JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
J. Int. Math. Virtual Inst., Vol. **10**(1)(2020), 59-74
DOI: 10.7251/JIMVI2001059D

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

CONTACT PSEUDO-SLANT SUBMANIFOLDS OF AN $(\epsilon)-$ PARA SASAKIAN SPACE FORMS

S. Dirik, M. Atçeken, and Ü. Yıldırım

Abstract. In this paper, we study the geometry of the contact pseudo-slant submanifolds of an $(\epsilon)-$para Sasakian space form. We give some results for totally umbilical pseudo-slant submanifolds of an $(\epsilon)-$ para Sasakian manifolds and an $(\epsilon)-$para Sasakian space forms. It is proved that such manifolds are η-Einstein. Finally, we have verified the theorems by providing an example of 5- dimensional proper contact pseudo-slant submanifold of $\mathbb{R}^9$ with it's usual almost paracontact metric structure.

1. Introduction

In [**22**] S. Tanno classified connected almost contact metric manifolds , as those automorphism groups posses maximum dimension. For such manifold, the sectional curvature of a plane sections containing ξ is a constant say k. This is classification is as;

(a) Homogeneous normal contact Riemannian manifolds with say $k > 0$.

(b) Global Riemannian product of a line (or a circle) and a Kaehlerian manifold with constant holomorphic sectional curvature if $k = 0$.

(c) A warped product space $R \times_f C^n$, if $k < 0$.

It is known that manifold of class (a) is characterized by some tensorial equations . It admits a Sasakian structure. The manifold of class (b) is characterized by a tensorial relation admitting a cosymplectic structure. The manifold of class (c) is characterized by some tensorial equations, attaining a Kenmotsu structure. An almost paracontact structure (φ, ξ, η) satisfying $\varphi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold was introduced by Sato [**21**] in 1976. Takahashi [**23**] in 1969,

2010 *Mathematics Subject Classification.* 53C15, 53C25, 53C17, 53D15, 53D10.
Key words and phrases. Para-Sasakian manifold, slant submanifold, contact pseudo-slant submanifold, (ϵ)-para-Sasakian space form.

gave the notion of almost contact manifold equipped with an associated pseudo-Riemannian metric. After, Tripathi et. al. [**24**] has drawn a relation between a semi-Riemannian metric (not necessarily Lorentzian) and an almost paracontact structure, and he named this indefinite almost paracontact metric structure an (ϵ)-almost paracontact structure, where the structure vector field ξ will be spacelike or timelike according as $\epsilon = 1$ or $\epsilon = -1$. Authors have discussed $(\epsilon)-$ almost paracontact manifolds and in particular $(\epsilon)-$ Sasakian manifolds in [**24**]. On the other hand, the differential geometry of slant submanifolds has shown an increasing development since Chen defined slant submanifolds in complex manifolds as a natural generalization of both invariant and anti-invariant submanifolds [**6, 7**]. Many research articles have been appeared on the existence of these submanifolds in different knows spaces. The slant submanifolds of an almost contact metric manifolds were defined and studied by Lotta [**12**]. After, such submanifolds were studied by Cabrerizo et al. of Sasakian manifolds [**4**]. Recently, in [**2, 8**], Atçeken et al. studied slant and pseudo-slant submanifold in various manifolds. The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by Papagiuc [**14**]. Cabrerizo [**5**] defined and studied bi-slant immersions in almost contact metric manifolds and simultaneously gave the notion of pseudo-slant submanifolds. Pseudo-slant submanifolds also have been studied by Khan et al. in [**11**]. The present paper is organized as follows.

This paper contains the analysis about contact pseudo-slant submanifolds of an $(\epsilon)-$para Sasakian space form. Necessary and sufficient conditions are given for a submanifold to be contact pseudo-slant submanifolds. Finally, we give some results for totally umbilical pseudo-slant submanifolds of an $(\epsilon)-$ para Sasakian manifolds and $(\epsilon)-$para Sasakian space forms.

2. Preliminaries

In this section, we give some notations used throughout this paper. We recall some necessary fact and formulas from the theory of $(\epsilon)-$ para Sasakian manifolds and their submanifols.

Let $\widetilde{M}$ be an n-dimensional almost paracontact manifold [**21**] equipped with an almost paracontact structure (φ, ξ, η) consisting of a tensor field φ of type (1, 1), a vector field ξ and a 1-form η satisfying

$$\varphi^2 X = X - \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \ \eta(X) = g(X, \xi) \tag{2.2}$$

and

$$\eta(\varphi) = 0 \tag{2.3}$$

for vector field X on $\widetilde{M}$. A semi-Riemannian metric g on a manifold $\widetilde{M}$, is a non degenerate symmetric tensor field g of type (0, 2). If this metric is of index 1 then it is called Lorentzian metric [**3**]. Let g be semi-Riemannian metric with index 1 in an n-dimensional almost paracontact manifold $\widetilde{M}$ such that,

$$g(\varphi X, \varphi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.4}$$

for all vector fields X and Y on $\widetilde{M}$. Where $\epsilon = +1$ or $\epsilon = -1$.

Then $\widetilde{M}$ is called an $(\epsilon)-$almost paracontact metric manifold equipped with an $(\epsilon)-$almost paracontact metric structure $(\varphi, \xi, \eta, g, \epsilon)$[**24**]. In particular, if index(g) = 1, then an $(\epsilon)-$almost paracontact metric manifold will be called a Lorentzian almost paracontact manifold. In particular, if the metric g is positive definite, then an $(\epsilon)-$ almost paracontact metric manifold is the usual almost paracontact metricmanifold [**21**].

From $(2.1), (2.2)$ and (2.4), we have

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{2.5}$$

and

$$g(X, \xi) = \epsilon\eta(X). \tag{2.6}$$

From (2.6), it can be easily observed that

$$g(\xi, \xi) = \epsilon,$$

all vector fields X and Y on $\widetilde{M}$.

