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A NOTE ON THE EXTENSION OF THE SOFT SUBSTRUCTURES OF A SOFT SEMIGROUP

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Abstract. In this paper, we introduce the notions of extended soft substructures for the soft substructures of a soft semigroup and study their lattice theoretic properties. Further, we show that the complete lattice of all (regular) soft substructures of a soft semigroup is complete (isomorphic) epimorphic to the complete lattice of all extended soft substructures for the soft substructures of a soft semigroup.

1. Introduction

In 1999, Molodtsov [**9**] introduced the notion of a soft set over a universal set as a mathematical tool for modelling uncertainties. Since its introduction, several mathematicians imposed various algebraic (sub) structures on them and studied some of their elementary properties. In 2010, Ali-Shabir-Shum [**3**] introduced the notions of soft semigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) over a semigroup and studied some of their properties.

In this paper we introduce the notions of extended soft substructures for the soft substructures of a soft semigroup. Further, we study the lattice theoretic properties of these extended soft substructures for the soft substructures of a soft semigroup and the relation between the (regular) soft substructures of a soft semigroup and the extended soft substructures for the soft substructures of a soft semigroup.

Notice that results of this paper play an important role in proving the results of Murthy-Maheswari [**13**], Representation of Soft Substructures of a Soft Semigroup by Products, namely, for any soft semigroup over a semigroup 1. there is a crisp

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semigroup such that the complete lattice of all soft substructures of the given soft semigroup is complete epimorphic to a complete lattice of certain substructures of the crisp semigroup and 2. there is a crisp semigroup such that the complete lattice of all regular soft substructures of the given soft semigroup is complete isomorphic to a complete lattice of certain substructures of the crisp semigroup.

2. Preliminaries

In this section we recall some basic definitions and elementary results in the theory of Lattices, Semigroups, Soft Sets and Soft Semigroups which are used in the main results later.

Definition 2.1. For any poset $(Q, \leqslant_Q)$ and for any subset P of Q, the binary relation $\leqslant_P$ on P defined by $\leqslant_P = \{(a, b)|a, b \in P \text{ and } (a, b) \in \leqslant_Q\} = (P \times P) \cap \leqslant_Q$ makes $(P, \leqslant_P)$ a subposet of $(Q, \leqslant_Q)$ and this is called the *induced partial ordering* from the super poset Q.

Lemma 2.1. *For any poset $(Q, \leqslant_Q)$, for any subposet P of Q with the induced partial ordering from the super poset Q and for any non-empty subset A of P, $\vee_Q A \leqslant_Q \vee_P A$ ($\wedge_P A \leqslant_Q \wedge_Q A$) whenever both of them exist. However, equality holds whenever $\vee_Q A \in P$ ($\wedge_Q A \in P$).*

Definition 2.2. For any non-empty subset S of a meet (join) complete poset L with the largest (least) element 1_L (0_L), one can define $\overline{\vee} S = \wedge\{\beta \in L|\alpha \wedge \beta = \alpha \text{ for all } \alpha \in S\}$ ($\overline{\wedge} S = \vee\{\beta \in L|\alpha \wedge \beta = \beta \text{ for all } \alpha \in S\}$) called the *meet (join) induced join (meet)* in L. Then L is a complete lattice with the $\overline{\vee}$ ($\overline{\wedge}$) called the *associated complete lattice* for the meet (join) complete poset L.

Definition 2.3. A set S together with a binary operation which is associative is called a *semigroup*.

Notice that the empty set is trivially a semigroup with the empty binary operation called the *empty semigroup*.

Definition 2.4. For any pair of subsets A, B of a semigroup S, the set AB is defined by $AB = \{ab \in S|a \in A \text{ and } b \in B\}$ and it is a subset of S.

Definition 2.5. For any subset A of a semigroup S,

(1) A is a *subsemigroup* of S iff $A^2 \subseteq A$. Notice that as in Grillet[**6**], the empty semigroup is trivially a subsemigroup of any semigroup;

(2) A is a *left (right) ideal* of S iff $SA \subseteq A$ ($AS \subseteq A$);

(3) A is an *ideal* of S iff $SA \cup AS \subseteq A$ iff it is both a left and a right ideal of S;

(4) A is a *quasi-ideal* of S iff $SA \cap AS \subseteq A$;

(5) A is a *bi-ideal* of S iff $AA \subseteq A$ and $ASA \subseteq A$.

