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NEW TECHNIQUE IN ASYMPTOTIC STABILITY FOR SECOND ORDER NONLINEAR DELAY INTEGRO DIFFERENTIAL EQUATIONS

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Abstract. The second order nonlinear integro-differential equation

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t))\dot{x}(t) + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} a_j(t,s) g_j(s, x(s))\, ds = 0,$$

with variable delays $\tau_j(t) \geqslant 0$, $1 \leqslant j \leqslant N$, is investigated with low restrictions on the delays. Omitting assumptions such as differentiability on τ_j or inversibility of functions $t - \tau_j(t)$, makes the variation of parameters method difficult to apply to the equation. To circumvent the difficulties we choose conditions for f, a_j, g_j and we, carefully, amplify space of functions so that the equation takes a suitable form that facilitates the inversion of the equation into an equivalent one from which we derive a fixed point mapping. The end result is not only conditions for existence and uniqueness of solutions of the equation, but also for boundedness and stability of the zero solution of that equation. We also provide conditions that make zero solution asymptotically stable. The technique we use here avoids many difficulties which we often encounter in studying any class of second order nonlinear equations with variables delays and offers, what we hope, a new way to investigate the stability by fixed point theory. Our work extends and improves previous results in the literature such as, D. Pi: Study the stability of solutions of functional differential equations via fixed points. *Nonlinear Analysis*, **74**(2)(2011), 639–651.

1. Introduction

The theory of stability for delays nonlinear neutral integro-differential equations makes possible the treatment of physical and biological phenomena systems such as nuclear reactors or neural networks systems. Such systems are often sources

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of instability and degradation in many control problems. This has motivated investigators to mathematically discuss problems of stability, instability and asymptotic stability of these systems of great interest. For more than 100 years, the Lyapunov direct method has been the ultimate key tool to study stability problems. The direct method was widely used to study the stability of solutions of ordinary differential equations and functional differential equations. Nevertheless, the pointwise nature of this method and the construction of the Lyapunov functionals have been, and still are, an arduous task (see [**7**]).

Recently, many authors have realized that the fixed points theory can be used to overcome the difficulties and to study the stability of the solutions (see [**1**]-[**8**], [**10**], [**12**], [**14**]-[**16**], [**18**]-[**20**]). Levin and Nohel [**13**] studied the following nonlinear system of differential equations of Liénard form

$$\ddot{x} + h\left(t, x, \dot{x}\right)\dot{x} + f\left(x\right) = a(t). \tag{1.1}$$

They obtained, by constructing a proper Lyapunov function, conditions under which all solutions of (1.1) tend to zero as $t \to \infty$. D. Pi (see [**14**]) studied the asymptotic stability of the equation

$$\ddot{x} + f\left(t, x, \dot{x}\right)\dot{x} + g\left(x\left(t - \tau\left(t\right)\right)\right) = 0. \tag{1.2}$$

D. Pi result requires the assumption that $\tau\left(\cdot\right)$ is continuous and its derivative has an inverse. D. Pi has considered other equations related to (1.2) (see [**15, 16**]), nevertheless his results rely on the introduction of an arbitrary and unknown continuous function which is contested by the public of this domain because there is no real success at finding such a function. Many other interesting results on fixed points and stability properties can be found in the references [**1**]-[**7**].

In this paper, we consider the equation

$$\ddot{x} + f\left(t, x, \dot{x}\right)\dot{x} + \sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t} a_j(t,s)g_j\left(s, x\left(s\right)\right)ds = 0, \tag{1.3}$$

for $t \geqslant 0$ where, for $j = \overline{1,N}$, functions

$$\tau_j : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \quad a_j(\cdot,\cdot) : \mathbb{R}_+ \times \left[-\tau_j\left(0\right), \infty\right) \longrightarrow \mathbb{R}, \quad f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+$$
$$\text{and} \quad g_j\left(\cdot,\cdot\right) : \left[-\tau_j\left(0\right), \infty\right) \times \mathbb{R} \longrightarrow \mathbb{R}$$

are all continuous functions. We assume that,

$$t - \tau_j\left(t\right) \longrightarrow \infty \text{ as } t \longrightarrow \infty,\ j = \overline{1,N}. \tag{1.4}$$

For each $t_0 \geqslant 0$, define

$$m_j\left(t_0\right) =: \inf\left\{s - \tau_j\left(s\right) : s \geqslant t_0\right\},\ j = \overline{1,N}$$

and let

$$m\left(t_0\right) := \min\left\{m_j\left(t_0\right),\ j = \overline{1,N}\right\}.$$

Let $\mathcal{C}\left(t_0\right) := \mathcal{C}\left(\left[m\left(t_0\right), t_0\right], \mathbb{R}\right)$ be the space of continuous functions endowed with supremum norm $\left\|\cdot\right\|$, that is, for $\psi \in \mathcal{C}\left(t_0\right)$,

$$\left\|\psi\right\| := \sup\left\{\left|\psi\left(s\right)\right| : m\left(t_0\right) \leqslant s \leqslant t_0\right\}.$$

We will also use $\|\varphi\| := \sup\{|\varphi(s)| : s \in [m(t_0), \infty)\}$ to express the supremum of continuous bounded functions on $[m(t_0), \infty)$ later. It is well known (see [**11**]) that, for a given continuous function ψ, there exists a solution for equation (1.3) on an interval $[m(t_0), T)$, and if the solution remains bounded, then $T = \infty$. We denote by $x(t)$ the solution $x(t, t_0, \psi)$. Now, let $G(t) := f((t, x(t), \dot{x}(t))$. We can rewrite equation (1.3) as

$$\ddot{x} + G(t)\dot{x} + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} a_j(t,s) g_j(s, x(s))\, ds = 0. \tag{1.5}$$

Our purpose is to give a necessary and sufficient condition ensuring that the zero solution of the above equation is asymptotically stable. Being free of the introduction of the unknown function used in Pi work (see [**15, 16**]), we hope that the present method will offers not only results of stability but a new way to investigate second order nonlinear integro-differential equations as well.

