_C e = \cap_{i\in I}\sigma_{C_i} e$ for all $e \in C$. Let $\delta_s\mathsf{C} = \mathsf{C} - \{0\} = \mathsf{F}$. Then $F = C$ and $\sigma_F e = \sigma_C e - \{0\}$ for all $e \in F$. Let $\delta_s\mathsf{C}_i = \mathsf{C}_i - \{0\} = \mathsf{G}_i$. Then $G_i = C_i$ and $\sigma_{G_i} e = \sigma_{C_i} e - \{0\}$ for all $e \in G_i$. Let $\cap_{i\in I}\mathsf{G}_i = \mathsf{G}$. Then $G = \cap_{i\in I} G_i$ and $\sigma_G e = \cap_{i\in I}\sigma_{G_i} e$ for all $e \in G$.

We show that $\mathsf{F} = \mathsf{G}$ or (i) $F = G$ (ii) $\sigma_F e = \sigma_G e$ for all $e \in F$.

(i): $F = C = \cap_{i\in I} C_i = \cap_{i\in I} G_i = G$.

(ii): Let $e \in F = G$ be fixed.

Now $\sigma_F e = \sigma_C e - \{0\} = (\cap_{i\in I}\sigma_{C_i} e) - \{0\} = (\cap_{i\in I}\sigma_{C_i} e) \cap \{0\}^c = \cap_{i\in I}(\sigma_{C_i} e \cap \{0\}^c) = \cap_{i\in I}(\sigma_{C_i} e - \{0\}) = \cap_{i\in I}\sigma_{G_i} e = \sigma_G e$.

Now (i) and (ii) imply $\mathsf{F} = \mathsf{G}$ or $\delta_s(\cap_{i\in I}\mathsf{C}_i) = \cap_{i\in I}\delta_s\mathsf{C}_i$.

(10): Let $\overline{\vee}_{i\in I}\mathsf{C}_i = \mathsf{C}$. Then $C = \cup_{i\in I} C_i$ and $\sigma_C e = (\cup_{i\in I_e}\sigma_{C_i} e)_{s,\,\sigma_{E_0} e}$ for all $e \in C$, where $I_e = \{i \in I / e \in C_i\}$. Let $\delta_s\mathsf{C} = \mathsf{C} - \{0\} = \mathsf{F}$. Then $F = C$ and $\sigma_F e = \sigma_C e - \{0\}$ for all $e \in F$. Let $\delta_s\mathsf{C}_i = \mathsf{C}_i - \{0\} = \mathsf{G}_i$. Then $G_i = C_i$ and $\sigma_{G_i} e = \sigma_{C_i} e - \{0\}$ for all $e \in G_i$. Let $\overline{\vee}_{i\in I}\mathsf{G}_i = \mathsf{G}$. Then $G = \cup_{i\in I} G_i$ and $\sigma_G e = (\cup_{i\in I_e}\sigma_{G_i} e)_{s,\,\sigma_E e}$ for all $e \in G$, where $I_e = \{i \in I / e \in G_i\}$.

We show that $\mathsf{F} = \mathsf{G}$ or (i) $F = G$ (ii) $\sigma_F e = \sigma_G e$ for all $e \in F$.

(i): $F = C = \cup_{i\in I} C_i = \cup_{i\in I} G_i = G$.

(ii): Let $e \in F = G$ be fixed.

Now

$\sigma_G e = (\cup_{i\in I_e}\sigma_{G_i} e)_{s,\,\sigma_E e} = (\cup_{i\in I_e}(\sigma_{C_i} e - \{0\}))_{s,\,\sigma_E e} = (\cup_{i\in I_e}(\sigma_{C_i} e \cap \{0\}^c))_{s,\,\sigma_E e}$ $= ((\cup_{i\in I_e}\sigma_{C_i} e) \cap \{0\}^c)_{s,\,\sigma_E e} = ((\cup_{i\in I_e}\sigma_{C_i} e) - \{0\})_{s,\,\sigma_E e} = (\cup_{i\in I_e}\sigma_{C_i} e)_{s,\,\sigma_{E_0} e} - \{0\}$ $= \sigma_C e - \{0\} = \sigma_F e$.

