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WEAK-INTERIOR IDEALS AND FUZZY WEAK-INTERIOR IDEALS OF $\Gamma-$SEMIRINGS

Marapureddy Murali Krishna Rao

Abstract. In this paper, we introduce the notion of weak-interior ideal and fuzzy weak-interior ideal of $\Gamma-$semiring. In addition, we characterize the simple $\Gamma-$semiring and the regular $\Gamma-$semiring in terms of fuzzy weak-interior ideals of $\Gamma-$ semiring.

1. Introduction

The notion of a one sided ideal of any algebraic structure is a generalization of the notion of an ideal. The quasi ideals are generalization of left ideals and right ideals whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [**2**] for semigroups. In 1976, the concept of interior-ideals was introduced by Lajos [**10**] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [**11**]. In 1956, Steinfeld [**47**] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [**4, 5, 6**] introduced the concept of quasi ideal for a semiring. Henriksen [**3**] studied ideals in semirings. Quasi ideals in $\Gamma-$semirings studied by Jagtap and Pawar [**8, 9**]. As a generalization of ideal, r-ideals, left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of $\Gamma-$semiring, M. M. Krishna Rao et al. [**28, 29, 30, 33, 34, 35, 31, 26, 36**] introduced and studied bi-interior ideals, quasi-interior ideal, bi-quasi-ideals, tri-ideals and bi-quasi- interior ideals in $\Gamma-$semigroups, semigroups, semirings and $\Gamma-$semirings. M. M. Krishna Rao [**38**] introduced and studied the notion of fuzzy quasi-interior ideal of $\Gamma-$semiring.

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He characterizes the regular $\Gamma-$semiring in terms of fuzzy quasi-interior ideal of $\Gamma-$semiring and studied some of their properties.

Semiring is the algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by Vandiver [**49**] in 1934 but non-trivial examples of semirings had appeared in the earlier studies on the theory of commutative ideals of rings by Dedekind in 19th century. In 1995, M. M. Krishna Rao [**17, 18, 24, 19, 22**] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [**16**] in 1964. The notion of a ternary algebraic system was introduced by Lehmer [**12**] in 1932. In 1971, Lister [**13**] introduced ternary ring. The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a $\Gamma-$semiring where $\Gamma = \mathbf{Z}$. Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. In 1981, Sen [**45**] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup.

The algebraic structure plays a prominent role in mathematics with wide range of applications. Generalization of ideals of algebraic structures and ordered algebraic structure plays a very remarkable role and also necessary for further advance studies and applications of various algebraic structures. During 1950-1980, the concepts of bi-ideals, quasi ideals and interior ideals were studied by many mathematicians and the applications of these ideals only studied by mathematicians in the period 1950-2019. Between 1980 and 2016 there have been no new generalization of these ideals of algebraic structures. Then the author [**28, 29, 30, 33, 34, 35, 31, 26**] introduced and studied bi quasi ideals, bi-interior ideals, tri-ideals, bi quasi interior ideals and quasi interior ideals of $\Gamma-$semirings,semirings, $\Gamma-$semigroups,semigroups as a generalization of bi-ideal, quasi ideal and interior ideal of algebraic structures and characterized regular algebraic structures as well as simple algebraic structures using these ideals.

The fuzzy set theory was developed by L. A. Zadeh [**50**] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by A. Rosenfeld [**44**] and he introduced the notion of fuzzy subgroups in 1971. N. Kuroki studied fuzzy semigroups. K.L. N. Swamy and U. M. Swamy [**48**] studied fuzzy prime ideals in rings in 1988. D. Mandal [**15**] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. M. M. Krishna Rao [**25, 20, 32, 27, 21, 23, 37, 38, 39, 40, 41, 42, 43**] studied fuzzy k-ideals, $T-$fuzzy ideals of ordered $\Gamma-$semirings.

In this paper, we introduce the notion of weak-interior ideal and fuzzy weak-interior ideal of $\Gamma-$semiring and characterize the simple $\Gamma-$ semiring and the regular $\Gamma-$semiring in terms of fuzzy weak-interior ideals of $\Gamma-$semiring.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

DEFINITION 2.1. ([**1**]) A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) Addition is a commutative operation.

(ii) Multiplication distributes over addition both from the left and from the right.

(iii) There exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

DEFINITION 2.2. Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$semigroup if it satisfies

(i) $x\alpha y \in M$, and

(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

DEFINITION 2.3. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. For a $\Gamma-$semigroup M it is said to be a $\Gamma-$semiring M if for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, it satisfies the following axioms:

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication.

DEFINITION 2.4. A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$.

EXAMPLE 2.1. Let M be the additive semi group of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers, then with respect to usual matrix multiplication M is a $\Gamma-$semiring.

DEFINITION 2.5. Let M be a $\Gamma-$semiring and A be a non-empty subset of M. A is called a $\Gamma-$subsemiring of $\Gamma-$semiring M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

DEFINITION 2.6. Let M be a $\Gamma-$semiring. A subset A of M is called a left (right) ideal of $\Gamma-$semiring M if A is closed under addition and $M\Gamma A \subseteq A$ ($A\Gamma M \subseteq A$). A is called an ideal of M if it is both a left ideal and a right ideal of M .

DEFINITION 2.7. Let M be a $\Gamma-$semiring. An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

DEFINITION 2.8. Let M be a $\Gamma-$semiring. If every element of M is a regular, then M is said to be regular $\Gamma-$semiring.

DEFINITION 2.9. An element $a \in M$ is said to be idempotent of M if $a = a\alpha a$ for some $\alpha \in \Gamma$.