Lemma 2.2. *In any semigroup S, the following are true:*

(1) *The empty semigroup is trivially a (left, right, quasi-, bi-) ideal of S;*

(2) *Arbitrary union of (left, right) ideals of S is a (left, right) ideal of S but arbitrary union of subsemigroups (quasi-ideals, bi-ideals) of S need not be a subsemigroup (quasi-ideal, bi-ideal) of S*

(3) *Arbitrary intersection of subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of S is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S;*

(4) *The intersection of all subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of S containing a given subset is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S which is unique and smallest with respect to the containment of the given subset;*

(5) *For any subset A of a semigroup S, whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the unique smallest subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) containing the given subset A defined as in (4) above is called the subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) generated by A and is denoted by $(A)_{s,\,S}$ $((A)_{*,\,S})$. Notice that $(\phi)_{*,\,S} = \phi$ and $A \neq \phi$ iff $(A)_{*,\,S} \neq \phi$.*

LEMMA 2.3. *Whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, for any subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) B of S and for any subset A of B, $(A)_{s,\,S}$ $((A)_{*,\,S})$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of B.*

DEFINITION 2.6. A semigroup S has a *zero element* 0_S iff $0_S \in S$ such that $0_S 0_S = 0_S s = s 0_S = 0_S$ for all $s \in S$.

Notice that a semigroup cannot have more than one zero element.

REMARK 2.1. Whenever S is a semigroup, an element 0 can be adjoined to S such that S is a subsemigroup of $S \cup \{0\}$ and $00 = 0s = s0 = 0$ for all $s \in S$.

DEFINITION 2.7. For any semigroup S, the semigroup $S \cup \{0\}$ such that S is a subsemigroup of $S \cup \{0\}$ is called the 0-*adjoined semigroup* and is denoted by S_0.

DEFINITION 2.8. For any semigroup S and for any subset B of the 0-adjoined semigroup S_0, $B - \{0\}$ is called the *0-contraction* of B in S.

Notice that the 0-contraction of S_0 is S and the 0-contraction of ϕ is ϕ itself.

LEMMA 2.4. *In any semigroup S, the following are true:*

(1) *If A is a subsemigroup of S then A is also a subsemigroup of S_0;*

(2) *If A is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S then A_0 is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S_0;*

(3) *If $\phi \neq B$ is a (left, right, quasi-, bi-) ideal of S_0 then $0 \in B$;*

(4) *If B is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S_0 then $B - \{0\}$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S;*

(5) *For any subset A of S, $(A)_{*,\,S} \cup \{0\} = (A \cup \{0\})_{*,\,S_0}$ for $* = s,\, q,\, b$.*

DEFINITION 2.9. ([**9**]) Let U be a universal set, $P(U)$ be the power set of U and E be a set of parameters. A pair (F, E) is called a *soft set* over U iff $F : E \to P(U)$ is a mapping defined by for each $e \in E$, $F(e)$ is a subset of U.

Notice that a collective presentation of all the notions algebras, soft sets, fuzzy soft sets, f-soft algebras, f-fuzzy soft algebras in the single paper, Murthy-Maheswari

[**10**] raised some serious notational conflicts and to fix the same we deviated from the above notation for a soft set and adapted the following notation for convenience as follows:

Let U be a universal set. A typical *soft set* over U is an ordered pair $\mathsf{E} = (\sigma_E, E)$, where E is a set of *parameters*, called the *underlying parameter set* for E, $P(U)$ is the power set of U and $\sigma_E : E \to P(U)$ is a map, called the *underlying set valued map* for E.

Definition 2.10. ([**4**]) The *empty soft set* over U is a soft set with the empty parameter set, denoted by $\Phi = (\sigma_\phi, \phi)$. Clearly, it is unique.

Definition 2.11. ([**4**]) A soft set E over U is said to be a *null soft set* iff $\sigma_E e = \phi$ for all $e \in E$.

Definition 2.12. ([**14**]) For any pair of soft sets A, B over U, A is a *soft subset* of B, denoted by $\mathsf{A} \subseteq \mathsf{B}$, iff (i) $A \subseteq B$ (ii) $\sigma_A a \subseteq \sigma_B a$ for all $a \in A$.