An almost paracontact metric structure (φ, ξ, η, g) on a $\widetilde{M}$ is an $(\epsilon)-$para-Sasakian manifold if and only if

$$(\widetilde{\nabla}_X\varphi)Y = -g(\varphi X, \varphi Y)\xi - \epsilon\eta(Y)\varphi^2 X, \tag{2.7}$$

for all vector fields X and Y on $\widetilde{M}$. $\widetilde{\nabla}$ is Levi-Civita connection of $\widetilde{M}$.

From equqtion (2.7), put $Y = \xi$, we have

$$\nabla_X\xi = \epsilon\varphi X. \tag{2.8}$$

An $(\epsilon)-$ para -Sasakian manifold $\widetilde{M}$ is said to be η-Einstein if its Ricci tensor S of type $(0, 2)$ is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

for any vector fields X, Y on $\widetilde{M}$. Where a and b are smooth functions on $\widetilde{M}$. In particular, if $b = 0$ in the equation above, $\widetilde{M}$ is an Einstein manifold.

Let $\widetilde{M}(k)$ be an $(\epsilon)-$ para-Sasakian space form with constant φ-paraholomorphic sectional curvature k. Then the curvature tensor $\widetilde{R}$ of $\widetilde{M}(k)$ is given by

$$\begin{aligned} g(\widetilde{R}(X, Y)Z, W) &= (\frac{k-3}{4})\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ &+(\frac{k+1}{4})\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ &+g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) \\ &+g(Y, \varphi Z)g(\varphi X, W) - g(X, \varphi Z)g(\varphi Y, W) \\ &+2g(\varphi X, Y)g(\varphi Z, W)\}, \end{aligned} \tag{2.9}$$

for any vector fields X, Y, Z, W on $\widetilde{M}(k)$.

Now, let M be a submanifold of an $(\epsilon)-$ para Sasakian manifold $\widetilde{M}$. Also, let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then the Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.10}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.11}$$

for any vector fields X, Y on M, where σ and A_V are the second fundamental form and the shape operator (corresponding to the normal vector field V), respectively, for the immersion of M into $\widetilde{M}$. The second fundamental form σ and shape operator A_V are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V), \tag{2.12}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The mean curvature vector H of M is given by $H = \frac{1}{m}\sum_{i=1}^{m}\sigma(e_i, e_i)$, where m is the dimension of M and $\{e_1, e_2, ..., e_m\}$ is a local orthonormal frame of M. A submanifold M of an $(\epsilon)-$para Sasakian manifold $\widetilde{M}$ is said to be totally umbilical if

$$\sigma(X, Y) = g(X, Y)H, \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$. A submanifold M is said to be totally geodesic if $\sigma = 0$ and M is said to be minimal if $H = 0$.
For any submanifold M of a Riemannian manifold $\widetilde{M}$, the equation of Gauss is given by

$$\widetilde{R}(X, Y)Z = R(X, Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X\sigma)(Y, Z) - (\widetilde{\nabla}_Y\sigma)(X, Z), \tag{2.14}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\widetilde{R}$ and R denote the Riemannian curvature tensor of $\widetilde{M}$ and M, respectively. The covariant derivative $\widetilde{\nabla}h$ of h is defined by

$$(\widetilde{\nabla}_X\sigma)(Y, Z) = \nabla^{\perp}_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(\nabla_X Z, Y). \tag{2.15}$$

The normal component of (2.14) is said to be the Codazzi equation and it is given by

$$(\widetilde{R}(X, Y)Z)^{\perp} = (\widetilde{\nabla}_X\sigma)(Y, Z) - (\widetilde{\nabla}_Y\sigma)(X, Z), \tag{2.16}$$

where $(\widetilde{R}(X, Y)Z)^{\perp}$ denotes the normal part of $\widetilde{R}(X, Y)Z$. If $(\widetilde{R}(X, Y)Z)^{\perp} = 0$, then M is said to be curvature-invariant submanifold of $\widetilde{M}$. The Ricci equation is given by

$$g(\widetilde{R}(X, Y)V, U) = g(\widetilde{R}^{\perp}(X, Y)V, U) + g([A_U, A_V]X, Y), \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$, where $\widetilde{R}^{\perp}$ denotes the Riemannian curvature tensor of the normal $T^{\perp}M$. If $\widetilde{R}^{\perp} = 0$, then the normal connection of the submanifold M is called flat.

Let M be a submanifold of an $(\epsilon)-$para Sasakian manifold $\widetilde{M}$. Then for any $X \in \Gamma(TM)$, we can write

$$\varphi X = TX + FX, \tag{2.18}$$

where TX is the tangential component and FX is the normal component of φX. Similarly for $V \in \Gamma(T^{\perp}M)$, we can write

$$\varphi V = tV + fV, \tag{2.19}$$

where tV is the tangential component and fV is also the normal component of φV.

A submanifold M is said to be *invariant* if F is identically zero, that is, $\varphi X \in \Gamma(TM)$ for all $X \in \Gamma(TM)$. On the other hand, M is said to be *anti- invariant* if T is identically zero, that is, $\varphi X \in \Gamma(T^{\perp}M)$ for all $X \in \Gamma(TM)$.

Taking into account (2.9) and (2.17), we have

$$\begin{aligned} g(\widetilde{R}^{\perp}(X,Y)V,U) &= g([A_V, A_U]X, Y) + (\frac{k+1}{4})\{g(\varphi X, U)g(Y, \varphi V) \\ &\quad -g(X, \varphi V)g(\varphi Y, U) + 2g(\varphi X, Y)g(\varphi V, U)\}, \end{aligned} \tag{2.20}$$

for any $X, Y \in \Gamma(TM)$ and $V, U \in \Gamma(T^{\perp}M)$. By using (2.9) and (2.14), the Riemannian curvature tensor R of an immersed submanifold M of an $(\epsilon)-$ para-Sasakian space form $\widetilde{M}(k)$ is given by

$$\begin{aligned} R(X,Y)Z &= (\frac{k-3}{4})\{g(Y,Z)X - g(X,Z)Y\} + (\frac{k+1}{4})\{\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &\quad -g(X,\varphi Z)\varphi Y + g(Y,\varphi Z)\varphi X + 2g(\varphi X,Y)\varphi Z\} \\ &\quad +A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y + (\widetilde{\nabla}_Y\sigma)(X,Z) - (\widetilde{\nabla}_X\sigma)(Y,Z). \end{aligned} \tag{2.21}$$