2. Preliminaries

Some asymptotic properties on integral equations are needed in this work. So, let f be a real or complex-valued function of the variable $t > 0$ and p be a real or a complex parameter such that $\Re(p) > 0$. We define the Laplace transform (see [**9**],[**17**]) of f as

$$F(p) = \mathfrak{L}(f(t))_{(p)} = \int_0^\infty e^{-pt} f(t)\, dt. \tag{2.1}$$

Also, recall that the Laplace transform (2.1) of power function t^γ is given by

$$\mathfrak{L}(t^\gamma)_{(p)} = \int_0^\infty e^{-pt} t^\gamma dt = \frac{\Gamma(\gamma+1)}{p^{\gamma+1}},\ \gamma > -1,\, p > 0,$$

with Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \mathfrak{L}\left(t^{z-1}\right)_{(1)},$$

which converges in the right half of the complex plane $\Re(z) > 0$. Now, let $-\infty \leqslant \alpha < \beta \leqslant +\infty$, $\varphi : [\alpha, \beta] \to \mathbb{R}$ and define, for $\lambda \in \mathbb{R}$, the integral

$$F(\lambda) = \int_\alpha^\beta e^{-\lambda\varphi(t)} f(t)\, dt.$$

We assume that there exists a constant $\lambda_0 > 0$ such that for every $\lambda \geqslant \lambda_0$ we have,

$$\int_\alpha^\beta e^{-\lambda\varphi(t)} |f(t)|\, dt < \infty.$$

The following theorem is crucial to reach our goal (see [**9**, Theorem VII.3.1]).

THEOREM 2.1. *Let $\varphi : [\alpha, \beta) \longrightarrow \mathbb{R}_+$ be a function such that φ is of class $\mathcal{C}^1$, $\varphi' > 0$ on $[\alpha, \beta)$. Assume that f is a continuous function at α with $f(\alpha) \neq 0$. Then,*

$$F(\lambda) \sim \frac{f(\alpha)}{\varphi'(\alpha)} \frac{1}{\lambda} e^{-\lambda\varphi(\alpha)}, \text{ as } \lambda \to +\infty. \tag{2.2}$$

PROOF. **(a)** To begin with, suppose $\varphi(t) = t$ and $\alpha = 0$. Then,

$$F(\lambda) = \int_0^\beta e^{-\lambda t} f(t)\, dt.$$

We check that $F(\lambda)$ satisfies the property (2.2). Indeed, since f is continuous at $\alpha = 0$, then, for any given $\varepsilon > 0$, one can choose η sufficiently small, such that

$$|f(t) - f(0)| \leqslant \varepsilon, \text{ for } 0 \leqslant t \leqslant \eta.$$

Next, we decompose $F(\lambda)$ in the following manner

$$F(\lambda) = f(0) \int_0^\eta e^{-\lambda t} dt + \int_0^\eta e^{-\lambda t} (f(t) - f(0))\, dt + \int_\eta^\beta e^{-\lambda t} f(t)\, dt. \tag{2.3}$$

From (2.3) we can establish the following estimates

$$\int_0^\eta e^{-\lambda t} dt = \frac{1}{\lambda}\left(1 - e^{-\lambda\eta}\right).$$

$$\int_0^\eta e^{-\lambda t} (f(t) - f(0))\, dt \leqslant \varepsilon \int_0^\infty e^{-\lambda t} dt = \frac{\varepsilon}{\lambda}.$$

For $t \geqslant \eta$ we have $(\lambda - \lambda_0)(t - \eta) \geqslant 0$. Thus, $-\lambda t \leqslant -\lambda t + (\lambda - \lambda_0)(t - \eta)$ and so

$$\int_\eta^\beta e^{-\lambda t} f(t)\, dt \leqslant e^{-\eta(\lambda - \lambda_0)} \int_\eta^\beta e^{-\lambda_0 t} f(t)\, dt.$$

(b) Let us return to the general case. For this purpose, consider the function

$$g : [\alpha, \beta) \longrightarrow [0, \beta_0), \quad t \longmapsto g(t) := \varphi(t) - \varphi(\alpha),$$

where $\beta_0 = \varphi(\beta) - \varphi(\alpha)$. We observe that g is bijective on $[\alpha, \beta)$. Denote the reciprocal function of g by

$$\psi : [0, \beta_0) \longrightarrow [\alpha, \beta), \quad u \longmapsto \psi(u).$$

The change of variables $t = \psi(u)$ yields the integral formula

$$F(\lambda) = e^{-\lambda\varphi(\alpha)} \int_0^{\beta_0} e^{-\lambda u} f(\psi(u))\, \psi'(u)\, du. \tag{2.4}$$

We see that

$$\frac{d\psi(u)}{dt} = \psi'(\varphi(t) - \varphi(\alpha))\, \varphi'(t) = 1 \text{ and } \psi'(0) = \frac{1}{\varphi'(\alpha)}.$$