Now (i) and (ii) imply $\mathsf{F} = \mathsf{G}$ or $\delta_s(\overline{\vee}_{i\in I}\mathsf{C}_i) = \overline{\vee}_{i\in I}\delta_s\mathsf{C}_i$.

(11): It follows from (1), (7) and (8)(i) above.

(12): It follows from (2), (9) and (10)(i) above.

(b): (1): It follows in a similar way as in (a)(1) above.

(2): Let $\mathsf{F} \in \mathcal{S}_q(\mathsf{E})$. Then $\mathsf{F}_0 \in \mathcal{S}_q(\mathsf{E}_0)$. Now $\delta_q\mathsf{F}_0 = \mathsf{F}_0 - \{0\} = \mathsf{G}$ implies $G = F_0$ and $\sigma_G e = \sigma_{F_0} e - \{0\}$ for all $e \in G$ which implies $G = F_0 = F$ and $\sigma_G e = \sigma_{F_0} e - \{0\} = (\sigma_F e \cup \{0\}) - \{0\} = \sigma_F e$ for all $e \in G$ implying $\mathsf{G} = \mathsf{F}$ or $\delta_q \mathsf{F}_0 = \mathsf{G} = \mathsf{F}$ or δ_q is onto.

(3) and (4): Follow in a similar way as in (a) (3) and (4) above.

(5): Case I: $\Phi \in \mathcal{S}_q(\mathsf{E}_0)$, $(\varepsilon_q \circ \delta_q)(\Phi) = \varepsilon_q(\delta_q(\Phi)) = \varepsilon_q(\Phi) = \Phi$.

Case II: Let $\mathcal{S}^n_q(\mathsf{E}_0) = \{\Phi_{A_0}/A_0 \subseteq E_0$ and $\sigma_{A_0} e = \phi$ for all $e \in A_0\}$ be the set of all null soft quasi-ideals of E_0. Clearly, $\mathcal{S}^n_q(\mathsf{E}_0) \subseteq \mathcal{S}_q(\mathsf{E}_0)$.

If $\mathsf{C} \in \mathcal{S}^n_q(\mathsf{E}_0)$ then $(\varepsilon_q \circ \delta_q)(\mathsf{C}) = \varepsilon_q(\delta_q(\mathsf{C})) = \varepsilon_q(\mathsf{C} - \{0\}) = \varepsilon_q(\mathsf{C}) = \mathsf{C}_0 \supset \mathsf{C}$.

Case III: If $\Phi \neq \mathsf{C} \in \mathcal{S}_q(\mathsf{E}_0) - \mathcal{S}^n_q(\mathsf{E}_0)$ then $(\varepsilon_q \circ \delta_q)(\mathsf{C}) = \varepsilon_q(\delta_q(\mathsf{C}))$.

Let $\delta_q\mathsf{C} = \mathsf{C} - \{0\} = \mathsf{F}$. Then $F = C$ and $\sigma_F e = \sigma_C e - \{0\}$ for all $e \in F$. Let $\varepsilon_q\mathsf{F} = \mathsf{F}_0$. Then $F_0 = F$ and $\sigma_{F_0} e = \sigma_F e \cup \{0\}$ for all $e \in F_0$.

We show that $\mathsf{F}_0 = \mathsf{C}$ or (i) $F_0 = C$ (ii) $\sigma_{F_0} e = \sigma_C e$ for all $e \in F_0$.

(i): $F_0 = F = C$.

(ii): Let $e \in F_0 = C$ be fixed.

Now $\sigma_{F_0} e = \sigma_F e \cup \{0\} = (\sigma_C e - \{0\}) \cup \{0\} = \sigma_C e$.

Now (i) and (ii) imply $\mathsf{F}_0 = \mathsf{C}$ or $(\varepsilon_q \circ \delta_q)(\mathsf{C}) = \varepsilon_q(\delta_q(\mathsf{C})) = \mathsf{C}$.