The normal part of (2.21), we have

$$\begin{aligned} (\widetilde{\nabla}_X\sigma)(Y,Z) - (\widetilde{\nabla}_Y\sigma)(X,Z) &= (\frac{k+1}{4})\{-g(X,TZ)FY \\ &\quad +g(Y,TZ)FX + 2g(TX,Y)FZ\}. \end{aligned} \tag{2.22}$$

Thus by using (2.1), (2.18) and (2.19), we obtain

$$T^2 = I - \eta \otimes \xi - tF, \quad FT + fF = 0 \tag{2.23}$$

and

$$f^2 = I - Ft, \quad Tt + tf = 0. \tag{2.24}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(FX, Y) = g(X, FY)$ and $V, U \in \Gamma(T^{\perp}M)$, we get $g(U, fV) = g(fU, V)$. These relations show that F and f are symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have

$$g(FX, V) = g(X, tV) \tag{2.25}$$

which gives the relation between F and t.

On the other hand, the covariant derivatives of the tensor fields T, F, t and f, respectively, defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.26}$$

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{2.27}$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla^{\perp}_X V \tag{2.28}$$

and

$$(\nabla_X f)V = \nabla^{\perp}_X fV - f\nabla^{\perp}_X V, \tag{2.29}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X, Y \in \Gamma(TM)$.

Since M is tangent to ξ, making use of $\widetilde{\nabla}_X \xi = \epsilon\varphi X$, (2.10) and (2.12) we obtain

$$\nabla_X \xi = \epsilon TX, \quad \sigma(X, \xi) = \epsilon FX, \quad A_V \xi = \epsilon tV, \tag{2.30}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$.

By direct calculations, we obtain the following

$$(\nabla_X T)Y = A_{FY}X + B\sigma(X, Y) + g(TX, TY)\xi + g(FX, FY)\xi + \epsilon\eta(Y)(X - \eta(X)\xi) \tag{2.31}$$

and

$$(\nabla_X F)Y = f\sigma(X, Y) - \sigma(X, TY). \tag{2.32}$$

Similarly, for any $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$, we obtain

$$(\nabla_X t)V = A_{fV}X - TA_V X + g(TX, tV)\xi - g(NX, fV)\xi \tag{2.33}$$

and

$$(\nabla_X f)V = -\sigma(tV, X) - FA_V X - \epsilon\eta(Y)V. \tag{2.34}$$

By direct calculations, (2.31), (2.32), (2.33) and (2.34) if takking $X = \xi$, we obtain the following formulas;

$$(\nabla_\xi T)Y = 2\epsilon tFY \tag{2.35}$$

and

$$(\nabla_\xi F)Y = -2\epsilon FtY. \tag{2.36}$$

Similarly, for any $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$, we obtain

$$(\nabla_\xi t)V = 2\epsilon tfV \tag{2.37}$$

and

$$(\nabla_\xi f)V = -2\epsilon FtV - \epsilon\eta(Y)V. \tag{2.38}$$

In contact geometry, A. Lotta introduced slant submanifolds as follows:

Let M be a submanifold of an (ϵ)-para-Sasakian manifold $(\bar{M}, \varphi, \xi, \eta, g)$. Then M is said to be a contact slant submanifold if the angle $\theta(X)$ between φX and $T_M(p)$ is constant at any point $p \in M$ for any X linearly independent of ξ. Thus the invariant and anti-invariant submanifolds are special class of slant submanifolds with slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. If the slant angle θ is neither zero nor $\frac{\pi}{2}$, then slant submanifold is said to be proper contact slant submanifold. The slant submanifolds of an almost contact metric manifold, the following theorem is well known [**12**].

THEOREM 2.1. *Let M be a submanifold of an (ϵ)-para-Sasakian manifold $\widetilde{M}$ such that ξ is tangent to M. Than M is a slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ shuch that*

$$T^2 = \lambda(I - \eta \otimes \xi). \tag{2.39}$$

Moreover, İf θ is the slant angle of M, then it satisfies $\lambda = \cos^2\theta$.

COROLLARY 2.1. *Let M be a slant submanifold of an $(\epsilon)-$para-Sasakian manifold $\widetilde{M}$ with slant angle θ. Then for any $X, Y \in \Gamma(TM)$, we have*

$$g(TX, TY) = \cos^2\theta \left\{ g(X,Y) - \epsilon\eta(X)\eta(Y) \right\} \tag{2.40}$$

and

$$g(FX, FY) = \sin^2\theta \left\{ g(X,Y) - \epsilon\eta(X)\eta(Y) \right\}. \tag{2.41}$$

PROOF. Taking account of T being symmetric and Theorem 2.1, direct calculation gives(2.40). To prove (2.41), it is enought to take into account (2.6) and (2.18). □

3. Contact Pseudo-Slant Submanifold of an (ϵ) -Para-Sasakian Manifold

In this section, we study the geometry of the contact pseudo-slant submanifolds of an (ϵ)- para-Sasakian manifold and obtain integrability conditions for the distributions on these submanifolds.

DEFINITION 3.1. Let M be a submanifold of an (ϵ)-para-Sasakian manifold $\widetilde{M}$. M is said to be *contact pseudo-slant submanifold* of $\widetilde{M}$ if there exist two orthogonal distributions $D^\perp$ and D_θ on M such that:

(i) The distribution D_θ is a slant, that is, the slant angle between of D_θ and $\varphi(D_\theta)$ is a constant.

(ii) The distribution $D^\perp$ is an anti-invariant i.e., $\varphi(D^\perp) \subset T^\perp M$.

(iii) ([**11**]) TM has the orthogonal direct decomposition $TM = D^\perp \oplus D_\theta$, $\xi \in \Gamma(D_\theta)$.

If $\theta = 0$ then, the submanifold becomes a semi-invariant submanifold.

Let d_1 =dim($D^\perp$) and d_2=dim(D_θ). We distinguish the following six cases.