Define

$$\tilde{f}(u) := f(\psi(u))\, \psi'(u).$$

Clearly, the function $\tilde{f}$ is continuous at 0. Moreover,

$$\tilde{f}(0) = f(\psi(0))\,\psi'(0) = \frac{f(\alpha)}{\varphi'(\alpha)}.$$

Thus, equation (2.4) is as

$$F(\lambda) = e^{-\lambda\varphi(\alpha)} \int_0^{\beta_0} e^{-\lambda u} \tilde{f}(u)\,du, \tag{2.5}$$

Now, by applying a similar argumentation as in **(a)** to the integral term of (2.5) we obtain

$$\int_0^{\beta_0} e^{-\lambda u} \tilde{f}(u)\,du \sim \tilde{f}(0)\,\frac{1}{\lambda} = \frac{f(\alpha)}{\varphi'(\alpha)}\frac{1}{\lambda}, \text{ as } \lambda \to +\infty.$$

Therefore,

$$F(\lambda) \sim \frac{f(\alpha)}{\varphi'(\alpha)}\frac{1}{\lambda}e^{-\lambda\varphi(\alpha)}, \text{ as } \lambda \to +\infty.$$

This ends the proof. □

Stability definitions, fixed point technique and more details on delay differential equations can be found in ([**11, 7**]).

Definition 2.1. The zero solution of (1.3) is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\psi \in \mathcal{C}(t_0),\ \|\psi\| < \delta]$ implies that $|x(t, t_0, \psi)| < \varepsilon$ for $t \geqslant t_0$.

Definition 2.2. The zero solution of (1.3) is asymptotically stable if it is stable and there is a $\delta_1 = \delta_1(t_0) > 0$ such that $[\psi \in \mathcal{C}(t_0),\ \|\psi\| < \delta_1]$ implies that $|x(t, t_0, \psi)| \longrightarrow 0$ as $t \longrightarrow \infty$.

3. Main Results

In this section, we will prove Theorem 3.1 and Theorem 3.2 on stability and asymptotic stability, respectively, for equation (1.3) by using the contraction mapping principle. But our equation is second order, nonlinear and has no non trivial edo term so the inversion of that equation needs some preparations to be domesticated. In fact, we have to transform (1.3) but, as we shall see, such a transformation is not simple to find nevertheless remains crucial for our work. Lemmas 3.1 and 3.2 are the subject of these esthetic works. More precisely, we give some conditions that will help to better rewrite equation (1.3) for the integration. We, then, use the variation of parameter to the given equation in Lemma 3.3 to invert it into an equivalent one and derive a fixed point mapping a solution of which is the solution of (1.3). For all that we shall need some functional preparations. First, let β be such that

$$\beta > \sigma := \sup \int_0^t e^{-\int_s^t a(v)dv} ds. \tag{3.1}$$

Let $X := \mathcal{C}([t_0,\infty),\mathbb{R}_+)$ to be the Banach space of real-valued bounded continuous functions on $[t_0,\infty)$ with the supremum norm $\|\cdot\|$. Next consider K_β the subset of X defined by

$$K_\beta := \left\{ \eta \in \mathcal{C}([t_0,\infty),\mathbb{R}_+) : \eta \leqslant \frac{1}{2\beta} \right\}.$$

Hence K_β is closed subset of X. Then, $(K_\beta,\|\cdot\|)$ is also a Banach space of continuous real-valued functions with supremum norm defined by

$$\|\varphi\| = \sup\{|\varphi(t)| : t \in [t_0,\infty)\}.$$

Next, an important special case of a first order differential equation is the first order nonlinear differential equation given by

$$\varphi' + a\varphi - \varphi^2 = b, \tag{A}$$

together with the initial condition

$$\varphi(t_0) = \varphi_{t_0}, \tag{B}$$

where a and b are arbitrary functions. Assume that $\varphi,\phi : [t_0,\infty) \longrightarrow \mathbb{R}_+$ are continuous with

$$\phi = a - \varphi. \tag{C}$$

In this section, we begin by studying such nonlinear equations involving multiple unknown functions. Next lemma shows that, under certain assumptions on a, the nonlinear equation (A) can always be solved in K_β. So, we will present some criteria for the existence of positive solutions of differential equations of first order (A)–(C).

Lemma 3.1. *Suppose that $a,b : [t_0,\infty) \longrightarrow \mathbb{R}_+$ are continuous functions and (3.1) holds. Assume that*

$$a(t) \geqslant \varphi_{t_0} + \frac{1}{2\beta}, \tag{3.3}$$

for $t \geqslant t_0$. Then, there exist two positive continuous functions ϕ,φ that satisfy (A)–(C).

Proof. Let $d(x,y) = \|x-y\|$ be the associate metric. Then, (K_β,d) is a complete metric space. Having in mind (A), we observe that

$$\left(\varphi e^{\int_{t_0}^t a(v)dv}\right)' = \left(\varphi^2 + b\right) e^{\int_{t_0}^t a(v)dv}. \tag{3.4}$$

Consequently, the integration of (3.4) from t_0 to t gives

$$\varphi(t) = \varphi_{t_0} e^{-\int_{t_0}^t a(v)dv} + \int_{t_0}^t e^{-\int_s^t a(v)dv} \varphi^2(s)\,ds + \int_{t_0}^t e^{-\int_s^t a(v)dv} b(s)\,ds.$$

Define the map T on K_β by the expression, for $\omega \in K_\beta$

$$(T\omega)(t) := \varphi_{t_0} e^{-\int_{t_0}^t a(v)dv} + \int_{t_0}^t e^{-\int_s^t a(v)dv} \omega^2(s)\,ds + \int_{t_0}^t e^{-\int_s^t a(v)dv} b(s)\,ds.$$