Definition 2.13. For any family of soft subsets $(\mathsf{A}_i)_{i\in I}$ of E,

(1) ([**5**]) the *soft union* of $(\mathsf{A}_i)_{i\in I}$, denoted by $\cup_{i\in I}\mathsf{A}_i$, is defined by the soft set A, where

(i) $A = \cup_{i\in I} A_i$

(ii) $\sigma_A a = \cup_{i\in I_a} \sigma_{A_i} a$ for all $a \in A$, where $I_a = \{i \in I/a \in A_i\}$;

(2) the *soft intersection* of $(\mathsf{A}_i)_{i\in I}$, denoted by $\cap_{i\in I}\mathsf{A}_i$, is defined by the soft set A, where

(i) $A = \cap_{i\in I} A_i$

(ii) $\sigma_A a = \cap_{i\in I} \sigma_{A_i} a$ for all $a \in A$.

Definition 2.14. ([**3**]) A soft set (F, A) over a semigroup S which is *neither* empty *nor* null is said to be a soft semigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) over S iff $F(a)$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of S for all $a \in A$ whenever $F(a) \neq \phi$.

3. Soft Substructures of a Soft Semigroup

In what follows from Murthy-Maheswari [**12**], we recall the notions of soft (sub) semigroup, soft (left, right, quasi-, bi-) ideal of a soft semigroup and some properties of them which are used in the due course. Notice that throughout this section U is a semigroup unless otherwise explicitly stated.

Definition 3.1. A soft set E over a semigroup U is said to be a *soft semigroup* over U iff $\sigma_E e$ is a subsemigroup of U for all $e \in E$. Consequently, for us the empty soft set Φ and the null soft set Φ_E over U are trivially soft semigroups over U.

Definition 3.2. For any soft subset A of a soft semigroup E over U, A is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E iff $\sigma_A a$ is a subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of $\sigma_E a$ for all $a \in A$.

Notice that the empty soft subset Φ and a null soft subset Φ_A of E are trivially soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of E.

Whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the set of *all soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals)* of E is denoted by $\mathcal{S}_s(\mathsf{E})$ ($\mathcal{S}_*(\mathsf{E})$).

Definition 3.3. For any soft semigroup E over U and for any soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) A of E, A is a *d-total (regular) soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal)* of E iff $A = E$ ($\sigma_A a \neq \phi$ for all $a \in A$).

Notice that the empty soft set Φ is trivially regular soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E. Whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the set of *all d-total (regular) soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals)* of E is denoted by $\mathcal{S}^d_*(\mathsf{E})$ ($\mathcal{S}^r_*(\mathsf{E})$) and the set of *all d-total and regular soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals)* of E is denoted by $\mathcal{S}^{d,\,r}_*(\mathsf{E})$.

Definition 3.4. For any soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) A of a soft semigroup E over U, the *support* of A, denoted by $\mathrm{Supp}(\mathsf{A})$, is defined by $\mathrm{Supp}(\mathsf{A}) = \{a \in A / \sigma_A a \neq \phi\}$. Notice that $\mathrm{Supp}(\mathsf{A}) \subseteq A$.

Lemma 3.1. *For any soft semigroup E over U, the following are true:*

(1) *For any pair of soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) A, B of E, A is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of B iff A is a soft subset of B;*

(2) *Arbitrary union of soft (left, right) ideals of E is always a soft (left, right) ideal of E but arbitrary union of soft subsemigroups (quasi-ideals, bi-ideals) of E need not be a soft subsemigroup (quasi-ideal, bi-ideal) of E;*

(3) *Arbitrary intersection of soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of E is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E;*

(4) *The intersection of all soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) containing a given soft subset is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) which is unique and smallest with respect to the containment of the given soft subset.*

Definition 3.5. For any soft subset A of a soft semigroup E over U, whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the unique smallest soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E containing A defined as in the Lemma 3.1(4) is called the *soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) generated by* A and is denoted by $(\mathsf{A})_{s,\,\mathsf{E}}$ ($(\mathsf{A})_{*,\,\mathsf{E}}$).

Lemma 3.2. *For any soft subset A of a soft semigroup E over U, whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) generated by A, $(\mathsf{A})_{s,\,\mathsf{E}}$ ($(\mathsf{A})_{*,\,\mathsf{E}}$), is given by C, where $C = A$ and $\sigma_C e = (\sigma_A e)_{s,\,\sigma_E e}$ ($(\sigma_A e)_{*,\,\sigma_E e}$) for all $e \in C$.*