(i) If $d_2 = 0$, then M is an anti-invariant submanifold.

(ii) If $d_1 = 0$ and $\theta = 0$, then M is invariant submanifold.

(iii) If $d_1 = 0$ and $\theta \in (0, \frac{\pi}{2})$, then M is a proper slant submanifold. (iv) If $\theta = \frac{\pi}{2}$ then, M is an anti-invariant submanifold.

(v) If $d_2 d_1 \neq 0$ and $\theta = 0$, then M is a semi-invariant submanifold.

(vi) If $d_2 d_1 \neq 0$ and $\theta \in (0, \frac{\pi}{2})$, then M is a contact pseudo-slant submanifold.

If μ is the invariant subspaces of the normal bundle $T^\perp M$, then in the case of contact pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M = \mu \oplus F(D^\perp) \oplus F(D_\theta).$$

The bases of the contact pseudo-slant submanifolds are given below. Also, let e_1, $e_2, ..., e_p$, $e_{p+1} = \sec\theta T e_1$, $e_{p+2} = \sec\theta T e_2, ..., e_{2p} = \sec\theta T e_p$, $e_{2p+1} = \xi$, e_{2p+2}, $e_{2p+3}, ..., e_{2p+q+1}$ be an orthonormal basis of $\Gamma(TM)$ such that $e_1, e_2, ..., e_p$, $e_{p+1} = \sec\theta T e_1$, $e_{p+2} = \sec\theta T e_2, ..., e_{2p} = \sec\theta T e_p$, $e_{2p+1} = \xi$ are tangent to $\Gamma(D_\theta)$ and $e_{2p+2}, e_{2p+3}, ..., e_{2p+q+1}$ are tangent to $\Gamma(D^\perp)$. Here, $\dim(M) = 2p + q + 1$ [**9**].

Theorem 3.1. *Let M be a contact pseudo-slant submanifold of an (ϵ)-para-Sasakian manifold $\widetilde{M}$. Then we obtain*

$$A_{FZ}W = A_{FW}Z,$$

for all $Z, W \in \Gamma(D^\perp)$.

Proof. For any $Z, W \in \Gamma(D^\perp)$ and $U \in \Gamma(TM)$, also by using (2.1), (2.7), (2.10) and (2.12), we have

$$\begin{aligned} g(A_{FZ}W - A_{FW}Z, U) &= g(\sigma(W, U), FZ) - g(\sigma(Z, U), FW) \\ &= g(\widetilde{\nabla}_U W, \varphi Z) - g(\widetilde{\nabla}_U Z, \varphi W) \\ &= g(\varphi\widetilde{\nabla}_U Z, W) - g(\widetilde{\nabla}_U \varphi Z, W) \\ &= -g((\widetilde{\nabla}_U \varphi)Z, W) \\ &= g(g(\varphi U, \varphi Z)\xi + \epsilon\eta(Z)\varphi^2 U, W) \\ &= g(g(FU, FZ)\xi, W) = 0. \end{aligned}$$

It follows that

$$A_{FZ}W = A_{FW}Z, \tag{3.1}$$

for any $Z, W \in \Gamma(D^\perp)$. □

Theorem 3.2. *Let M be a contact pseudo-slant submanifold of an (ϵ)-para sasakian manifold $\widetilde{M}$. Then anti-invariant distribution $D^\perp$ is always integrable.*

PROOF. For any $Z, W \in \Gamma(D^{\perp})$, we have

$$\begin{aligned} -g(FZ, FW)\xi &= \widetilde{\nabla}_Z \varphi W - \varphi \widetilde{\nabla}_Z W \\ &= -A_{FW}Z + \nabla^{\perp}_Z FW - T\nabla_Z W - t\sigma(Z, W) \\ &\quad -F\nabla_Z W - f\sigma(Z, W) \end{aligned}$$

which implies that

$$A_{FW}Z = -T\nabla_Z W - t\sigma(Z, W) + g(FZ, FW)\xi.$$

Thus we have

$$T\,[W, Z] = A_{FZ}W - A_{FW}Z \tag{3.2}$$

From (3.1) and (3.2), we conclude that $T\,[W, Z] = 0$, i.e., $[W, Z] \in \Gamma(D^{\perp})$. The proof is completes. □

THEOREM 3.3. *Let M be a contact pseudo-slant submanifold of an (ϵ)-para-Sasakian manifold $\widetilde{M}$. Then we obtain*

$$A_{FZ}U + T\nabla_U Z + t\sigma(U, Z) = 0$$

or

$$A_{FZ}U + T\nabla_U Z + t\sigma(U, Z) \in D^{\perp}.$$

PROOF. For any $Z \in \Gamma(D^{\perp})$ and $X, U \in \Gamma(D_\theta)$, from (2.5), (2.7), (2.10), (2.11), (2.18)and (2.25), we obtain

$$\begin{aligned} g(A_{\varphi Z}U, X) &= -g(\widetilde{\nabla}_U \varphi Z, X) \\ &= -g((\widetilde{\nabla}_U \varphi)Z, X) - g(\varphi \widetilde{\nabla}_U Z, X) \\ &= -g(g(\varphi U, \varphi Z)\xi - \epsilon\eta(Z)\varphi^2 U, X) + g(\widetilde{\nabla}_U Z, \varphi X) \\ &= g(\varphi U, \varphi Z)\eta(X) + g(\widetilde{\nabla}_U Z, \varphi X) \\ &= g(FU, FZ)\eta(X) + g(\nabla_U Z, TX) + g(\sigma(U, Z), FX) \\ &= g(g(FU, FZ)\xi, X) - g(T\nabla_U Z, X) - g(t\sigma(U, Z), X) \\ &= g(\sin^2\theta g(U, Z)\xi, X) - g(T\nabla_U Z, X) - g(t\sigma(U, Z), X) \\ &= -g(T\nabla_U Z, X) - g(t\sigma(U, Z), X) \end{aligned}$$

which proves the assertion. □

THEOREM 3.4. *Let M be contact pseudo-slant submanifold of an (ϵ)-para Sasakian manifold $\bar{M}$. Then the slant distribution D_θ is integrable if and only if*

$$g(A_{FTY}X - A_{FTX}Y, Z) = 2g(A_{FZ}TY, X),$$

for all $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$.