From the above three Cases, we get that $(\varepsilon_q \circ \delta_q)(\mathsf{X}) \supseteq \mathsf{X}$ for all $\mathsf{X} \in \mathcal{S}_q(\mathsf{E}_0)$ or $\varepsilon_q \circ \delta_q \supseteq 1_{\mathcal{S}_q(\mathsf{E}_0)}$, where $1_{\mathcal{S}_q(\mathsf{E}_0)}$ is the identity map on $\mathcal{S}_q(\mathsf{E}_0)$.

(6)-(12): Follow in a similar way as in (a) (6)-(12) above.

For $* = b$, the proof follows in a similar way as in (b) above.

(c): (1): It is straightforward.

(2): It follows in a similar way as in (b)(2) above.

(3) and (4): Follow in a similar way as in (a) (3) and (4) above.

(5): It follows in a similar way as in (b)(5) above.

(6) and (7): Follow in a similar way as in (a) (6) and (7) above.

(8): Let $\cup_{i\in I}\mathsf{A}_i = \mathsf{A}$. Then $A = \cup_{i\in I} A_i$ and $\sigma_A e = \cup_{i\in I_e}\sigma_{A_i} e$ for all $e \in A$, where $I_e = \{i \in I/e \in A_i\}$. Let $\varepsilon_l\mathsf{A} = \mathsf{A}_0$. Then $A_0 = A$ and $\sigma_{A_0} e = \sigma_A e \cup \{0\}$ for all $e \in A_0$. Let $\varepsilon_l\mathsf{A}_i = \mathsf{B}_{i0}$. Then $B_{i0} = A_i$ and $\sigma_{B_{i0}} e = \sigma_{A_i} e \cup \{0\}$ for all $e \in B_{i0}$. Let $\cup_{i\in I}\mathsf{B}_{i0} = \mathsf{B}_0$. Then $B_0 = \cup_{i\in I} B_{i0}$ and $\sigma_{B_0} e = \cup_{i\in I_e}\sigma_{B_{i0}} e$ for all $e \in B_0$, where $I_e = \{i \in I/e \in B_{i0}\}$.

We show that $\mathsf{A}_0 = \mathsf{B}_0$ or (i) $A_0 = B_0$ (ii) $\sigma_{A_0} e = \sigma_{B_0} e$ for all $e \in A_0$.

(i): $A_0 = A = \cup_{i\in I} A_i = \cup_{i\in I} B_{i0} = B_0$.

(ii): Let $e \in A_0 = B_0$ be fixed.

Now $\sigma_{B_0} e = \cup_{i\in I_e}\sigma_{B_{i0}} e = \cup_{i\in I_e}(\sigma_{A_i} e \cup \{0\}) = (\cup_{i\in I_e}\sigma_{A_i} e) \cup \{0\} = \sigma_A e \cup \{0\} = \sigma_{A_0} e$.

Now (i) and (ii) imply $\mathsf{A}_0 = \mathsf{B}_0$ or $\varepsilon_l(\cup_{i\in I}\mathsf{A}_i) = \cup_{i\in I}\varepsilon_l\mathsf{A}_i$.

(9): It follows in a similar way as in (a)(9) above.

(10): Let $\cup_{i\in I}\mathsf{C}_i = \mathsf{C}$. Then $C = \cup_{i\in I}C_i$ and $\sigma_C e = \cup_{i\in I_e}\sigma_{C_i}e$ for all $e \in C$, where $I_e = \{i \in I/e \in C_i\}$. Let $\delta_l\mathsf{C} = \mathsf{C} - \{0\} = \mathsf{F}$. Then $F = C$ and $\sigma_F e = \sigma_C e - \{0\}$ for all $e \in F$. Let $\delta_l\mathsf{C}_i = \mathsf{C}_i - \{0\} = \mathsf{G}_i$. Then $G_i = C_i$ and $\sigma_{G_i}e = \sigma_{C_i}e - \{0\}$ for all $e \in G_i$. Let $\cup_{i\in I}\mathsf{G}_i = \mathsf{G}$. Then $G = \cup_{i\in I}G_i$ and $\sigma_G e = \cup_{i\in I_e}\sigma_{G_i}e$ for all $e \in G$, where $I_e = \{i \in I/e \in G_i\}$.