Observe that, if we choose φ_{t_0} such that

$$0 \leqslant \varphi_{t_0} e^{-\int_{t_0}^{t} a(v)dv} + \int_{t_0}^{t} e^{-\int_{s}^{t} a(v)dv} b(s)\,ds < \frac{1}{4\beta}, \tag{3.5}$$

then, one can show that T maps K_β into itself. Indeed, let $\omega \in K_\beta$, by making use of (3.1) and (3.1), we have

$$\begin{aligned} |T\omega(t)| &\leqslant \varphi_{t_0} e^{-\int_{t_0}^{t} a(v)dv} + \int_{t_0}^{t} e^{-\int_{s}^{t} a(v)dv} b(s)\,ds + \int_{t_0}^{t} e^{-\int_{s}^{t} a(v)dv} \omega^2(s)\,ds \\ &\leqslant \frac{1}{4\beta} + \|\omega\|^2 \int_{t_0}^{t} e^{-\int_{s}^{t} a(v)dv} ds \\ &\leqslant \frac{1}{4\beta} + \left(\frac{1}{2\beta}\right)^2 \beta \leqslant \frac{1}{2\beta}. \end{aligned}$$

That is, $T\omega \in K_\beta$. Further, T is a contraction on K_β, since for $\phi, \varphi \in K_\beta$

$$\begin{aligned} |T\phi(t) - T\varphi(t)| &\leqslant \int_{t_0}^{t} e^{-\int_{s}^{t} a(v)dv} \left|\phi^2(s) - \varphi^2(s)\right| ds \\ &\leqslant \|\phi - \varphi\| \int_{t_0}^{t} e^{-\int_{s}^{t} a(v)dv} |\phi(s) + \varphi(s)|\,ds \\ &\leqslant \frac{2}{2\beta} \|\phi - \varphi\| \int_{t_0}^{t} e^{-\int_{s}^{t} a(v)dv} ds \\ &\leqslant \frac{\sigma}{\beta} \|\phi - \varphi\|, \end{aligned}$$

where σ is defined in (3.1). Consequently, T has, by the contraction mapping principle, a unique solution φ^* which satisfies the following inequality

$$a(t) - \varphi^*(t) \geqslant a(t) - \frac{1}{2\beta} \geqslant \frac{1}{2\beta} - \frac{1}{2\beta} = 0,$$

from which the proof of the lemma becomes immediate. □

We now turn our attention to study the asymptotic stability of (1.3) and we begin first by transforming it into an equivalent equation which conserves the same properties.

LEMMA 3.2. *Suppose all conditions in Lemma 3.1 hold. Then, there exist $p, q : [t_0, \infty) \longrightarrow \mathbb{R}_+$ such that the equation (1.3) is equivalent to*

$$\begin{aligned} &\frac{d}{dt}(\dot{x}(t) + p(t)x(t)) + q(t)(\dot{x}(t) + p(t)x(t)) \\ &+ \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} a_j(t,s) g_j(s, x(s))\,ds = 0. \end{aligned} \tag{3.6}$$

for $t \in [t_0, \infty)$.

PROOF. It is clear that by differentiating the first term in brackets of equation (3.6) and by comparing it with the equation (1.3), we see that p is a solution of equation $p'+qp=0$ and $p+q=G$. Substituting $q=G-p$ into equation $p'+qp=0$, we obtain $p'+Gp=p^2$. By Lemma 3.1 we observe that $p,q:[t_0,\infty)\longrightarrow\mathbb{R}_+$. $\square$

LEMMA 3.3. *If $x(t)$ is a solution of equation (1.3) on an interval $[t_0,T)$ satisfying the initial condition $x(t)=\psi(t)$ on $[m(t_0),t_0]$ and $\dot{x}(t_0)=\dot{x}_{t_0}$, then $x(t)$ is a solution of the following integral equation*

$$\begin{aligned} x(t) &= x_{t_0}e^{-\int_{t_0}^{t}p(v)dv}+(\dot{x}_{t_0}+p_0x_{t_0})\int_{t_0}^{t}e^{-\int_s^t p(v)dv}e^{-\int_{t_0}^{s}q(v)dv}ds \\ &-\sum_{j=1}^{N}\int_{t_0}^{t}e^{-\int_s^t p(v)dv}\int_{t_0}^{u}e^{-\int_u^s q(v)dv}\int_{u-\tau_j(u)}^{u}a_j(u,v)g_j(v,x(v))\,dvduds, \end{aligned} \tag{3.7}$$

on $[t_0,T)$. Conversely, if a continuous function $x(\cdot)$ is equal to $\psi(\cdot)$ for $t\in[m(t_0),t_0]$ and is the solution of above integral equation on an interval $[t_0,T_1)$, then $x(\cdot)$ is a solution of (1.3) on $[t_0,T_1)$.