Proof. By using (2.4), (2.7), (2.18), (2.11), (2.27) and (2.41), we obtain

$$\begin{aligned}
g([X,Y],Z) &= g(\nabla_X Y, Z) - g(\nabla_Y X, Z) = g(\widetilde{\nabla}_Y Z, X) - g(\widetilde{\nabla}_X Z, Y)\\
&= g(\varphi\widetilde{\nabla}_Y Z, \varphi X) - \epsilon\eta(\widetilde{\nabla}_Y Z)\eta(X) - g(\varphi\widetilde{\nabla}_X Z, \varphi Y) + \epsilon\eta(\widetilde{\nabla}_X Z)\eta(Y)\\
&\quad g(\widetilde{\nabla}_Y \varphi Z, \varphi X) - g((\widetilde{\nabla}_Y\varphi)Z, \varphi X) - g(\widetilde{\nabla}_X \varphi Z, \varphi Y) + g((\widetilde{\nabla}_X\varphi)Z, \varphi Y)\\
&= g(\widetilde{\nabla}_Y \varphi Z, \varphi X) - g(g(\varphi Y, \varphi Z)\xi + \epsilon\eta(Z)\varphi^2 Y, \varphi X)\\
&\quad -g(\widetilde{\nabla}_X \varphi Z, \varphi Y) + g(g(\varphi X, \varphi Z)\xi + \epsilon\eta(Z)\varphi^2 X, \varphi Y)\\
&= g(\widetilde{\nabla}_Y \varphi Z, TX) + g(\widetilde{\nabla}_Y \varphi Z, FX) - g(\widetilde{\nabla}_X \varphi Z, TY) - g(\widetilde{\nabla}_X \varphi Z, FY)\\
&= -g(A_{\varphi Z} TX, Y) + g(A_{\varphi Z} TY, X) + g(\nabla^{\perp}_Y FZ, F\ X) - g(\nabla^{\perp}_X FZ, FY)\\
&= -g(A_{\varphi Z} TX, Y) + g(A_{\varphi Z} TY, X) + g((\nabla_Y F)Z + F\nabla_Y FZ, FX)\\
&\quad -g((\nabla_X F)Z + F\nabla_X Z, FY)\\
&= g(A_{\varphi Z} TY, X) - g(A_{\varphi Z} TX, Y) - g(F\nabla_X Z, FY) + g(F\nabla_Y Z, FX)\\
&\quad +g(\sigma(X,Z), fFY) - g(\sigma(Y,Z), fFX)\\
&= g(A_{\varphi Z} TY + TA_{\varphi Z} Y, X) - \sin^2\theta g(\nabla_X Z, Y) + \sin^2\theta g(\nabla_Y Z, X)\\
&\quad +g(\sigma(X,Z), fFY) - g(\sigma(Y,Z), fFX)\\
&= g(TA_{\varphi Z} Y + A_{\varphi Z} TY, X) + \sin^2\theta g([X,Y],Z)\\
&\quad +g(A_{fFY} X - A_{fFX} Y, Z),
\end{aligned}$$

for all $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$. Consequently, we reach at

$$\cos^2\theta g([X,Y],Z) = g(TA_{\varphi Z} Y + A_{\varphi Z} TY, X) + g(A_{FTX} Y - A_{FTY} X, Z)$$

which proves our assertion. □

4. Contact Pseudo-Slant Submanifolds of an $(\epsilon)-$ Para-Sasakian Space Form

In this section, we study pseudo-slant submanifolds in an (ϵ)-para-Sasakian space form $\widetilde{M}(k)$ with constant φ-sectional curvature k. We obtain some results for such submanifolds in terms of curvature tensor.

Theorem 4.1. *Let M be a contact pseudo-slant submanifold of an (ϵ)-para Sasakian space form $\widetilde{M}(k)$ of constant curvature k with $k \neq -1$. Then,*

$$\widetilde{R}(D_\theta,\ D_\theta,\ D^{\perp}, D^{\perp}) = 0, \tag{4.1}$$

where $D^{\perp}$ denotes the orthogonal complemantary distribution to D_θ on M and $\widetilde{R}$ denotes the curvature tensor of the (ϵ)-para-Sasakian space form $\widetilde{M}(k)$.

Proof. Let M be a contact pseudo-slant submanifold. Then on making use of the formula (2.9), one may easily obtain the equation (4.1). □

Theorem 4.2. *Let M be a contact pseudo-slant submanifold of an (ϵ)-para-Sasakian space form $\widetilde{M}(k)$ such that $k \neq -1$. If M is a contact pseudo-slant curvature-invariant submanifold. Then,*

i) either M is invariant

ii) or M *anti-invariant,*
iii) or $\dim(M) = 1$.

PROOF. Assume that M is a pseudo-slant curvature-invariant submanifold of an $(\epsilon)-$ para- Sasakian space form $\widetilde{M}(k)$ such that $k \neq -1$. Then from (2.16) and (2.22), we have

$$-g(TX, Z)FY + g(Y, TZ)FX + 2g(TX, Y)FZ = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. If we put, $X = Z$ and $Y = TZ$ then we have, $g(TZ, TZ)FZ = 0$. Here, by equation (2.40), we obtain

$$\cos^2\theta\left\{g(Z, Z) - \epsilon\eta^2(Z)\right\}^2 FZ = 0$$

or

$$\sin 2\theta(g(Z, Z) - \epsilon\eta^2(Z)) = 0$$

which implies that, either M is invariant or anti-invariant submanifold or $\dim(M) = 1$. □

THEOREM 4.3. *Let* M *be a contact pseudo-slant submanifold of an* (ϵ)*-para Sasakian space form* $\widetilde{M}(k)$ *with flat normal connection such that* $k \neq -1$*. If* $TA_V = A_V T$ *for any vector* V *normal to* M*, then* M *is either anti- invariant or a generic submanifold of* $\widetilde{M}(k)$*.*