DEFINITION 2.10. A non-empty subset A of a Γ-semiring M is called

(i) a Γ-subsemiring of M if $(A,+)$ is a subsemigroup of $(M,+)$ and $A\Gamma A \subseteq A$.
(ii) a quasi ideal of M if A is a Γ-subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) a bi-ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.
(iv) an interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.
(v) a left (right) ideal of M if A is a Γ-subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(vi) an ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.
(vii) a $k-$ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x+y \in A, y \in A$ then $x \in A$.
(viii) a bi-interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.
(ix) a left bi-quasi ideal (right bi-quasi ideal) of M if A is a subsemigroup of $(M,+)$ and

$$M\Gamma A \cap A\Gamma M\Gamma A \subseteq A\ (A\Gamma M \cap A\Gamma M\Gamma A \subseteq A).$$

(x) a bi-quasi ideal of M if B is a $\Gamma-$subsemiring of M and $B$is a left bi-quasi ideal and a right bi-quasi ideal of M.
(xi) a left quasi-interior ideal (right quasi-interior ideal) of M if A is a Γ-subsemiring of M and

$$M\Gamma A\Gamma M\Gamma A \subseteq A\ (A\Gamma M\Gamma A\Gamma M \subseteq A).$$

(xii) a quasi-interior of M if B is a $\Gamma-$subsemiring of M and $B$is a left quasi-interior ideal and a right quasi-interior ideal of M.
(xiii) a bi-quasi-interior ideal of M if A is a Γ-subsemiring of M and

$$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B.$$

(xiv) a left tri- ideal (right tri- ideal) of M if A is a Γ-subsemiring of M and

$$A\Gamma M\Gamma A\Gamma A \subseteq A\ (A\Gamma A\Gamma M\Gamma A \subseteq A).$$

(xv) a tri- ideal of M if A is a Γ-subsemiring of M and

$$A\Gamma M\Gamma A\Gamma A \subseteq A \text{ and } A\Gamma A\Gamma M\Gamma A \subseteq A.$$

DEFINITION 2.11. A $\Gamma-$semiring M is a left (right) simple $\Gamma-$semiring if M has no proper left (right) ideal of M A $\Gamma-$semiring M is a bi-quasi simple $\Gamma-$semiring if M has no proper bi-quasi ideal of M A $\Gamma-$semiring M is said to be simple $\Gamma-$semring if M has no proper ideals.

DEFINITION 2.12. Let M be a non-empty set.A mapping $f : M \to [0,1]$ is called a fuzzy subset of $\Gamma-$semiring M. If f is not a constant function then f is called a non-empty fuzzy subset

DEFINITION 2.13. Let f be a fuzzy subset of a non-empty set M, for $t \in [0,1]$ the set $f_t = \{x \in M \mid f(x) \geqslant t\}$ is called a level subset of M with respect to f.

DEFINITION 2.14. Let M be a $\Gamma-$semiring.A fuzzy subset μ of M is said to be fuzzy $\Gamma-$subsemiring of M if it satisfies the following conditions
(*i*) $\mu(x+y) \geqslant min\ \{\mu(x), \mu(y)\}$
(*ii*) $\mu(x\alpha y) \geqslant min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

DEFINITION 2.15. A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left (right) ideal of M if for all $x, y \in M, \alpha \in \Gamma$ it satisfies the following conditions
(*i*) $\mu(x+y) \geqslant min\{\mu(x), \mu(y)\}$
(*ii*) $\mu(x\alpha y) \geqslant \mu(y)\ (\mu(x))$, for all $x, y \in M, \alpha \in \Gamma$.

DEFINITION 2.16. A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy ideal of M if it satisfies the following conditions
(*i*) $\mu(x+y) \geqslant min\{\mu(x), \mu(y)\}$
(*ii*) $\mu(x\alpha y) \geqslant max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

DEFINITION 2.17. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leqslant \mu(x)$ for all $x \in M$.

DEFINITION 2.18. ([**8**]) Let f and g be fuzzy subsets of a $\Gamma-$semiring M. Then $f \circ g, f+g, f \cup g, f \cap g$, are defined by

$$f \circ g(z) = \begin{cases} \sup\limits_{z=x\alpha y} \{min\{f(x), g(y)\}\}, \\ 0, \quad \text{otherwise.} \end{cases} ;\ f+g(z) = \begin{cases} \sup\limits_{z=x+y} \{min\{f(x), g(y)\}\}, \\ 0, \quad \text{otherwise} \end{cases}$$

$$f \cup g(z) = max\{f(z), g(z)\}\ ;\ f \cap g(z) = min\{f(z), g(z)\}$$

for all $x, yz \in M, \alpha \in \Gamma$..

DEFINITION 2.19. A function $f : R \to M$ where R and M are $\Gamma-$semirings is said to be $\Gamma-$semiring homomorphism if $f(a+b) = f(a) + f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R, \alpha \in \Gamma$.

DEFINITION 2.20. Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M, defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

3. Weak-interior ideals of $\Gamma-$semirings

In this section, we introduce the notion of weak-interior ideal as a generalization of quasi-ideal and interior ideal of $\Gamma-$semiring and study the properties of weak-interior ideal of $\Gamma-$semiring.Throughout this paper$M$ is a $\Gamma-$semiring with unity element.

DEFINITION 3.1. A non-empty subset B of a $\Gamma-$semiring M is said to be left weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and $M\Gamma B\Gamma B \subseteq B$.

DEFINITION 3.2. A non-empty subset B of a $\Gamma-$semiring M is said to be right weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and $B\Gamma B\Gamma M \subseteq B$.

Definition 3.3. A non-empty subset B of a $\Gamma-$semiring M is said to be weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and $B$is a left weak-interior ideal and a right weak-interior ideal of M.

Remark: A weak-interior ideal of a $\Gamma-$semiring M need not be quasi-ideal, interior ideal, bi-interior ideal. and bi-quasi ideal of $\Gamma-$semiring M.

Example 3.1. Let Q be the set of all rational numbers,

$$M = \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in Q \right\}$$

be the additive semigroup of matrices and $\Gamma = M$. The ternary operation $A\alpha B$ is defined as usual matrix multiplication of A, α, B, for all $A, \alpha, B \in M$.Then M is a $\Gamma-$semiring If $R = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq b \in Q \right\}$ then R is a left weak interior ideal of the $\Gamma-$semiring M and R is neither a left ideal nor a right ideal, not a weak interior ideal and not a interior ideal of the $\Gamma-$semiring M.