Proof. Clearly, C is a soft subsemigroup of E. Let B be a soft subsemigroup of E such that $\mathsf{A} \subseteq \mathsf{B}$. Then $A \subseteq B$ and $\sigma_A e \subseteq \sigma_B e$ for all $e \in A$ implies $C = A \subseteq B$ and by the Lemma 2.3, $\sigma_C e = (\sigma_A e)_{s,\,\sigma_E e} \subseteq \sigma_B e$ for all $e \in C = A$ or $\mathsf{C} \subseteq \mathsf{B}$

or C is the smallest soft subsemigroup of E containing A or $\mathsf{C} = (\mathsf{A})_{s,\mathsf{E}}$. For $* = l, r, i, q, b$, the proofs follow in a similar way as above. □

Notation: Whenever $* = s\,(l, r, i, q, b)$, for any soft semigroup E over U and for any pair of soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) A, B of E, $\mathsf{A} \leqslant_s \mathsf{B}$ ($\mathsf{A} \leqslant_* \mathsf{B}$) iff A is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of B.

THEOREM 3.1. *For any soft semigroup E over U, whenever $* = s, q, b, l, r, i$, the set $\mathcal{S}_*(\mathsf{E})$ is a complete lattice with*

(1) *the partial ordering defined by: for any $\mathsf{A}, \mathsf{B} \in \mathcal{S}_*(\mathsf{E})$, $\mathsf{A} \leqslant \mathsf{B}$ iff $\mathsf{A} \leqslant_* \mathsf{B}$;*

(2) *the largest and the least elements in $\mathcal{S}_*(\mathsf{E})$ are E and Φ respectively;*

(3) *for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_*(\mathsf{E})$, $\wedge_{i\in I}\mathsf{A}_i = \cap_{i\in I}\mathsf{A}_i$;*

(4) *for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_*(\mathsf{E})$, however,*

(i) *for $* = s, q, b$, $\vee_{i\in I}\mathsf{A}_i = \overline{\vee}_{i\in I}\mathsf{A}_i$, where $\overline{\vee}$ is the meet induced join in $\mathcal{S}_*(\mathsf{E})$ and $\overline{\vee}_{i\in I}\mathsf{A}_i = \mathsf{A}$, where $A = \cup_{i\in I}A_i$ and $\sigma_A e = (\cup_{i\in I_e}\sigma_{A_i}e)_{*,\sigma_E e}$ for all $e \in A$, where $I_e = \{i \in I / e \in A_i\}$, and*

(ii) *for $* = l, r, i$, $\vee_{i\in I}\mathsf{A}_i = \cup_{i\in I}\mathsf{A}_i$.*

PROOF. (1): Clearly, $\mathcal{S}_s(\mathsf{E})$ is a poset with the partial ordering defined by: for any $\mathsf{A}, \mathsf{B} \in \mathcal{S}_s(\mathsf{E})$, $\mathsf{A} \leqslant \mathsf{B}$ iff A is a soft subsemigroup of B iff $\mathsf{A} \subseteq \mathsf{B}$ by the Lemma 3.1(1).

(2): $\Phi \subseteq \mathsf{A} \subseteq \mathsf{E}$ for all $\mathsf{A} \in \mathcal{S}_s(\mathsf{E})$ implies $\Phi \leqslant \mathsf{A} \leqslant \mathsf{E}$ for all $\mathsf{A} \in \mathcal{S}_s(\mathsf{E})$ implying the largest and the least elements in $\mathcal{S}_s(\mathsf{E})$ are E and Φ respectively.

(3): Let $(\mathsf{A}_i)_{i\in I}$ be a subset of $\mathcal{S}_s(\mathsf{E})$. By the Lemma 3.1(3), we have $\cap_{i\in I}\mathsf{A}_i = \mathsf{A} \in \mathcal{S}_s(\mathsf{E})$.

(i) $\cap_{i\in I}\mathsf{A}_i = \mathsf{A} \subseteq \mathsf{A}_i$ for all $i \in I$ as $A = \cap_{i\in I}A_i \subseteq A_i$ for all $i \in I$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i}e \subseteq \sigma_{A_i}e$ for all $e \in A$ and for all $i \in I$ implies $\cap_{i\in I}\mathsf{A}_i = \mathsf{A} \leqslant \mathsf{A}_i$.