PROOF. If the normal connection of M is flat, then from (2.20), we have

$$\begin{aligned} g([A_U, A_V]X, Y) &= (\frac{k+1}{4})\{g(\varphi X, U)g(Y, \varphi V) - g(X, \varphi V)g(\varphi Y, U) \\ &\quad +2g(\varphi X, Y)g(\varphi V, U)\}, \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$. Here, choosing $U = fV$ and $Y = TX$, by direct calculations, we can state

$$\begin{aligned} g([A_V, A_{CV}]X, PX) &= -(\frac{k+1}{2})\{g(\varphi X, TX)g(\varphi V, fV)\}, \\ &= -(\frac{k+1}{2})\{g(TX, TX)g(fV, fV)\}, \end{aligned}$$

that is,

$$g(A_{CV}A_V TX - A_V A_{CV}TX, X) = -(\frac{k+1}{2})\left\{g(TX, TX)g(fV, fV)\right\},$$

from which

$$tr(A_{CV}A_V T) - tr(A_V A_{CV}T) = -(\frac{k+1}{2})tr(T^2)g(fV, fV).$$

If $TA_V = A_V T$, then we conclude that $tr(A_{fV}A_V T) = tr(A_V A_{fV}T)$ and thus

$$tr(T^2)g(fV, fV) = 0,$$

from here $dim(M) = 2p + q + 1$, then we can easily to see that

$$(2p + q + 1)cos^2\theta g(fV, fV) = 0.$$

Thus θ is either $\frac{\pi}{2}$ or $f=0$. This implies that M is either anti-invariant or it is a generic submanifold. □

THEOREM 4.4. *Let M be a contact pseudo-slant submanifold of an (ϵ)-para Sasakian space form $\widetilde{M}(k)$. Then the Ricci tensor S of M is given by*

$$\begin{aligned} S(X,W) &= \left\{(\frac{k-3}{4})(2p+q)-(\frac{k+1}{4})(2\epsilon+\epsilon^2\cos^2\theta)\right\}g(X,W) \\ &-(\frac{k+1}{4})(-4+2p+q+2\epsilon^2+\epsilon+\epsilon\cos^2\theta)\eta(X)\eta(W) \\ &+(2p+q+1)g(\sigma(X,W),H)-\sum_{m=1}^{2p+q+1}g(\sigma(e_m,W),\sigma(X,e_m)), \end{aligned} \tag{4.2}$$

for any $X,W\in\Gamma(TM)$.

PROOF. For any $X,Y,Z,W\in\Gamma(TM)$, by using (2.21), we have

$$\begin{aligned} g(R(X,Y)Z,W) &= (\frac{k-3}{4})\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\} \\ &+(\frac{k+1}{4})\{\eta(X)\eta(Z)g(Y,W)-\eta(Y)\eta(Z)g(X,W) \\ &+\eta(Y)\eta(W)g(X,Z)-\eta(X)\eta(W)g(Y,Z) \\ &-g(X,\varphi Z)g(\varphi Y,W)+g(Y,\varphi Z)g(\varphi X,W) \\ &+2g(\varphi X,Y)g(\varphi Z,W)\}+g(\sigma(X,W),\sigma(Y,Z)) \\ &-g(\sigma(Y,W),\sigma(X,Z)). \end{aligned}$$

Now, let e_1, e_2, ..., e_p, $e_{p+1}=\sec\theta Te_1$, $e_{p+2}=\sec\theta Te_2$, ..., $e_{2p}=\sec\theta Te_p$, $e_{2p+1}=\xi$, e_{2p+2}, e_{2p+3}, ..., e_{2p+q+1} be an orthonormal basis of $\Gamma(TM)$ such that e_1, e_2, ..., e_p, $e_{p+1}=\sec\theta Te_1$, $e_{p+2}=\sec\theta Te_2$, ..., $e_{2p}=\sec\theta Te_p$, $e_{2p+1}=\xi$ are tangent to $\Gamma(D_\theta)$ and $e_{2p+2}, e_{2p+3}, ..., e_{2p+q+1}$ are tangent to $\Gamma(D^\perp)$.

Therefore, taking $Y=Z=e_i$, e_j, e_k and $1\leqslant i\leqslant p$, $1\leqslant j\leqslant p$, ξ, $2p+2\leqslant l\leqslant 2p+q+1$ then, we have

$$\begin{aligned} S(X,W) &= \sum_{i=1}^{p}g(R(X,e_i)e_i,W)+\sum_{j=p+1}^{2p}g(R(X,\sec\theta Te_j)\sec\theta Te_j,W) \\ &+g(R(X,\xi)\xi,W)+\sum_{l=2p+2}^{2p+q+1}g(R(X,e_l)e_l,W). \end{aligned}$$

Followed by

$$\begin{aligned}
S(X,W) &= \left\{(\frac{k-3}{4})(2p+q)+(\frac{k+1}{4})(-2\epsilon-\epsilon^2\cos^2\theta)\right\}g(X,W)\\
&+(\frac{k+1}{4})(4-2p-q+2\epsilon^2-\epsilon-\epsilon\cos^2\theta)\eta(X)\eta(W)\\
&+(2p+q+1)g(h(X,W),H)-\sum_{i=1}^{p}g(\sigma(e_i,W),\sigma(X,e_i))\\
&-\sum_{j=p+1}^{2p}g(\sigma(\sec\theta Te_j,W),\sigma(X,\sec\theta Te_j))\\
&-g(\sigma(\xi,W),\sigma(X,\xi))-\sum_{l=2p+2}^{2p+q+1}g(\sigma(e_l,W),\sigma(X,e_l)).
\end{aligned}$$

From here

$$\begin{aligned}
\sum_{m=1}^{2p+q+1}g(\sigma(e_m,W),\sigma(X,e_m)) &= \sum_{i=1}^{p}g(\sigma(e_i,W),\sigma(X,e_i))\\
&+\sum_{j=p+1}^{2p}g(\sigma(\sec\theta Te_j,W),\sigma(X,\sec\theta Te_j))\\
&+g(\sigma(\xi,W),\sigma(X,\xi))\\
&+\sum_{l=2p+2}^{2p+q+1}g(\sigma(e_l,W),\sigma(X,e_l)).
\end{aligned}$$