In the following theorem, we mention some important properties and we omit the proofs since they are straight forward.

Theorem 3.1. *Let M be a $\Gamma-$semring. Then the following are hold.*

(1) *Every left ideal is a left weak-interior ideal of M.*
(2) *Every right ideal is a right weak-interior ideal of M.*
(3) *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M.If $M\Gamma M\Gamma B \subseteq B$ and$B\Gamma M\Gamma M \subseteq B$ then B is a weak-interior ideal of M.*
(4) *Every ideal is a weak-interior ideal of M.*
(5) *If B is a weak-interior ideal and T is an interior ideal of M then $B \cap T$ is a weak-interior ideal of ring M.*
(6) *If L is a $\Gamma-$subsemiring of M and R is a right ideal of a $\Gamma-$semiring M then $B = L\Gamma R$ is a weak-interior ideal of M.*
(7) *Let B be a $\Gamma-$subsemiring of a $\Gamma-$semiring M. If $M\Gamma M\Gamma M\Gamma B \subseteq B$ then B is a left weak-interior ideal of M.*

Theorem 3.2. *If B be an interior ideal of a $\Gamma-$semiring M, then B is a left weak-interior ideal of M.*

Proof. Suppose B is an interior ideal of the $\Gamma-$semiring M. Then $M\Gamma B\Gamma B \subseteq M\Gamma B\Gamma M \subseteq B$. Hence B is a left weak-interior ideal of M. □

Corollary 3.1. *If B be an interior ideal of a $\Gamma-$semiring M, then B is a right weak-interior ideal of M.*

Corollary 3.2. *If B be an interior ideal of a $\Gamma-$semiring M, then B is a weak-interior ideal of M.*

Theorem 3.3. *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M. B is a weak-interior ideal of M if and only if there exists left ideal L such that $L\Gamma B \subseteq B \subseteq L$.*

Proof. Suppose B is a weak-interior ideal of the $\Gamma-$semiring M. Then

$$M\Gamma B\Gamma B \subseteq B.$$

Let $L = M\Gamma B$. Then L is a left ideal of M. Therefore $L\Gamma B \subseteq B \subseteq L$.

Conversely suppose that there exists left ideal L of M such that $L\Gamma B \subseteq B \subseteq L$. Then $M\Gamma B\Gamma B \subseteq M\Gamma(L)\Gamma(B) \subseteq L\Gamma(B) \subseteq B$. Hence B is a left weak-interior ideal of M. □

Corollary 3.3. *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M. B is a right weak-interior ideal of M if and only if there exist right ideal R such that $B\Gamma R \subseteq B \subseteq R$. .*

Theorem 3.4. *The intersection of a left weak-interior ideal B of a $\Gamma-$semiring M and a left ideal A of M is always a left weak-interior ideal of M.*

Proof. Suppose $C = B \cap A$. Then $M\Gamma C\Gamma C \subseteq M\Gamma B\Gamma B \subseteq B$ and $M\Gamma C\Gamma C \subseteq M\Gamma A\Gamma A \subseteq A$ since A is a left ideal of M. Therefore $M\Gamma C\Gamma C \subseteq B \cap A = C$. Hence, the intersection of a left weak-interior ideal B of a $\Gamma-$semiring M and a left ideal A of M is always a left weak-interior ideal of M. □

Corollary 3.4. *The intersection of a right weak-interior ideal B of a $\Gamma-$semiring M and a right ideal A of M is always a right weak-interior ideal of M.*

Corollary 3.5. *The intersection of a weak-interior ideal B of a $\Gamma-$semiring M and an ideal A of M is always a weak-interior ideal of M.*

Theorem 3.5. *Let A and C be $\Gamma-$subsemirings of a $\Gamma-$semiring M and $B = A\Gamma C$ and B is an additively subsemigroup of M. If A is the left ideal of M then B is a quasi-interior ideal of M.*

Proof. Let A and C be $\Gamma-$subsemirings of M and $B = A\Gamma C$. Suppose A is the left ideal of M. Then $B\Gamma B = A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B$. Thus $M\Gamma B\Gamma B = M\Gamma A\Gamma C\Gamma A\Gamma C$ and $M\Gamma B\Gamma B \subseteq A\Gamma C = B$. Hence B is a left weak-interior ideal of M. □

Corollary 3.6. *Let A and C be $\Gamma-$subsemirings of a $\Gamma-$semiring M and $B = A\Gamma C$and B is an additively subsemigroup of M. If C is a right ideal then B is a right weak-interior ideal of M.*

Theorem 3.6. *If B is a left weak-interior ideal of a $\Gamma-$semiring M, $B\Gamma T$ is an additively subsemigroup of M and $T \subseteq B$ then $B\Gamma T$ is a left weak-interior ideal of M.*

Proof. Suppose B is a left weak-interior ideal of the $\Gamma-$semiring M, $B\Gamma T$ is an additively subsemigroup of M and $T \subseteq B$. Then $B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma T$. Hence $B\Gamma T$ is a $\Gamma-$subsemiring of M. We have $M\Gamma B\Gamma T\Gamma B\Gamma T \subseteq M\Gamma B\Gamma B\Gamma T \subseteq B\Gamma T$. Hence $B\Gamma T$ is a left weak-interior ideal of the $\Gamma-$semiring M. □

Theorem 3.7. *Let B and I be a left weak interior ideals of a $\Gamma-$semiring M. Then $B \cap I$ is a left weak-interior ideal of M.*

Proof. Suppose $B$and I are left weak interior ideals of M. Obviously $B \cap I$ is a $\Gamma-$subsemiring of M. Then

$$M\Gamma(B \cap I)\Gamma(B \cap I) \subseteq M\Gamma B\Gamma B \subseteq B$$
$$M\Gamma(B \cap I)\Gamma(B \cap I) \subseteq M\Gamma I\Gamma I \subseteq I$$