PROOF. By Lemma 3.2, equation (1.3) can be written as

$$\begin{aligned} &\frac{d}{dt}(\dot{x}(t)+p(t)x(t))+q(t)(\dot{x}(t)+p(t)x(t)) \\ &+\sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t}a_j(t,s)g_j(s,x(s))\,ds \;=\; 0. \end{aligned} \tag{3.8}$$

By letting $z(t):=\dot{x}(t)+p(t)x(t)$ for $t\geqslant t_0$, equation (3.8) can be expressed as

$$\dot{z}+q(t)z=-\sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t}a_j(t,s)g_j(s,x(s))\,ds.$$

Multiplying both sides of the above equation by $e^{\int_{t_0}^{t}q(v)dv}$ and integrating from t_0 to t, we obtain

$$\begin{aligned} \dot{x}(t)+p(t)x(t) &= (\dot{x}_{t_0}+p_0x_{t_0})\,e^{-\int_{t_0}^{t}q(v)dv} \\ &-\sum_{j=1}^{N}\int_{t_0}^{t}e^{-\int_u^t q(v)dv}\int_{u-\tau_j(u)}^{u}a_j(u,v)g_j(v,x(v))\,dvdu. \end{aligned} \tag{3.9}$$

Similarly, multiplying both sides of (3.9) by $e^{\int_{t_0}^{t}p(v)dv}$ and integrating from t_0 to t, we find exactly (3.7). Conversely, suppose that a continuous function $x(\cdot)$ is equal to $\psi(\cdot)$ on $[m(t_0),t_0]$ and satisfies (3.7) on an interval $[t_0,T_1)$. Then it is twice differentiable on $[t_0,T_1)$. Differentiating (3.7) with the aid of Leibniz's rule, we obtain (1.3). $\square$

Next, we shall define a mapping directly from (3.7) such that a fixed point of this map will be a solution of (3.7) and, hence, a solution of equation (1.3) by Lemma 3.3. To obtain stability of the zero solution of (1.3), we need the mapping defined by (3.7) to map bounded functions into bounded functions. For that, we let $(\mathcal{C}, \|\cdot\|)$ to be the Banach space of real-valued bounded continuous functions on $[m(t_0), \infty)$ with the supremum norm $\|\cdot\|$, that is for $\varphi \in \mathcal{C}$

$$\|\varphi\| := \sup \{|\varphi(t)| ;\ t \in [m(t_0), \infty)\} .$$

Our investigations will be carried out on the complete metric space $(\mathcal{C}, \rho)$, where ρ is the uniform metric. That is, for $\varphi, \phi \in \mathcal{C}$ we set $\rho(\varphi, \phi) := \|\varphi - \phi\|$.

Let $\psi \in \mathcal{C}([m(t_0), t_0], \mathbb{R})$ be fixed and define

$$S_\psi := \{\varphi : [m(t_0), \infty) \to \mathbb{R} \mid \varphi \in \mathcal{C},\ \varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0]\}. \tag{3.10}$$

Being closed in $\mathcal{C}$, (S_ψ, ρ) is itself complete. There is no confusion if we use the norm $\|\cdot\|$ on $[m(t_0), t_0]$ or on $[m(t_0), \infty)$.

Below we want to force the mapping suggested by (3.7) and explicitly defined in the next lemma to map S_ψ into itself. For that reason we assume that the followings conditions hold.

i.

$$\liminf_{t \longrightarrow \infty} \int_{t_0}^{t} p(s) ds > -\infty. \tag{3.11}$$

ii. There exists some functions $R_j(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}_+)$ such that, for x_1, $x_2 \in \mathbb{R}$

$$|g_j(t, x_1) - g_j(t, x_2)| \leqslant R_j(t) |x_1 - x_2|,\ j = 1, ..., N \text{ for all } t \in \mathbb{R}, \tag{3.12}$$

$$g_j(t, 0) = 0,\ j = 1, ..., N \quad \text{for } t \in \mathbb{R}_+. \tag{3.13}$$

iii. For $t \geqslant t_0$, there is a constant $\alpha > 0$ satisfying

$$\sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t p(v)dv} \int_{t_0}^{s} e^{-\int_u^s q(v)dv} \int_{u-\tau_j(u)}^{u} |a_j(u, v)|\, R_j(v)\, dvduds \leqslant \alpha. \tag{3.14}$$

There exist constants $a_0 > 0$, $\gamma > 0$, $Q_0 > 0$ such that, for each $t \geqslant u \geqslant Q_0 \geqslant t_0$ we have

$$\int_{u}^{t} p(v)\, dv + \int_{t_0}^{u} q(v)\, dv \geqslant a_0 u^\gamma + b,\ b \in \mathbb{R}. \tag{3.15}$$

Lemma 3.4. *Define the mapping P on S_ψ as follows, for $\varphi \in S_\psi$,*

$$(P\varphi)(t) = \psi(t) \text{ if } t \in [m(t_0), t_0],$$

while for $t > t_0$

$$P\varphi(t) = x(t_0)\, e^{-\int_{t_0}^t p(v)dv} + (\dot{x}(t_0) + p_0 x(t_0)) \int_{t_0}^{t} e^{-\int_s^t p(v)dv} e^{-\int_{t_0}^s q(v)dv} ds$$

$$- \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t p(v)dv} \int_{t_0}^{s} e^{-\int_u^s q(v)dv} \int_{u-\tau_j(u)}^{u} a_j(u, v) g_j(v, \varphi(v))\, dvduds. \tag{3.16}$$

Suppose that the conditions, (3.12), (3.13), (3.14) and (3.15) hold true. Then $P : S_\psi \to S_\psi$.