(ii) $\mathsf{B} \in \mathcal{S}_s(\mathsf{E})$ such that $\mathsf{B} \leqslant \mathsf{A}_i$ for all $i \in I$ implies $\mathsf{B} \subseteq \mathsf{A}_i$ for all $i \in I$ implies $B \subseteq A_i$ for all $i \in I$ and $\sigma_B e \subseteq \sigma_{A_i}e$ for all $e \in B$ and for all $i \in I$ which implies $B \subseteq \cap_{i\in I}A_i = A$ and $\sigma_B e \subseteq \cap_{i\in I}\sigma_{A_i}e = \sigma_A e$ for all $e \in B$ which imply $\mathsf{B} \subseteq \mathsf{A} = \cap_{i\in I}\mathsf{A}_i$ implying $\mathsf{B} \leqslant \mathsf{A} = \cap_{i\in I}\mathsf{A}_i$.

Now (i) and (ii) imply $\wedge_{i\in I}\mathsf{A}_i = \cap_{i\in I}\mathsf{A}_i$.

(4): Let $(\mathsf{A}_i)_{i\in I}$ be a subset of $\mathcal{S}_s(\mathsf{E})$. Now we claim that $\overline{\vee}_{i\in I}\mathsf{A}_i = \mathsf{A}$, where $A = \cup_{i\in I}A_i$ and $\sigma_A e = (\cup_{i\in I_e}\sigma_{A_i}e)_{s,\sigma_E e}$ for all $e \in A$, where $I_e = \{i \in I / e \in A_i\}$.

Let us recall that $(\mathsf{A})_{s,\mathsf{E}} = \cap_{\mathsf{A}\subseteq\mathsf{B},\,\mathsf{B}\leqslant_s\mathsf{E}}\mathsf{B} = \cap_{\mathsf{A}\subseteq\mathsf{B},\,\mathsf{B}\in\mathcal{S}_s(\mathsf{E})}\mathsf{B} = \wedge_{\mathsf{A}\subseteq\mathsf{B},\,\mathsf{B}\in\mathcal{S}_s(\mathsf{E})}\mathsf{B}$. For any $\mathsf{B} \in \mathcal{S}_s(\mathsf{E})$, $\cup_{i\in I}\mathsf{A}_i \subseteq \mathsf{B}$ iff $\mathsf{A}_i \subseteq \mathsf{B}$ for all $i \in I$ iff A_i is a soft subsemigroup of B for all $i \in I$ by the Lemma 3.1(1) iff $\mathsf{A}_i \leqslant \mathsf{B}$ for all $i \in I$.

Now $(\cup_{i\in I}\mathsf{A}_i)_{s,\mathsf{E}} = \wedge_{\cup_{i\in I}\mathsf{A}_i\subseteq\mathsf{B},\,\mathsf{B}\in\mathcal{S}_s(\mathsf{E})}\mathsf{B} = \wedge_{\mathsf{A}_i\leqslant\mathsf{B}\text{ for all }i\in I,\,\mathsf{B}\in\mathcal{S}_s(\mathsf{E})}\mathsf{B} = \overline{\vee}_{i\in I}\mathsf{A}_i$.

On the other hand, if $\mathsf{C} = \cup_{i\in I}\mathsf{A}_i$ then $C = \cup_{i\in I}A_i$ and $\sigma_C e = \cup_{i\in I_e}\sigma_{A_i}e$ for all $e \in C$, where $I_e = \{i \in I / e \in A_i\}$. Now by the Lemma 3.2, $\overline{\vee}_{i\in I}\mathsf{A}_i = (\cup_{i\in I}\mathsf{A}_i)_{s,\mathsf{E}} = (\mathsf{C})_{s,\mathsf{E}} = \mathsf{D}$, where $D = C$ and $\sigma_D e = (\sigma_C e)_{s,\sigma_E e}$ for all $e \in D$.

We show that $\mathsf{D} = \mathsf{A}$ or (i) $D = A$ (ii) $\sigma_D e = \sigma_A e$ for all $e \in D$.

(i): $D = C = \cup_{i\in I}A_i = A$.

(ii): Let $e \in D = A$ be fixed. Now $\sigma_D e = (\sigma_C e)_{s,\sigma_E e} = (\cup_{i\in I_e}\sigma_{A_i}e)_{s,\sigma_E e} = \sigma_A e$.

Now (i) and (ii) imply $\mathsf{D} = \mathsf{A}$ or $\overline{\vee}_{i\in I}\mathsf{A}_i = \mathsf{A}$ as required.

From (1)-(4) and by the Definition 2.2, we get that $\mathcal{S}_s(\mathsf{E})$ is a complete lattice.