Therefore

$$M\Gamma(B \cap I)\Gamma(B \cap I) \subseteq B \cap I.$$

Hence $B \cap I$ is a left weak-interior ideal of M. □

Theorem 3.8. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ left weak-interior ideals of a $\Gamma-$semiring M is a left weak-interior ideal of M.*

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then B is a $\Gamma-$subsemiring of M. Since B_λ is a left weak-interior ideal of M, we have $M\Gamma B_\lambda \Gamma B_\lambda \subseteq B_\lambda$ for all $\lambda \in A$. Then $M\Gamma \cap B_\lambda \cap B_\lambda \subseteq \cap B_\lambda$. Thus $M\Gamma B\Gamma B \subseteq B$. Hence B is a left weak-interior ideal of M. □

Corollary 3.7. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ right weak-interior ideals of a $\Gamma-$semiring M is a right weak-interior ideal of M.*

Corollary 3.8. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ weak-interior ideals of a $\Gamma-$semiring M is a weak-interior ideal of M.*

Theorem 3.9. *Let B be a right weak-interior ideal of a $\Gamma-$semiring M ,$e \in B$ and e be $\beta-$idempotent such that $e\Gamma B \subseteq B$. Then $e\Gamma B$ is a right weak-interior ideal of M.*

Proof. Let B be a right weak-interior ideal of the $\Gamma-$semiring M. Suppose $x \in B \cap e\Gamma M$. Then $x \in B$ and $x = e\alpha y, \alpha \in \Gamma, y \in M$.

$$x = e\alpha y = e\beta e\alpha y = e\beta(e\alpha y) = e\beta x \in e\Gamma B.$$

Therefore $B \cap e\Gamma M \subseteq e\Gamma B$ and $e\Gamma B \subseteq B$ and $e\Gamma B \subseteq e\Gamma M$. Thus $e\Gamma B \subseteq B \cap e\Gamma M$ and $e\Gamma B = B \cap e\Gamma M$. Hence $e\Gamma B$ is a right weak-interior ideal of M. □

Corollary 3.9. *Let M be a $\Gamma-$semiring M and e be $\alpha-$idempotent. Then $e\Gamma M$ and $M\Gamma e$ are right weak-interior ideal and left weak-interior ideal of M respectively.*

Theorem 3.10. *Let M be a $\Gamma-$semiring. If $M = M\Gamma a$, for all $a \in M$. Then every left weak-interior ideal of M is a quasi ideal of M.*

Proof. Let B be a left weak-interior ideal of the $\Gamma-$semiring M and $a \in B$. Then $M\Gamma a \subseteq M\Gamma B$ and $M \subseteq M\Gamma B \subseteq M$. Thus $M\Gamma B = M$ and $B\Gamma M = B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B\Gamma B \subseteq B$. Now $M\Gamma B \cap B\Gamma M \subseteq M \cap B\Gamma M \subseteq B\Gamma M \subseteq B$. Therefore B is a quasi ideal of M. Hence the theorem. □

Theorem 3.11. *B is a left weak-interior ideal of a $\Gamma-$semiring M if and only if B is a left ideal of some left ideal of a $\Gamma-$semiring M.*

Proof. Suppose B is a left ideal of left ideal L of the $\Gamma-$semiring M.Then $L\Gamma B \subseteq B, M\Gamma L \subseteq L$ and $M\Gamma B\Gamma B \subseteq M\Gamma L\Gamma B \subseteq L\Gamma B \subseteq B$. Therefore B is a left weak-interior ideal of the $\Gamma-$semiring M .

Conversely suppose that B is a left weak-interior ideal of the $\Gamma-$semiring M. Then $M\Gamma B\Gamma B \subseteq B$. Therefore B is a left ideal of left ideal $M\Gamma B$ of the $\Gamma-$semiring M. □

Corollary 3.10. *B is a right weak-interior ideal of a $\Gamma-$semiring M if and only if B is a right ideal of some right ideal of a $\Gamma-$semiring M.*

Corollary 3.11. *B is a weak-interior ideal of a $\Gamma-$semiring M if and only if B is an ideal of some ideal of a $\Gamma-$semiring M.*

4. Weak-interior simple $\Gamma-$semirhing

In this section, we introduce the notion of left(right) weak-interior simple $\Gamma-$semiring and characterize the left weak-interior simple $\Gamma-$semiring using left weak-interior ideals of $\Gamma-$semiring and study the properties of minimal left weak-interior ideals of $\Gamma-$semiring..

Definition 4.1. A $\Gamma-$semiring M is a left (right) simple $\Gamma-$semiring if M has no proper left (right) ideals of M.

Definition 4.2. A $\Gamma-$semiring M is said to be simple $\Gamma-$semiring if M has no proper ideals of M.

Definition 4.3. A $\Gamma-$semiring M is said to be left (right) weak-interior simple $\Gamma-$semiring if M has no left (right) weak-interior ideal other than M itself.

Definition 4.4. A $\Gamma-$semiring M is said to be weak-interior simple $\Gamma-$semiring if M has no weak-interior ideal other than M itself.

Theorem 4.1. *If M is a division $\Gamma-$semiring then M is a right weak-interior simple $\Gamma-$semiring.*

Proof. Let B be a proper right weak-interior ideal of the division $\Gamma-$semiring M, $x \in M$ and $0 \neq a \in B$. Since M is a division $\Gamma-$semiring, there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exists $\beta \in \Gamma$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. Therefore $x \in B\Gamma M$ and $M \subseteq B\Gamma M$. We have $B\Gamma M \subseteq M$. Hence $M = B\Gamma M$. Mow $M = B\Gamma M = B\Gamma B\Gamma M \subseteq B$. Therefore $M = B$. Hence division $\Gamma-$semiring M has no proper right-quasi-interior ideals. □

Corollary 4.1. *If M is a division $\Gamma-$semiring then M is a left weak-interior simple $\Gamma-$semiring.*

Corollary 4.2. *If M is a division $\Gamma-$ semiring then M is a weak-interior left simple $\Gamma-$semiring.*