Proof. First, due to condition (3.11) one can define

$$M = \sup_{t \geqslant t_0} e^{-\int_{t_0}^{t} p(v)dv}. \tag{3.17}$$

Obviously, if φ is continuous then, due to the definition of P, $P\varphi$ is continuous and agrees with ψ on $[m(t_0), t_0]$. For $t > t_0$, note that from (3.11), (3.12), (3.13) and (3.14), it follows that

$$|P\varphi(t)| \leqslant \|\psi\| M + (|\dot{x}(t_0)| + p_0 \|\psi\|) \int_{t_0}^{t} e^{-\int_u^t p(v)dv} e^{-\int_{t_0}^{u} q(v)dv} du + \alpha \|\varphi\|.$$

To prove that $P : S_\psi \to S_\psi$ it is necessary to show that the term

$$\int_{t_0}^{t} e^{-\int_u^t p(v)dv} e^{-\int_{t_0}^{u} q(v)dv} du$$

is bounded. For that, we decompose the last integral term in the following manner

$$\begin{aligned} \int_{t_0}^{t} e^{-\int_u^t p(v)dv} e^{-\int_{t_0}^{u} q(v)dv} du = & \int_{t_0}^{J} e^{-\int_u^t p(v)dv} e^{-\int_{t_0}^{u} q(v)dv} du \\ & + \int_{J}^{t} e^{-\int_u^t p(v)dv} e^{-\int_{t_0}^{u} q(v)dv} du, \end{aligned} \tag{3.18}$$

for some $J \geqslant Q_0$. The first term on the right hand side of (3.18) is obviously bounded. For the second term on the right hand side of (3.18), we use (3.15) to have

$$\int_{J}^{t} e^{-\int_u^t p(v)dv} e^{-\int_{t_0}^{u} q(v)dv} du \leqslant e^{-b} \int_{J}^{t} e^{-a_0 u^\gamma} du. \tag{3.19}$$

Now, put

$$F(J) := \int_{J}^{\infty} e^{-a_0 u^\gamma} du. \tag{3.20}$$

Performing the change of variables $u = \theta^{\frac{1}{\gamma}}$, we obtain

$$F(J) = \frac{1}{\gamma} \int_{J^\gamma}^{\infty} e^{-a_0 \theta} \theta^{\frac{1}{\gamma}-1} d\theta \leqslant \frac{1}{\gamma} \int_{0}^{\infty} e^{-a_0 \theta} \theta^{\frac{1}{\gamma}-1} d\theta = \frac{\Gamma(1/\gamma)}{\gamma a_0^{1/\gamma}}. \tag{3.21}$$

Then, $F(J)$ is bounded for $\gamma > 0$. Consequently, $|P\varphi(t)| < +\infty$ and thus $P\varphi \in S_\psi$. □

Seizing upon Lemma 3.3 and Lemma 3.4 we built an existence and uniqueness result. Under the conditions of the next theorem, we prove that for a given continuous function $\psi : [m(t_0), t_0] \longrightarrow \mathbb{R}$ there exists a unique continuous function x which is solution of (1.3) on $[m(t_0), \infty)$ and coincides with ψ on $[m(t_0), t_0]$. We also prove that the zero solution of (1.3) have the property of Definition 2.1.

Theorem 3.1. *Suppose all hypotheses of Lemma 3.4 hold with $\alpha \in (0, 1)$ in (3.14). Then, for each initial continuous function $\psi : [m(t_0), t_0] \longrightarrow \mathbb{R}$, there is a unique continuous function $x : [m(t_0), \infty) \to \mathbb{R}$ with $x(t) = \psi(t)$ on $[m(t_0), t_0]$ that* *satisfies (1.3) on* $[t_0, \infty)$. *Moreover,* $x(\cdot)$ *is bounded on* $[m(t_0), \infty)$. *Furthermore, the zero solution of (1.3) is stable at* $t = t_0$.

PROOF. Consider S_ψ the space defined by the initial continuous function $\psi : [m(t_0), t_0] \to \mathbb{R}$ given by (3.10). By Lemma 3.4 we know that $P : S_\psi \to S_\psi$. In fact, P is a contraction with constant $\alpha < 1$ too. To see this, let $\varphi, \phi \in S_\psi$. Making use of condition (3.14) we obtain

$$\begin{aligned}
&\|P\varphi - P\phi\| \\
&\leqslant \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t p(v)dv} \int_{t_0}^{s} e^{-\int_u^s q(v)dv} \int_{u-\tau_j(u)}^{u} |a_j(u,v)| \, R_j(v) \, |\varphi(u) - \varphi(v)| \, dvduds \\
&\leqslant \alpha \|\varphi - \phi\| .
\end{aligned}$$

for $t > t_0$. Trivially, this inequality also holds on $[m(t_0), t_0]$. Therefore, P is a contraction mapping on the complete metric space (S_ψ, ρ) since we have supposed $\alpha < 1$. By the contraction mapping principle, P possesses a unique fixed point x in S_ψ which is bounded continuous function. Due to Lemma 3.4, this is a solution of (1.3) and hence a solution of (1.3) on $[m(t_0), \infty)$. It follows that x is the only bounded function satisfying (1.3) on $[m(t_0), \infty)$ and the initial function ψ.

It remains to show that the zero solution of (1.3) is stable. Toward this, put

$$L := \sup_{t \geqslant t_0} \int_{t_0}^{t} e^{-\int_u^t p(v)dv} e^{-\int_{t_0}^u q(v)dv} du. \tag{3.22}$$

Let $\varepsilon > 0$ be given. Choose $|\dot{x}(t_0)|$ and $\psi : [m(t_0), t_0] \longrightarrow \mathbb{R}$ such that $\|\psi\| < \delta$ ($\delta \leqslant \varepsilon$), and

$$M\delta + (|\dot{x}(t_0)| + p_0\delta) L \leqslant (1 - \alpha) \varepsilon. \tag{3.23}$$