We show that $\mathsf{F} = \mathsf{G}$ or (i) $F = G$ (ii) $\sigma_F e = \sigma_G e$ for all $e \in F$.

(i): $F = C = \cup_{i\in I}C_i = \cup_{i\in I}G_i = G$.

(ii): Let $e \in F = G$ be fixed.

Now

$\sigma_G e = \cup_{i\in I_e}\sigma_{G_i}e = \cup_{i\in I_e}(\sigma_{C_i}e - \{0\}) = \cup_{i\in I_e}(\sigma_{C_i}e \cap \{0\}^c)$
$= (\cup_{i\in I_e}\sigma_{C_i}e) \cap \{0\}^c = (\cup_{i\in I_e}\sigma_{C_i}e) - \{0\} = \sigma_C e - \{0\} = \sigma_F e.$

Now (i) and (ii) imply $\mathsf{F} = \mathsf{G}$ or $\delta_l(\cup_{i\in I}\mathsf{C}_i) = \cup_{i\in I}\delta_l\mathsf{C}_i$.

(11): It follows from (1), (7) and (8)(ii).

(12): It follows from (2), (9) and (10)(ii).

For $* = r$, i, the proofs follow in a similar way as in (c) above. □

The following Example shows that in the above Theorem whenever $* = s$, q, b, l, r, i, (1) the map ε_* is *not* onto and (2) the map δ_* is *not* one-one.

EXAMPLE 4.1. Let S and S_0 be the semigroups same as in the Example 2.3, $\mathsf{E} = (\{(e, S)\}, \{e\})$ and $\mathsf{E}_0 = (\{(e, S_0)\}, \{e\})$ be the whole soft semigroups over S and S_0 respectively.

(1) For $* = s$, q, b, l, r, i, let $\Phi_{E_0} = (\{(e, \phi)\}, \{e\}) \in \mathcal{S}_*(\mathsf{E}_0)$. Clearly, there is *no* $\mathsf{A} \in \mathcal{S}_*(\mathsf{E})$ such that $\varepsilon_*\mathsf{A} = \Phi_{E_0}$. Therefore ε_* is *not* onto.

(2) For $* = s$, q, b, l, r, i, let $\mathsf{C} = (\{(e, \phi)\}, \{e\})$ and $\mathsf{D} = (\{(e, \{0\})\}, \{e\}) \in \mathcal{S}_*(\mathsf{E}_0)$. Then $\delta_*\mathsf{C} = (\{(e, \phi)\}, \{e\}) = \delta_*\mathsf{D}$ but $\mathsf{C} \neq \mathsf{D}$. Therefore δ_* is *not* one-one.

THEOREM 4.2. *For any soft semigroup E over U, the maps $\varepsilon : \mathcal{S}_s(\mathsf{E}) \to \mathcal{S}_s(\mathsf{E}_1)$ defined by for any $\mathsf{A} \in \mathcal{S}_s(\mathsf{E})$, $\varepsilon\mathsf{A} = \mathsf{A}_1$, where $A_1 = A$ and $\sigma_{A_1}e = \sigma_A e \cup \{1\}$ for all $e \in A_1$, and $\delta : \mathcal{S}_s(\mathsf{E}_1) \to \mathcal{S}_s(\mathsf{E})$ defined by for any $\mathsf{C} \in \mathcal{S}_s(\mathsf{E}_1)$, $\delta\mathsf{C} = \mathsf{C} - \{1\} = \mathsf{F}$, where $F = C$ and $\sigma_F e = \sigma_C e - \{1\}$ for all $e \in F$, satisfy the following properties:*