Theorem 4.2. *Let M be a left simple $\Gamma-$semiring.Then M is a left weak-interior simple $\Gamma-$semiring .*

Proof. Let M be a left simple $\Gamma-$semiring and B be a left weak-interior ideal of M. Then $M\Gamma B\Gamma B \subseteq B$ and $M\Gamma B\Gamma B$ is a left ideal of M. Since M is a simple $\Gamma-$semiring, we have $M\Gamma B\Gamma B = M$. Therefore $M\Gamma B = M\Gamma B\Gamma B\Gamma B \subseteq B$. Thus $M\Gamma B \subseteq B$. Hence the theorem. □

Corollary 4.3. *Let M be a right simple $\Gamma-$semiring. Then M is a right weak-interior simple $\Gamma-$semiring .*

Corollary 4.4. *Let M be a simple $\Gamma-$semiring. Then M is a weak-interior simple $\Gamma-$semiring .*

Theorem 4.3. *Let M be a $\Gamma-$semiring. M is a left weak-interior simple $\Gamma-$semiring if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest left weak-interior ideal generated by a.*

Proof. Let M be a $\Gamma-$semiring. Suppose M is the left weak-interior simple $\Gamma-$semiring, $a \in M$ and $B = M\Gamma a$. Then B is a left ideal of M. Therefore, by Theorem[3.1], B is a left weak-interior ideal of M. Further on, $B = M$. Hence $M\Gamma a = M$, for all $a \in M$. we have $M\Gamma a \subseteq < a > \subseteq M$. Thus $M \subseteq < a > \subseteq M$. Therefore, $M =< a >$.

Conversely suppose that $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest left weak-interior ideal generated by a.and $a \in A$. Therefore $A = M$. Hence M is a left weak-interior simple $\Gamma-$semiring. □

Theorem 4.4. *Let M be a $\Gamma-$semiring. Then M is a left weak-interior simple $\Gamma-$semiring if and only if $M\Gamma a\Gamma a = M$, for all $a \in M$.*

Proof. Suppose M is the left-weak interior simple $\Gamma-$semiring and $a \in M$. Then $M\Gamma a\Gamma a$ is a weak-interior ideal of M. Hence $M\Gamma a\Gamma a = M$, for all $a \in M$.

Conversely, suppose that $M\Gamma a\Gamma a = M$, for all $a \in M$. Let B be a left weak-interior ideal of the $\Gamma-$semiring M and $a \in B$. Then $M = M\Gamma a\Gamma a \subseteq M\Gamma B\Gamma B \subseteq B$. Therefore $M = B$. So, M is a left weak-interior simple $\Gamma-$semiring. □

Corollary 4.5. *Let M be a $\Gamma-$semiring. Then M is a right weak-interior simple $\Gamma-$semrring if and only if $a\Gamma a\Gamma M = M$, for all $a \in M$.*

Corollary 4.6. *Let M be a $\Gamma-$semiring. Then M is a weak-interior simple $\Gamma-$semrring if and only if $a\Gamma a\Gamma M = M$ and $M\Gamma a\Gamma a = M$, for all $a \in M$.*

Theorem 4.5. *Let M be a $\Gamma-$semiring and B be a left weak-interior ideal of M. Then B is a minimal left weak-interior ideal of M if and only if B is a left weak-interior simple $\Gamma-$subsemiring of M.*

Proof. Let B be a minimal left weak-interior ideal of a $\Gamma-$semiring M and C be a left weak-interior ideal of B. Then $B\Gamma C\Gamma C \subseteq C$ and $B\Gamma C\Gamma C$ is a left weak-interior ideal of M. Since C is a left weak-interior ideal of B, we have $B\Gamma C\Gamma C = B$ and $B = B\Gamma C\Gamma C \subseteq C$. Thus $B = C$.

Conversely, suppose that B is the left weak-interior simple $\Gamma-$subsemiring of M. Let C be a left weak-interior ideal of M and $C \subseteq B$. Then $B\Gamma C\Gamma C \subseteq M\Gamma C\Gamma C \subseteq M\Gamma B\Gamma B \subseteq B$. Therefore C is a left weak-interior of B. Thus $B = C$ since

B is a left weak-interior simple $\Gamma-$subsemiring of M. Hence B is a minimal left weak-interior ideal of M. □

Corollary 4.7. *Let M be a $\Gamma-$semiring and B be a right weak-interior ideal of M. Then B is a minimal right weak-interior ideal of M if and only if B is a right weak-interior simple $\Gamma-$subsemiring of M.*

Corollary 4.8. *Let M be a $\Gamma-$semiring and B be a weak-interior ideal of M. Then B is a minimal weak-interior ideal of M if and only if B is a weak-interior simple $\Gamma-$subsemiring of M.*

Theorem 4.6. *Let M be a $\Gamma-$semiring and $B = R\Gamma L$, where, R is an ideal and L is a minimal left ideal of M. Then B is a minimal left weak-interior ideal of M.*

Proof. Obviously $B = R\Gamma L$ is a left weak-interior ideal of M. Let A be a left weak-interior ideal of M such that $A \subseteq B$. We have $M\Gamma A\Gamma A$ is a left ideal of M. Then $M\Gamma A \subseteq M\Gamma B$ and $M\Gamma A = M\Gamma R\Gamma L$. Thus $M\Gamma A \subseteq L$. since L is a left ideal of M. Therefore $M\Gamma A\Gamma A = L$ since L is a minimal left ideal of M.. Hence $B \subseteq M\Gamma A\Gamma A \subseteq M\Gamma B\Gamma B \subseteq M\Gamma R\Gamma L\Gamma R\Gamma L \subseteq L = M\Gamma A\Gamma A \subseteq A$. Therefore $A = B$. Hence B is a minimal left weak-interior ideal of M. □

Corollary 4.9. *Let M be a $\Gamma-$semiring and $B = R\Gamma L$, where R is a minimal right ideal and L is a left ideal of M. Then B is a minimal right weak-interior ideal of M.*