If $x(t)$ is a solution of (1.3) with $\dot{x}(t_0) = \dot{x}_{t_0}$ then, $x(\cdot) = (Px)(\cdot)$ as defined in (3.16). Notice that with such a δ, $|x(s)| = |\psi(s)| < \varepsilon$ on $[m(t_0), t_0]$. We claim that $|x(t)| < \varepsilon$ for all $t \geqslant t_0$. If x is a solution with initial function ψ then, as consequence of (3.16), we have

$$\begin{aligned}
|x(t)| &\leqslant M\delta + (|\dot{x}(t_0)| + p_0\delta) L + \varepsilon\alpha \\
&\leqslant (1 - \alpha) \varepsilon + \varepsilon\alpha \leqslant \varepsilon.
\end{aligned} \tag{3.24}$$

Therefore, the zero solution is stable at $t = t_0$. □

THEOREM 3.2. *Under the hypotheses of Theorem 3.1 the zero solution of (1.3) is asymptotically stable if and only if*

$$\int_0^t p(s)ds \longrightarrow \infty, \text{ as } t \longrightarrow \infty. \tag{3.25}$$

PROOF. First, suppose that (3.25) holds. We wish the solutions of (1.3) to tend to zero whenever condition (3.25) holds. For this, we will modify S_ψ in order to receipt functions that tends to zero as $t \longrightarrow \infty$. So, we let

$$S_\psi^0 := \{\varphi \in [m(t_0),\infty) \to \mathbb{R} \mid \varphi \in \mathcal{C},\\ \varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0] \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Since S_ψ^0 is closed in S_ψ and (S_ψ, ρ) is complete, then the metric space $\left(S_\psi^0, \rho\right)$ is also complete. We begin by proving that $P\varphi(t) \to 0$ as $t \to \infty$ whenever $\varphi \in S_\psi^0$. To this end, denote the three terms on the right hand side of (3.16) by I_1, I_2, I_3 respectively and let $\varphi \in S_\psi^0$ be fixed. Since $\int_0^t p(s)ds \longrightarrow \infty$, as $t \longrightarrow \infty$, by condition (3.25), we see obviously that the first term I_1 of (3.16) tends to zero as $t \longrightarrow \infty$. We check that $I_2 \longrightarrow 0$ as $t \longrightarrow \infty$. To do this, we have to prove that the two right hand side terms of the decomposition expression (3.18) go to zero at infinity. But the first term of that decomposition is as

$$\int_{t_0}^{J} e^{-\int_u^t p(v)dv} e^{-\int_{t_0}^u q(v)dv} du = e^{-\int_J^t p(v)dv} \int_{t_0}^{J} e^{-\int_u^J p(v)dv} e^{-\int_{t_0}^u q(v)dv} du,$$

which tends to 0 as $t \longrightarrow \infty$ by condition (3.25). Nevertheless, the second term on the right had side of (3.18) needs some more details to prove that it converges to zero. To overcome the difficulties, remember that, upon replacing u by $J\theta$ in (3.20), we get

$$F(J) = J \int_1^\infty e^{-(a_0 J^\gamma)\theta^\gamma} d\theta.$$

The function $G(\lambda) := \int_1^\infty e^{-\lambda\theta^\gamma} d\theta$ satisfies the conditions of Theorem 2.1 where

$$\lambda = a_0 J^\gamma,\ \alpha = 1,\ \varphi(\theta) = \theta^\gamma,\ f \equiv 1,\ \varphi'(\alpha) = \gamma\alpha^{\gamma-1} = \gamma,\ f(\alpha) = 1.$$

It follows that

$$G(\lambda) \sim \frac{f(\alpha)}{\varphi'(\alpha)} \frac{1}{\lambda} e^{-\lambda\varphi(\alpha)} = \frac{1}{\gamma}\frac{1}{\lambda} e^{-\lambda}, \quad (\lambda \longrightarrow +\infty).$$

Thus we can write

$$F(J) \sim \frac{1}{\gamma a_0} J^{1-\gamma} e^{-a_0 J^\gamma},\ (J \longrightarrow +\infty).$$

It is enough to make $z = a_0 J^\gamma$ and a straightforward computation gives

$$\frac{1}{\gamma a_0} J^{1-\gamma} e^{-a_0 J^\gamma} = \frac{1}{\gamma a_0^{1/\gamma}} z^{\frac{1}{\gamma}-1} e^{-z} \leqslant \frac{1}{\gamma a_0^{1/\gamma}} z^m e^{-z} \longrightarrow 0 \text{ as } z \longrightarrow \infty.$$

where $m := [1/\gamma] + 1$. Thus, for every $\varepsilon > 0$ we can find a $J^* \gg Q_0$ large enough such that for every $J \geqslant J^*$

$$\frac{e^{-b}}{\gamma a_0} J^{1-\gamma} e^{-a_0 J^\gamma} \leqslant \varepsilon.$$

Clearly, the expansion (3.18) is valid if J is replaced by J^*. So, the last term tends towards zero when $t \longrightarrow \infty$. Hence the second term I_2 in (3.16) tends to zero as

$t \longrightarrow \infty$. We, now turn to I_3. To simplify, we define

$$V(s) := \sum_{j=1}^{N} \int_{t_0}^{s} e^{-\int_u^s q(v)dv} \int_{u-\tau_j(u)}^{u} |a_j(u,v)| R_j(v)\, dvdu. \tag{3.26}$$