Hence, the proof follows from the above relation. □

THEOREM 4.5. *Let M be a contact pseudo-slant submanifold of an (ϵ)-para-Sasakian space form $\widetilde{M}(k)$. Then the scalar curvature τ of M is given by*

$$\begin{aligned}
\tau &= \left\{(\frac{k-3}{4})(2p+q)+(\frac{k+1}{4})(-2\epsilon-\epsilon^2\cos^2\theta)\right\}(2p+q+1)\\
&+(\frac{k+1}{4})(4-2p-q+2\epsilon^2-\epsilon-\epsilon\cos^2\theta)+(2p+q+1)^2\left\|H)\right\|^2-\left\|\sigma\right\|^2. \qquad (4.3)
\end{aligned}$$

PROOF. From equation (4.2) by using $X=W=e_m$, we have $\tau=\sum_{m=1}^{2p+q+1}S(e_m,e_m)$ which gives (4.3). Thus, the proof is complete. □

THEOREM 4.6. *Let M be a totally umbilical contact pseudo-slant submanifold of an (ϵ)-para- Sasakian space form $\widetilde{M}(k)$. Then the Ricci tensor S of M is given* *by*

$$S(X,W) = \left\{(\frac{k-3}{4})(2p+q) - (\frac{k+1}{4})(2\epsilon + \epsilon^2\cos^2\theta)\right\} g(X,W)$$

$$-(\frac{k+1}{4})(-4+2p+q+2\epsilon^2+\epsilon+\epsilon\cos^2\theta)\eta(X)\eta(W), \tag{4.4}$$

for any $X, W \in \Gamma(TM)$.

Proof. From equation (4.2) by using (2.13), we obtain

$$\begin{aligned} S(X,W) &= \left\{(\frac{k-3}{4})(2p+q) - (\frac{k+1}{4})(2\epsilon + \epsilon^2\cos^2\theta)\right\} g(X,W) \\ &\quad -(\frac{k+1}{4})(-4+2p+q+2\epsilon^2+\epsilon+\epsilon\cos^2\theta)\eta(X)\eta(W) \\ &\quad +(2p+q+1)g(\sigma(X,W),H) - \sum_{m=1}^{2p+q+1} g(\sigma(e_m,W),\sigma(X,e_m)). \end{aligned}$$

Thus, the proof follows from the above relations, which proves the theorem completely. □

Thus, we have the following corollary.

Corollary 4.1. *Every totally umbilical contact pseudo-slant submanifold* M *of an* (ϵ)*-para-Sasakian space form* $\widetilde{M}(k)$ *is an* η*-Einstein submanifold.*

Theorem 4.7. *Let* M *be a totally umbilical contact pseudo-slant submanifold of an* (ϵ)*-para- Sasakian space form* $\widetilde{M}(c)$*. Then the scalar curvature* τ *of* M *is given by*

$$\tau = \left\{(\frac{k-3}{4})(2p+q) + (\frac{k+1}{4})(-2\epsilon - \epsilon^2\cos^2\theta)\right\}(2p+q+1)$$

$$+(\frac{k+1}{4})(4-2p-q+2\epsilon^2-\epsilon-\epsilon\cos^2\theta). \tag{4.5}$$

Proof. From equation (4.4), by using $X = W = e_m$, we have $\tau = \sum_{m=1}^{2p+q+1} S(e_m, e_m)$ which gives (4.5). Thus the proof is complete. □

Example 4.1. Let M be a submanifold of $\mathbb{R}^9$ defined by the following equation

$$\chi(u,v,w,t,z) = (2u\sin\alpha, -v\cos\alpha, -u\sin\alpha, v\cos\alpha, -w\cos t, \cos t, w\sin t, -\sin t, z).$$

We can easily to see that the tangent bundle of M is spanned by the tangent vectors

$$\begin{aligned} e_1 &= 2\sin\alpha\frac{\partial}{\partial x_1} - \sin\alpha\frac{\partial}{\partial x_2},\ e_2 = -\cos\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial y_2}, e_5 = \xi = \frac{\partial}{\partial z}. \\ e_3 &= -\cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial x_4},\ e_4 = w\sin t\frac{\partial}{\partial x_3} - \sin t\frac{\partial}{\partial y_3} + w\cos t\frac{\partial}{\partial\ x_4} - \cos t\frac{\partial}{\partial y_4}. \end{aligned}$$

For the almost paracontact metric structure φ of $\mathbb{R}^9$, whose coordinate systems $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, z)$, choosing

$$\begin{aligned} \varphi(\frac{\partial}{\partial x_i}) &= \frac{\partial}{\partial y_i}, \varphi(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_{\ j}}, 1 \leqslant i, j \leqslant 4 \\ \varphi(\frac{\partial}{\partial z}) &= 0,\ \xi = \frac{\partial}{\partial z},\ \eta = dz, \\ g &= \eta \otimes \eta + \sum_{i=1}^{4} d_{x_i} \otimes d_{x_i} - \sum_{i=1}^{4} d_{y_i} \otimes d_{y_i}, \end{aligned}$$

then we have

$$\begin{aligned} \varphi e_1 &= 2\sin\alpha\frac{\partial}{\partial y_1} - \sin\alpha\frac{\partial}{\partial y_2}, \varphi e_2 = \cos\alpha\frac{\partial}{\partial x_1} - \cos\alpha\frac{\partial}{\partial x_2}, \\ \varphi e_3 &= -\cos t\frac{\partial}{\partial y_3} + \sin t\frac{\partial}{\partial y_4}, \end{aligned}$$

and

$$\varphi e_4 = w\sin t\frac{\partial}{\partial y_3} + \sin t\frac{\partial}{\partial x_3} + w\cos t\frac{\partial}{\partial y_4} + \cos t\frac{\partial}{\partial x_4}.$$

By direct calculations, we can infer $D_\theta = span\{e_1, e_2\}$ is a slant distribution with slant angle

$$\cos\theta = \frac{g(e_1, \varphi e_2)}{\|e_1\|\,\|\varphi e_2\|} = \frac{3\sin\alpha.\cos\alpha}{\sqrt{5\sin^2\alpha}\sqrt{2\cos^2\alpha}} = \frac{3\sqrt{10}}{10},$$

$$\theta = arc\cos(\frac{3\sqrt{10}}{10}).$$

Since $g(\varphi e_3, e_i) = 0,\ \ i = 1,2,4,5$ and $g(\varphi e_4, e_j) = 0,\ j = 1,2,3,5$ are orthogonal to M, $D^\perp = span\{e_3, e_4, e_5\}$ is an anti-invariant distribution. Thus M is a 5-dimensional proper contact pseudo-slant submanifold of $\mathbb{R}^9$ with it's usual almost paracontact metric structure.