Theorem 4.7. *M is regular $\Gamma-$semiring if and only if $A\Gamma B = A \cap B$ for any right ideal A and left ideal B of $\Gamma-$semiring M.*

Theorem 4.8. *Let M be a $\Gamma-$semiring. If $M\Gamma B\Gamma B = B$. for all left weak-interior ideals B of M. then M is a regular $\Gamma-$semiring.*

Proof. Suppose $M\Gamma B\Gamma B = B$ for all left weak-interior ideals$B$ of M. Let $B = R \cap L$ and $C = R\Gamma L$, where R , L are ideal and left ideal of M respectively. Then $BandC$ are left weak-interior ideals of M. Therefore $M\Gamma(R \cap L)\Gamma(R \cap L) = R \cap L$. Now $R \cap L = M\Gamma(R \cap L)\Gamma(R \cap L)$ and $R \cap L \subseteq M\Gamma R\Gamma L$ and $R \cap L \subseteq R\Gamma L$. Thus $R \cap L \subseteq R \cap L$ since $R\Gamma L \subseteq L$ and $R\Gamma L \subseteq R$). Therefore $R \cap L = R\Gamma L$. So, M is a regular $\Gamma-$semiring. □

Theorem 4.9. *Let M be a commutative $\Gamma-$semiring.If B is a weak-interior ideal of a regular $\Gamma-$semiring M then $B\Gamma B\Gamma M = B$ for all weak-interior ideals B of M.*

Proof. Suppose M is the commutative regular semiring, B is a weak-interior ideal of M and $x \in B$. Then $M\Gamma B\Gamma B \subseteq B$, $y \in M$, $\alpha, \beta \in \Gamma$ such that

$$x = x\alpha y\beta x = y\alpha x\beta x \in M\Gamma B\Gamma B.$$

Therefore $x \in M\Gamma B\Gamma B$. Hence $M\Gamma B\Gamma B = B$. □

5. Fuzzy weak-interior ideals of $\Gamma-$semiring M

In this section, we introduce the notion of fuzzy right (left) weak-interior ideal as a generalization of fuzzy interior-ideal of a $\Gamma-$semiring and study the properties of fuzzy right (left) weak-interior ideals.

Definition 5.1. A fuzzy subset μ of a $\Gamma-$ semiring M is called a fuzzy right(left)weak-interior ideal if
(i) $\mu(x+y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$.
(ii) $\mu \circ \mu \circ \chi_M \subseteq \mu\big(\chi_M \circ \mu \circ \mu \subseteq \mu\big)$.
A fuzzy subset μ of $\Gamma-$semiring M is called a fuzzy weak- interior ideal if it is both left and right weak- interior ideal ideal of M.

Example 5.1. Let Q be the set of all rational numbers,

$$M = \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in Q \right\}$$

be the additive semigroup of M matrices and $\Gamma = M$. The ternary operation $A\alpha B$ is defined as usual matrix multiplication of A, α, B, for all $A, \alpha, B \in M$.Then M is a $\Gamma-$semiring If $R = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq b \in Q \right\}$ then R is a left weak interior ideal of the $\Gamma-$semiring M and R is neither a left ideal nor a right ideal, not a weak interior ideal and not a interior ideal of the $\Gamma-$semiring M .

Example 5.2. Let Q be the set of all rational numbers and

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}.$$

Then M is a $\Gamma-$semiring with respect to usual addition of matrices and ternary operation is defined as the usual matrix multiplication. If

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\},$$

then A is a right weak- interior ideal ideal but not a bi-ideal of the $\Gamma-$ semiring M. Define $\mu : M \to [0, 1]$ such that $\mu(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$ Then μ is a fuzzy right weak- interior ideal ideal of M.

Theorem 5.1. *Every fuzzy right ideal of a $\Gamma-$semiring M is a fuzzy right weak-interior ideal ideal of M.*

Proof. Let μ be a fuzzy right ideal of the $\Gamma-$ semiring M and $x \in M$. Then $\mu \circ \chi_M(x) = \sup\limits_{x=a\alpha b} \min\{\mu(a), \chi_M(b)\}$ for $a, b \in M, \alpha \in \Gamma$. and $\mu \circ \chi_M(x) = \sup\limits_{x=a\alpha b} \mu(a)$. Now $\mu \circ \chi_M(x) \leqslant \sup\limits_{x=a\alpha b} \mu(a\alpha b)$ and $\mu \circ \chi_M(x) = \mu(x)$. Therefore $\mu \circ \chi_M(x) \leqslant \mu(x)$. Now $\mu \circ \mu \circ \chi_M(x) = \sup\limits_{x=u\alpha v\beta s} \min\{\mu(u\alpha v), \mu \circ \chi_M(s)\}$ and $\mu \circ \mu \circ \chi_M(x) \leqslant \sup\limits_{x=u\alpha v\beta s} \min\{\mu(u\alpha v), \mu(s)\}$. So, $\mu \circ \mu \circ \chi_M(x) = \mu(x)$. Hence μ is a fuzzy right weak- interior ideal ideal of the $\Gamma-$semiring M. $\square$

Corollary 5.1. *Every fuzzy left ideal of a* $\Gamma-$*semiring* M *is a fuzzy left weak-interior ideal ideal of* M.

Corollary 5.2. *Every fuzzy ideal of a* $\Gamma-$*semiring* M *is a fuzzy weak- interior ideal ideal of* M.

Theorem 5.2. *Let* M *be a* $\Gamma-$*semiring and* μ *be a non-empty fuzzy subset of* M. *A fuzzy subset* μ *is a fuzzy left weak- interior ideal ideal of a* $\Gamma-$*semiring* M *if and only if the level subset* μ_t *of* μ *is a left weak-interior ideal ideal of a* $\Gamma-$*semiring* M *for every* $t \in [0,1]$, *where* $\mu_t \neq \phi$.