For $* = q$, b, the proofs follow in a similar way as above and for $* = l$, r, i, the proofs of (1)-(3) follow in a similar way as (1)-(3) above and the proof of (4) follows in a similar way as (3) above. □

Lemma 3.3. *For any soft semigroup E over U and for any family of soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) $(\mathsf{A}_i)_{i\in I}$ of E, the following are true:*

(1) *$Supp(\cap_{i\in I}\mathsf{A}_i) \subseteq \cap_{i\in I}Supp(\mathsf{A}_i)$;*

(2) *$Supp(\overline{\vee}_{i\in I}\mathsf{A}_i) = Supp(\cup_{i\in I}\mathsf{A}_i) = \cup_{i\in I}Supp(\mathsf{A}_i)$.*

Theorem 3.2. *For any soft semigroup E over U, whenever $* = s$, q, b, the set $\mathcal{S}^r_*(\mathsf{E})$ is a join complete subposet of the complete lattice $\mathcal{S}_*(\mathsf{E})$ and also itself is a complete lattice with*

(1) *the induced partial ordering from the super poset $\mathcal{S}_*(\mathsf{E})$;*

(2) *the largest and the least elements in $\mathcal{S}^r_*(\mathsf{E})$ are L, where $L = Supp(\mathsf{E})$ and $\sigma_L e = \sigma_E e$ for all $e \in L$, and Φ respectively*

for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}^r_*(\mathsf{E})$;

(3) *$\wedge_{i\in I}\mathsf{A}_i = \sqcap_{i\in I}\mathsf{A}_i$, where $\sqcap_{i\in I}\mathsf{A}_i = \mathsf{A}$ such that $A = Supp(\cap_{i\in I}\mathsf{A}_i)$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i}e$ for all $e \in A$;*

(4) *$\vee_{i\in I}\mathsf{A}_i = \overline{\sqcup}_{i\in I}\mathsf{A}_i = \overline{\vee}_{i\in I}\mathsf{A}_i$, where $\overline{\sqcup}$ and $\overline{\vee}$ are $\sqcap$ and $\wedge$ induced joins in $\mathcal{S}^r_*(\mathsf{E})$ and $\mathcal{S}_*(\mathsf{E})$ respectively.*

Proof. (1): It follows from the Definition 2.1.

(2): It follows in a similar way as in the Theorem 3.1(2).

(3): Let $(\mathsf{A}_i)_{i\in I}$ be a subset of $\mathcal{S}^r_s(\mathsf{E})$. Define $\sqcap_{i\in I}\mathsf{A}_i$ by A, where $A = Supp(\cap_{i\in I}\mathsf{A}_i)$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i}e$ for all $e \in A$. Clearly, $\mathsf{A} \in \mathcal{S}^r_s(\mathsf{E})$.

Now we claim that, in $\mathcal{S}^r_s(\mathsf{E})$, $\wedge_{i\in I}\mathsf{A}_i = \mathsf{A}$.

(i) If $A = \phi$ then $\mathsf{A} = \Phi \in \mathcal{S}^r_s(\mathsf{E})$ and $\mathsf{A} = \Phi \leqslant \mathsf{A}_i$ for all $i \in I$ in $\mathcal{S}^r_s(\mathsf{E})$. If $A \neq \phi$ then $\mathsf{A} \leqslant \cap_{i\in I}\mathsf{A}_i \leqslant \mathsf{A}_i$ for all $i \in I$ in $\mathcal{S}_s(\mathsf{E})$ implies $\mathsf{A} \leqslant \mathsf{A}_i$ for all $i \in I$ in $\mathcal{S}^r_s(\mathsf{E})$.

(ii) $\mathsf{B} \leqslant \mathsf{A}_i$ in $\mathcal{S}^r_s(\mathsf{E})$ implies $\mathsf{B} \leqslant \mathsf{A}_i$ in $\mathcal{S}_s(\mathsf{E})$ implies $\mathsf{B} \leqslant \cap_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_s(\mathsf{E})$ implies $\mathsf{B} \subseteq \cap_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_s(\mathsf{E})$ implies $B \subseteq \cap_{i\in I}A_i$ and $\phi \neq \sigma_B e \subseteq \cap_{i\in I}\sigma_{A_i}e$ for all $e \in B$ implies $B \subseteq \mathrm{Supp}(\cap_{i\in I}\mathsf{A}_i) = A$ and $\phi \neq \sigma_B e \subseteq \cap_{i\in I}\sigma_{A_i}e = \sigma_A e$ for all $e \in B$ implying $\mathsf{B} \subseteq \mathsf{A}$. If $A = \phi$ then $B = \phi$ and so $\mathsf{B} = \Phi \leqslant \mathsf{A}$ in $\mathcal{S}^r_s(\mathsf{E})$.