References

[1] B. E. Acet, S. Y. Perktas and E. Kilic. Symmetric lightlike hypersurfaces of a para-Sasakian space form. *An. St. Univ. Al. I. Cuza. Univ. Iasi.*, **LXII** (2)(Vol. 3)(2016), 915–926.

[2] M. Atçeken and S. K. Hui. Slant and pseudo-slant submanifolds in $(LCS)_n$-manifolds. *Czechoslovak Math. J.*, **63**(1)(2013), 177–190.

[3] J. K. Beem and P. E. Ehrlich. *Global Lorentzian Geometry.* New York: Marcel Dekker, Pure Appl. Math. 67, 1981.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fern ández and M. Fernández. Slant submanifolds in Sasakian manifolds. *Glasgow Math. J.*, **42**(2000), 125–138.

[5] J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández. Semi-slant submanifolds of a Sasakian manifold. *Geomeatriae Dedicata*, **78**(2)(1999),183–199.

[6] B. Y. Chen. *Geometry of slant submanifolds.* Katholieke Universiteit Leuven, Leuven, Belgium, View at Zentralblatt Mathematic, 1990.

[7] B.- Y. Chen. Slant immersions. *Bull. Austral. Math. Soc.*, **41**(1)(1990),135–147.

[8] S. Dirik and M. Atçeken. Pseudo-slant submanifolds in Cosymplectic space forms. *Acta Univ. Sapientiae mathematica*, **8**(1)(2016), 53–74.

[9] S. Dirik, M. Atçeken and Ü. Yıldırim. Contact pseudo-slant submanifolds of a Kenmotsu manifold. *J. Math. Computer Sci.*, **16**(3)(2016), 386–394.
[10] S. Kaneyuki and M. Konzai. Paracomplex structure and affine symmetric spaces. *Tokyo J. Math.*, **8**(1)(1985), 81–98.
[11] V. A. Khan and M. A. Khan. Pseudo-slant submanifolds of a Sasakian manifold. *Indian J. Prue Appl. Math.*, **38**(1)(2007), 31–42.
[12] A. Lotta. Slant submanifolds in contact geometry. *Bulletin Mathematical Society Roumanie*, **39**(87)(1996), 183–198.
[13] B. O. Neill. *Semi-Riemannian Geometry with Application to Relativity.* Academic Press, 1983.
[14] N. Papaghuic. Semi-slant submanifolds of a Kaehlarian manifold. *An. St. Univ. Al. I. Cuza. Univ. Iasi*, **40**(1994), 55–61.
[15] S. Y. Perktaş, E. Kiliç and B. E. Acet. Lightlike hypersurfaces of a para-Sasakian space form. *Gulf J. Math.*, **2**(2)(2014), 7–18.
[16] S. Y. Perktaş, E. Kiliç, M. M. Tripathi and S. Keleş. On (ϵ)-para Sasakian 3-manifolds. *Int. J. Pure Appl. Math.*, **77**(4)(2012),485–499.
[17] B. Sahin. Warped product submanifolds of Kaehler manifolds with a slant factor. *Ann. Pol. Math.*, **95**(3)(2009), 207–226.
[18] S. Uddin, Z. Ahsan and A.H. Yaakub. Classification of totally umbilical slant submanifolds of a Kenmotsu manifold. *Filomat*, **30**(9)(2016), 2405-2412.
[19] S. Uddin, C. Ozel , M. A. Khan and Singh K. Some classification result on totally umbilical proper slant and hemi slant submanifolds of a nearly Kenmotsu manifold. *Int. J. Physical Sci.*, **7**(40)(2012), 5538–5544.
[20] S. Uddin , C. Ozel and V. A. Khan. A classification of a totally umbilical slant submanifold of Cosymplectic manifolds. *Abscract and applied analysis*, **2012**(2012), Article ID 716967, 8 pages.
[21] I. Sato. On a structure similar to the almost contact structure. *Tensor N.S.*, **30**(3)(1976), 219–224.
[22] S. Tanno. The automorphism groups of almost contact Riemannian manifolds. *Tohoku Math. J.*, **21**(1)(1969), 21–38.
[23] T. Takahashi. Sasakian manifold with Pseudo-Riemannian metric. *Tohoku Math. J.*, **21**(2)(1969), 644–653.
[24] M. M. Tripathi, E. Kiliç, S. Y. Perktaş and S. Keleş. Indefinite almost paracontactmetric manidolds. *Int. J. Math. Math. Sci.*, **2010**(2010), Article ID 846195, 19 pages.
[25] S. Zamkovoy. Para-Sasakian manifolds with a constant paraholomorphic section curvature. 2009, arXiv:0812.1676v2.

Received by editors 31.05.2019; Revised version 05.09.2019; Available online 16.09.2019.

Department of Statistics, University of Amasya, 05250, Amasya, TURKEY
E-mail address: suleyman.dirik@amasya.edu.tr

Department of Mathematics, University of Gaziosmanpasa, 6000, Tokat, TURKEY
E-mail address: mehmet.atceken@gop.edu.tr

Department of Mathematics, University of Gaziosmanpasa, 6000, Tokat, TURKEY
E-mail address: umit.yildirim@gop.edu.tr