Proof. Let M be a $\Gamma-$semiring and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy left weak-interior ideal ideal of a $\Gamma-$semiring M, $\mu_t \neq \phi, t \in [0,1]$ and $a, b \in \mu_t$. Then $\mu(a) \geqslant t, \mu(b) \geqslant t$ and $\mu(a+b) \geqslant \min\{\mu(a), \mu(b)\} \geqslant t$. So, $a+b \in \mu_t$.

Let $x \in M\Gamma\mu_t\Gamma\mu_t$. Then $x = b\alpha a\beta d$, where $b \in M, a, c \in \mu_t, \alpha, \beta \in \Gamma$. Thus $\chi_M \circ \mu \circ \mu(x) \geqslant t$ and $\mu(x) \geqslant \chi_M \circ \mu \circ \mu(x) \geqslant t$. Therefore $x \in \mu_t$. So, μ_t is a left weak-interior ideal ideal of the $\Gamma-$ semiring M.

Conversely, suppose that μ_t is a left weak-interior ideal ideal of the $\Gamma-$semiring M, for all $t \in Im(\mu)$. Let $x, y \in M, \alpha \in \Gamma, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geqslant t_2$. Then $x, y \in \mu_{t_2}$ and $x + y \in \mu_{t_2}$ and $x\alpha y \in \mu_{t_2}$ Thus $\mu(x+y) \geqslant t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$. Therefore $\mu(x+y) \geqslant t_2 = \min\{\mu(x), \mu(y)\}$.

We have $M\Gamma\mu_l\Gamma\mu_l \subseteq \mu_t$, for all $l \in Im(\mu)$. Suppose $t = \min\{Im(\mu)\}$. Then $M\Gamma\mu_t\Gamma\mu_t \subseteq \mu_t$. Therefore $\chi_M \circ \mu \circ \mu \subseteq \mu$. Hence μ is a fuzzy left weak-interior ideal ideal of the $\Gamma-$semiring M. □

Corollary 5.3. *Let* M *be a* $\Gamma-$*semiring and* μ *be a non-empty fuzzy subset of* M. *A fuzzy subset* μ *is a fuzzy right weak-interior ideal ideal of a* $\Gamma-$*semiring if and only if the level subset* μ_t *of* μ *is a right weak-interior ideal ideal of a* $\Gamma-$*semiring* M *for every* $t \in [0,1]$, *where* $\mu_t \neq \phi$.

Theorem 5.3. *Let* I *be a non-empty subset of a* $\Gamma-$*semiring* M *and* χ_I *be the characteristic function of* I. *Then* I *is a right weak-interior ideal ideal of a* $\Gamma-$*semiring* M *if and only if* χ_I *is a fuzzy right weak- interior ideal ideal of a* $\Gamma-$*semiring* M.

Proof. Let I be a non-empty subset of a $\Gamma-$semiring M and χ_I be the characteristic function of I. Suppose I is a right weak- interior ideal ideal of the $\Gamma-$semiring M.

Obviously χ_I is a fuzzy $\Gamma-$subsemiring of M. We have $I\Gamma I\Gamma M \subseteq I$. Then $\chi_I \circ \chi_I \circ \chi_M = \chi_{I\Gamma I\Gamma M}$ and $\chi_I \circ \chi_I \circ \chi_M = \chi_{I\Gamma I\Gamma M}$. So, we have $\chi_I \circ \chi_I \circ \chi_M \subseteq \chi_I$. Therefore χ_I is a fuzzy right weak- interior ideal ideal of the $\Gamma-$semiring M.

Conversely suppose that χ_I is a fuzzy right weak- interior ideal ideal of M. Then I is a $\Gamma-$subsemiring of M. We have $\chi_I \circ \chi_I \circ \chi_M \subseteq \chi_I$ and $\chi_{I\Gamma I\Gamma M} \subseteq \chi_I$. Therefore $I\Gamma I\Gamma M \subseteq I$. Hence I is a right weak- interior ideal ideal of the $\Gamma-$semiring M. □

Theorem 5.4. *f* μ *and* λ *are fuzzy left weak- interior ideals of a* $\Gamma-$ *semiring* M, *then* $\mu \cap \lambda$ *is a fuzzy left weak- interior ideal ideal of a* Γ − *semiring* M.

PROOF. Let μ and λ be fuzzy left weak- interior ideals of the $\Gamma-$ semiring M. $x, y \in M$ and $\alpha \in \Gamma$,$\beta \in \Gamma$,Then

$$\begin{aligned}
\mu \cap \lambda(x+y) &= \min\{\mu(x+y), \lambda(x+y)\} \\
&\geqslant \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\
&= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\
&= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y)\} \\
\chi_M \circ \mu \cap \lambda(x) &= \sup_{x=a\alpha b} \min\{\chi_M(a), \mu \cap \lambda(b)\} \\
&= \sup_{x=a\alpha b} \min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\} \\
&= \sup_{x=a\alpha b} \min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\sup_{x=a\alpha b} \min\{\chi_M(a), \mu(b)\}, \sup_{x=a\alpha b} \min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\chi_M \circ \mu(x).\chi_M \circ \lambda(x)\} \\
&= \chi_M \circ \mu \cap \chi_M \circ \lambda(x).
\end{aligned}$$

Therefore

$$\chi_M \circ \mu \cap \lambda \circ \mu \cap \lambda = \chi_M \circ \mu \circ \mu \cap \chi_M \circ \lambda \circ \lambda.$$

Hence

$$\chi_M \circ \mu \cap \lambda \circ \mu \cap \lambda = \chi_M \circ \mu \circ \mu \cap \chi_M \circ \lambda \circ \lambda \subseteq \mu \cap \lambda.$$

Thusd $\mu \cap \lambda$ is a left fuzzy weak-interior ideal of M. Hence the theorem. □

THEOREM 5.5. *A $\Gamma-$semiring M is a regular if and only if $\lambda \circ \mu = \lambda \cap \mu$, for any fuzzy right ideal λ and fuzzy left ideal μ of M.*