So, for the given $\varepsilon > 0$, there exists a $T^* > t_0$ such that $s \geqslant T^*$ implies $|\varphi(v)| < \varepsilon$ for $j = \overline{1,N}$. It is clear that $|g_j(v, \varphi(v))| < R_j(v)\varepsilon$. Thus, for $t \geqslant T^*$, by making use of conditions (3.12) and (3.13) the term I_3 satisfies

$$\begin{aligned} I_3 &\leqslant \sup_{\zeta \geqslant m(t_0)} |\varphi(\zeta)| \int_{t_0}^{T^*} V(s)\, e^{-\int_s^t p(v)dv} ds + \varepsilon \int_{T^*}^{t} V(s)\, e^{-\int_s^t p(v)dv} ds \\ &\leqslant \alpha\varepsilon + \sup_{\zeta \geqslant m(t_0)} |\varphi(\zeta)| \int_{t_0}^{T^*} V(s)\, e^{-\int_s^t p(v)dv} ds. \end{aligned}$$

Also, the conditions (3.25) implies that, there exists $T^{**} > T^*$ such that for $t \geqslant T^{**}$ we have

$$e^{-\int_{T^{**}}^t p(v)dv} \sup_{\zeta \geqslant m(t_0)} |\varphi(\zeta)| \int_{T^*}^{t} V(s)\, e^{-\int_s^{T^{**}} p(v)dv} ds \leqslant \varepsilon.$$

So, $I_3 \longrightarrow 0$ as $t \longrightarrow \infty$. Consequently, $(P\varphi)(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, P maps S_ψ^0 into itself. Also, P is still a contraction on S_ψ^0 with a unique fixed point x. By Lemma 3.3, x is a solution of (1.3) on $[t_0, \infty)$. We conclude that $x(t)$ is the only continuous solution of (1.3) agreeing with the initial function ψ. As $x \in S_\psi^0$, $x(t) \to 0$ as $t \to \infty$. Therefore, the zero solution is asymptotically stable, since it is stable by Theorem 3.1 and we have just concluded that $|x(t)| \longrightarrow 0$ as $t \longrightarrow \infty$ if condition (3.25) holds.

Conversely, we shall prove that $\int_{t_0}^{\infty} p(v)dv = \infty$. Contrary to this, there exists a sequence $\{t_n\}_{n\geqslant 1}$ with $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and such that $\int_{t_0}^{t_n} p(v)dv = l$ for a certain finite number $l \in \mathbb{R}_+$. By condition (3.11), we may also choose $\mu > 0$ that satisfies the inequality, $-\mu \leqslant \int_{t_0}^{t_n} p(v)dv \leqslant \mu$, for all $n \geqslant 1$. Recalling (3.26) and using condition (3.14) we have

$$\int_{t_0}^{t_n} e^{-\int_{t_0}^{t_n} p(v)dv} V(s)\, ds \leqslant \alpha,$$

we deduce

$$e^{-\int_{t_0}^{t_n} p(v)dv} \int_{t_0}^{t_n} e^{\int_{t_0}^{s} p(v)dv} V(s)\, ds \leqslant \alpha.$$

This yields

$$\int_{t_0}^{t_n} e^{\int_{t_0}^{s} p(v)dv} V(s)\, ds \leqslant \alpha e^{\mu}. \tag{3.27}$$

The inequality (3.27) leads to the fact that the sequence

$$\int_{t_0}^{t_n} e^{\int_{t_0}^{s} p(v)dv} V(s)\, ds, \tag{3.28}$$

is bounded, so it has a convergent subsequence. For brevity, we assume that

$$\lim_{n\longrightarrow\infty}\int_0^{t_n} e^{\int_0^s p(v)dv}V(s)\,ds=\sigma>0.$$

Then, we can choose a positive integer n_0 large enough such that

$$\int_{t_{n_0}}^{t_n} e^{\int_{t_0}^s p(v)dv}V(s)\,ds<\frac{\delta_0}{2M}.$$

for $n\geqslant n_0$, where $\varepsilon>\delta_0>0$ satisfies

$$|\psi(t_{n_0})|\,M+\left(\left(|\dot{x}(t_{n_0})|+|p_0x(t_{t_0})|\right)\right)L\leqslant(1-\alpha).$$

Now, we consider the solution $x(t)=x(t,\psi,\dot{x}(t_{n_0}))$ of equation (1.3), for the initial values ψ and $\dot{x}(t_{n_0})$ such that

$$x(t_{n_0})=\dot{x}(t_{n_0})=\delta_0,\quad p_0=\frac{1}{4\beta}$$
$$|\psi(s)|\leqslant\delta_0,\qquad s\leqslant t_{n_0}.$$

By a similar argument as in (3.24) and by replacing ε by 1, this implies that $|x(t)|\leqslant 1$. Having in mind the fact that x is a fixed point of P, we have, for $n\geqslant n_0$

$$\begin{aligned}|x(t_n)|&\geqslant\delta_0e^{-\int_{t_{n_0}}^{t_n}p(v)dv}-\left|\int_{t_{n_0}}^{t_n}e^{-\int_s^{t_n}p(v)dv}V(s)ds\right|\\&\geqslant e^{-\int_{t_{n_0}}^{t_n}p(v)dv}\left[\delta_0-\left(e^{-\int_0^{t_{n_0}}p(v)dv}\right)\int_{t_{n_0}}^{t_n}e^{\int_0^s p(v)dv}V(s)ds\right]\\&\geqslant e^{-\int_{t_{n_0}}^{t_n}p(v)dv}\left[\delta_0-\frac{\delta_0}{2}\right]\geqslant\frac{\delta_0}{2}e^{-2\mu}>0.\end{aligned}$$

On the other hand, if the zero solution is asymptotically stable, then $x(t)=x(t,\psi,\dot{x}(t_{n_0}))\to 0$, as $t\to\infty$. But this leads leads to a contradiction. This completes the proof of our claim. □

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