If $A \neq \phi$ then $\mathsf{B} \subseteq \mathsf{A}$ in $\mathcal{S}^r_s(\mathsf{E})$ and so $\mathsf{B} \leqslant \mathsf{A}$ in $\mathcal{S}^r_s(\mathsf{E})$.

Now (i) and (ii) imply $\wedge_{i\in I}\mathsf{A}_i = \mathsf{A} = \sqcap_{i\in I}\mathsf{A}_i$.

(4): Let $(\mathsf{A}_i)_{i\in I}$ be a subset of $\mathcal{S}^r_s(\mathsf{E})$. Now we claim that $\overline{\vee}_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_s(\mathsf{E}) = \overline{\sqcup}_{i\in I}\mathsf{A}_i$ in $\mathcal{S}^r_s(\mathsf{E}) = \sqcap_{\mathsf{A}_i \leqslant \mathsf{B} \text{ for all } i\in I \text{ in } \mathcal{S}^r_s(\mathsf{E})}\mathsf{B}$. $\mathsf{A} = \overline{\vee}_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_s(\mathsf{E})$ implies $A = \cup_{i\in I}A_i$ and $\sigma_A e = (\cup_{i\in I_e}\sigma_{A_i}e)_{s,\,\sigma_E e}$ for all $e \in A$, where $I_e = \{i \in I / e \in A_i\}$. Clearly, $\mathsf{A} \in \mathcal{S}^r_s(\mathsf{E})$.

(i) $\mathsf{A} \in \mathcal{S}^r_s(\mathsf{E})$ and $\mathsf{A}_i \leqslant \mathsf{A}$ for all $i \in I$ in $\mathcal{S}^r_s(\mathsf{E})$ implies $\overline{\sqcup}_{i\in I}\mathsf{A}_i \subseteq \mathsf{A} = \overline{\vee}_{i\in I}\mathsf{A}_i$.

(ii) $\mathsf{B} \in \mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{A}_i \leqslant \mathsf{B}$ for all $i \in I$ in $\mathcal{S}^r_s(\mathsf{E})$ implies $\mathsf{A}_i \subseteq \mathsf{B}$ for all $i \in I$ in $\mathcal{S}^r_s(\mathsf{E})$ implies $A_i \subseteq B$ for all $i \in I$ and $\sigma_{A_i}e \subseteq \sigma_B e$ for all $e \in A_i$ and for all $i \in I$ in $\mathcal{S}^r_s(\mathsf{E})$ implies $A = \cup_{i\in I}A_i \subseteq B$ and $\cup_{i\in I_e}\sigma_{A_i}e \subseteq \sigma_B e$ for all $e \in A$ in $\mathcal{S}^r_s(\mathsf{E})$ implies $A \subseteq B$ and by the Lemma 2.3, $\sigma_A e = (\cup_{i\in I_e}\sigma_{A_i}e)_{s,\,\sigma_E e} \subseteq \sigma_B e$ for all

$e \in A$ in $\mathcal{S}_s^r(\mathsf{E})$ implies $\mathsf{A} \subseteq \mathsf{B}$ implies $\overline{\vee}_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_s(\mathsf{E}) = \mathsf{A} \subseteq \sqcap_{\mathsf{A}_i \leqslant \mathsf{B} \text{ for all } i\in I \text{ in } \mathcal{S}_s^r(\mathsf{E})}\mathsf{B}$ $= \sqcup_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_s^r(\mathsf{E})$.

Now (i) and (ii) imply $\sqcup_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_s^r(\mathsf{E}) = \overline{\vee}_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_s(\mathsf{E})$ or $\mathcal{S}_s^r(\mathsf{E})$ is a join complete subposet of $\mathcal{S}_s(\mathsf{E})$.