THEOREM 5.6. *Let M be a regular $\Gamma-$semiring. Then μ is a fuzzy left weak-interior ideal ideal of M if and only if μ is a fuzzyideal of M.*

PROOF. Let μ be a fuzzy left weak-interior ideal ideal of the $\Gamma-$semiring M and $x \in M$. Then $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Suppose $\chi_M \circ \mu(x) > \mu(x)$ and $\mu \circ \chi_M(x) > \mu(x)$. Since M is a regular, there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Thus

$$\mu \circ \chi_M(x) = \sup_{x=x\alpha y\beta x} \min\{\mu(x), \chi_M(y\beta x)\} = \sup_{x=x\alpha y\beta x} \min\{\mu(x), 1\} = \sup_{x=x\alpha y\beta x} \mu(x)$$

and finally $\mu \circ \chi_M(x) > \mu(x)$. Now

$$\begin{aligned}
\mu \circ \chi_M \circ \mu \circ \chi_M(x) &= \sup_{x=x\alpha y\beta x} \min\{\mu \circ \chi_M(x), \mu \circ \chi_M(y\beta x)\}) \\
&> \sup_{x=x\alpha y\beta x} \min\{\mu(x), \mu(y\beta x)\} \\
&= \mu(x).
\end{aligned}$$

This is a contradiction. Hence μ is a fuzzyi deal of M. By Theorem 4.15, converse is true. □

COROLLARY 5.4. *Let M be a regular $\Gamma-$semiring. Then μ is a fuzzy right weak-interior ideal ideal of M if and only if μ is a fuzzy ideal of M.*

THEOREM 5.7. *Let M be a $\Gamma-$semiring. Then M is a regular if and only if $\mu = \chi_M \circ \mu \circ \mu$, for any fuzzy left weak-interior ideal μ of a $\Gamma-$semiring M.*

PROOF. Let μ be a fuzzy left weak-interior ideal of the regular $\Gamma-$semiring M and $x, y \in M, \alpha, \beta \in \Gamma$. Now Then $\chi_M \circ \mu \circ \mu \subseteq \mu$.

$$\begin{aligned}\chi_M \circ \mu \circ \mu(x) &= \sup_{x=x\alpha y\beta x}\{\min\{\chi_M \circ \mu(x), \mu(y\beta x)\}\} \\ &\geqslant \sup_{x=x\alpha y\beta x}\{\min\{\mu(x), \mu(y\beta x)\}\} \\ &= \mu(x).\end{aligned}$$

Therefore $\mu \subseteq \chi_M \circ \mu \circ \mu$. Hence $\chi_M \circ \mu \circ \mu = \mu$.

Conversely suppose that $\mu = \chi_M \circ \mu \circ \mu$, for any fuzzy weak- interior ideal ideal μ of the $\Gamma-$semiring M. Let B be a weak- interior ideal of the $\Gamma-$semiring M. Then by Theorem 3.8, χ_B be a fuzzy weak- interior ideal ideal of the $\Gamma-$ semiring M. Therefore $\chi_B = \chi_M \circ \chi_B \circ \chi_B$ and $\chi_B = \chi_{M\Gamma B\Gamma B}$. Thus $\chi_B = M\Gamma B\Gamma B$. By Theorem [3.19], M is a regular $\Gamma-$semiring. □

THEOREM 5.8. *Let M be a $\Gamma-$semiring. Then M is a regular if and only if $\mu \cap \gamma \subseteq \gamma \circ \mu \circ \mu$, for every fuzzy left weak-interior ideal μ and every fuzzy ideal γ of $\Gamma-$semiring M.*

PROOF. Let M be a regular $\Gamma-$semiring and $x \in M$. Then there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$.

$$\begin{aligned}\gamma \circ \mu \circ \mu(x) &= \sup_{x=x\alpha y\beta x}\{\min\{\gamma(x\alpha y), \mu \circ \mu(x)\} \\ &\geqslant \min\{\sup_{x=x\alpha y\beta x}\{\min\{\gamma(x), \{\mu(x), \mu(y\beta x)\}\}\}\} \\ &\geqslant \min\{\min\{\mu(x), \gamma(x)\}, min\{\mu(x), \gamma(x)\}\} \\ &= \min\{\mu(x), \gamma(x)\} = \mu \cap \gamma(x).\end{aligned}$$

Hence $\mu \cap \gamma \subseteq \mu \circ \gamma \circ \gamma$.

Conversely suppose that the condition holds. Let μ be a fuzzy left weak- interior ideal ideal ofsemiring M. Then $\mu \cap \chi_M \subseteq \chi_M \circ \mu \circ \mu$ and $\mu \subseteq \chi_M \circ \mu \circ \mu$. By Theorem 5.8, M is a regular semiring. □

COROLLARY 5.5. *Let M be a $\Gamma-$semiring. Then M is a regular if and only if $\mu \cap \gamma \subseteq \mu \circ \mu \circ \gamma$, for every fuzzy right weak-interior ideal μ and every fuzzy ideal γ of $\Gamma-$semiring M.*

6. Conclusion

In this paper, as a further generalization of ideals, we introduced the notion of weak-interior ideal of $\Gamma-$semiring as a generalization of ideal, left ideal, right ideal, quasi ideal and interior ideal of $\Gamma-$semiring and studied some of their properties. We introduced the notion of weak-interior simple $\Gamma-$semiring and characterized the weak-interior simple $\Gamma-$semiring, regular $\Gamma-$semiring using weak-interior ideal ideals of $\Gamma-$semiring. In continuity of this paper, we study prime, maximal and minimal weak-interior ideals of $\Gamma-$semiring. In addition, we introduced the notion of fuzzy right (left) weak-interior ideal ideal of a $\Gamma-$semiring and characterized the regular $\Gamma-$semiring in terms of fuzzy right(left) weak- interior ideals of a $\Gamma-$semiring and studied some of their algebraical properties.

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M. M. Krishna Rao: Department of Mathematics, GIT, GITAM University, Visakhapatnam - 530 045, A.P., India.

E-mail address: mmarapureddy@gmail.com