(1) *The map ε is one-one;*

(2) *The map δ is onto;*

(3) *For any A, $\mathsf{B} \in \mathcal{S}_s(\mathsf{E})$, $\mathsf{A} \leqslant \mathsf{B}$ implies $\varepsilon\mathsf{A} \leqslant \varepsilon\mathsf{B}$;*

(4) *For any C, $\mathsf{D} \in \mathcal{S}_s(\mathsf{E}_1)$, $\mathsf{C} \leqslant \mathsf{D}$ implies $\delta\mathsf{C} \leqslant \delta\mathsf{D}$;*

(5) *$\varepsilon \circ \delta \supseteq 1_{\mathcal{S}_s(\mathsf{E}_1)}$, where $1_{\mathcal{S}_s(\mathsf{E}_1)}$ is the identity map on $\mathcal{S}_s(\mathsf{E}_1)$;*

(6) *$\delta \circ \varepsilon = 1_{\mathcal{S}_s(\mathsf{E})}$, where $1_{\mathcal{S}_s(\mathsf{E})}$ is the identity map on $\mathcal{S}_s(\mathsf{E})$.*

For any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_s(\mathsf{E})$,

(7) *$\varepsilon(\cap_{i\in I}\mathsf{A}_i) = \cap_{i\in I}\varepsilon\mathsf{A}_i$;*

(8) *$\varepsilon(\overline{\vee}_{i\in I}\mathsf{A}_i) = \overline{\vee}_{i\in I}\varepsilon\mathsf{A}_i$.*

For any family $(\mathsf{C}_i)_{i\in I}$ in $\mathcal{S}_s(\mathsf{E}_1)$,

(9) *$\delta(\cap_{i\in I}\mathsf{C}_i) = \cap_{i\in I}\delta\mathsf{C}_i$;*

(10) *$\delta(\overline{\vee}_{i\in I}\mathsf{C}_i) = \overline{\vee}_{i\in I}\delta\mathsf{C}_i$.*

(11) *The map ε is a complete monomorphism;*

(12) *The map δ is a complete epimorphism.*

PROOF. It follows in a similar way as in the case of $* = s$ of the Theorem 4.1. $\square$

The following Example shows that in the above Theorem (1) the map ε is *not* onto and (2) the map δ is *not* one-one.

EXAMPLE 4.2. Let S and S_1 be the semigroups same as in the Example 2.2, $\mathsf{E} = (\{(e, S)\}, \{e\})$ and $\mathsf{E}_1 = (\{(e, S_1)\}, \{e\})$ be the whole soft semigroups over S and S_1 respectively.

(1) Let $\Phi_{E_1} = (\{(e, \phi)\}, \{e\}) \in \mathcal{S}_s(\mathsf{E}_1)$. Clearly, there is *no* $\mathsf{A} \in \mathcal{S}_s(\mathsf{E})$ such that $\varepsilon\mathsf{A} = \mathsf{A}_1 = \Phi_{E_1}$. Therefore ε is *not* onto.

(2) Let $\mathsf{C} = (\{(e, \phi)\}, \{e\})$ and $\mathsf{D} = (\{(e, \{1\})\}, \{e\})$ be in $\mathcal{S}_s(\mathsf{E}_1)$. Then $\delta\mathsf{C} = (\{(e, \phi)\}, \{e\}) = \delta\mathsf{D}$ but $\mathsf{C} \neq \mathsf{D}$. Therefore δ is *not* one-one.

REMARK 4.2. Observe that from the Example 2.2, it is clear that adjoining of 1 does *not* preserve soft (left, right, quasi-, bi-) ideals. Consequently, the above Theorem has *no* analogues in the case of soft (left, right, quasi-, bi-) ideals.

Conclusion: These results are crucial in the Representation of Soft Substructures of a Soft Semigroup as certain type of Crisp Substructures of a crisp Semigroup.

Acknowledgments: The second author expresses her indebtedness to the first author for his enormous patience and meticulous supervision without which this paper would not have been what it is now.

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Received by editors 12.11.2019; Revised version 20.03.2020; Available online 30.03.2020.

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