From (1)-(4) and by the Definition 2.2, we get that $\mathcal{S}_s^r(\mathsf{E})$ is a complete lattice. For $* = q$, b, the proofs follow in a similar way as above. □

Theorem 3.3. *For any soft semigroup E over U, whenever $* = l$, r, i, the set $\mathcal{S}_*^r(\mathsf{E})$ is a join complete subposet of the complete lattice $\mathcal{S}_*(\mathsf{E})$ with*

(1) *the induced partial ordering from the super poset $\mathcal{S}_*(\mathsf{E})$;*

(2) *the largest and the least elements in $\mathcal{S}_*^r(\mathsf{E})$ are L (cf.Theorem 3.2(2)) and Φ respectively;*

(3) *for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_*^r(\mathsf{E})$, $\vee_{i\in I}\mathsf{A}_i = \cup_{i\in I}\mathsf{A}_i$.*

(4) *Further, $\mathcal{S}_*^r(\mathsf{E})$ is a complete lattice with the join induced meet $\overline{\wedge}$ given by for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_*^r(\mathsf{E})$, $\overline{\wedge}_{i\in I}\mathsf{A}_i = \mathsf{A}$, where $A = Supp(\cap_{i\in I}\mathsf{A}_i)$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i}e$ for all $e \in A$.*

Proof. (1) and (2) follows in a similar way as in the Theorem 3.2 (1) and (2).

(3): $\mathsf{A} = \vee_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_l(\mathsf{E})$ implies $\mathsf{A} = \cup_{i\in I}\mathsf{A}_i$ in $\mathcal{S}_l(\mathsf{E})$ implies $A = \cup_{i\in I}A_i$, $\sigma_A e$ $= \cup_{i\in I_e}\sigma_{A_i}e$ for all $e \in A$, where $I_e = \{i \in I | e \in A_i\}$, and $\sigma_A e \neq \phi$ as $\sigma_{A_i}e \neq \phi$ for all $e \in A_i$ and for all $i \in I$ implying $\mathsf{A} \in \mathcal{S}_l^r(\mathsf{E})$ or $\mathcal{S}_l^r(\mathsf{E})$ is a join complete subposet of $\mathcal{S}_l(\mathsf{E})$.

(4): Let $(\mathsf{A}_i)_{i\in I}$ be a subset of $\mathcal{S}_l^r(\mathsf{E})$. Now we claim that $\overline{\wedge}_{i\in I}\mathsf{A}_i = \mathsf{A}$, where $A = \text{Supp}(\cap_{i\in I}\mathsf{A}_i)$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i}e$ for all $e \in A$. Clearly, $\mathsf{A} \in \mathcal{S}_l^r(\mathsf{E})$.

Let us recall that $\overline{\wedge}_{i\in I}\mathsf{A}_i = \vee_{\mathsf{B}\leqslant\mathsf{A}_i \text{ for all } i\in I,\, \mathsf{B}\in\mathcal{S}_l^r(\mathsf{E})}\mathsf{B} = \mathsf{C}$.

We show that $\mathsf{C} = \mathsf{A}$.

(i) Since $\mathsf{A} \in \mathcal{S}_l^r(\mathsf{E})$ and $\mathsf{A} \leqslant \mathsf{A}_i$ for all $i \in I$, $\mathsf{A} \subseteq \vee_{\mathsf{B}\leqslant\mathsf{A}_i \text{ for all } i\in I,\, \mathsf{B}\in\mathcal{S}_l^r(\mathsf{E})}\mathsf{B} = \mathsf{C}$.

(ii) $\mathsf{B} \in \mathcal{S}_l^r(\mathsf{E})$ such that $\mathsf{B} \leqslant \mathsf{A}_i$ for all $i \in I$ implies $\mathsf{B} \subseteq \mathsf{A}_i$ for all $i \in I$ implies $B \subseteq A_i$ for all $i \in I$ and $\phi \neq \sigma_B e \subseteq \sigma_{A_i}e$ for all $e \in B$ and for all $i \in I$ implies $B \subseteq \cap_{i\in I}A_i$ and $\phi \neq \sigma_B e \subseteq \cap_{i\in I}\sigma_{A_i}e$ for all $e \in B$ which implies $B \subseteq$ $\text{Supp}(\cap_{i\in I}\mathsf{A}_i) = A$ and $\sigma_B e \subseteq \cap_{i\in I}\sigma_{A_i}e = \sigma_A e$ for all $e \in B$ implying $\mathsf{B} \subseteq \mathsf{A}$ or C $= \vee_{\mathsf{B}\leqslant\mathsf{A}_i \text{ for all } i\in I,\, \mathsf{B}\in\mathcal{S}_l^r(\mathsf{E})}\mathsf{B} \subseteq \mathsf{A}$.

Now (i) and (ii) imply $\mathsf{A} = \mathsf{C}$.

From (1)-(4) and by the Definition 2.2, we get that $\mathcal{S}_l^r(\mathsf{E})$ is a complete lattice.

For $* = r$, i, the proofs follow in a similar way as above. □

4. Extended Soft Substructures

In this section we introduce the notions of extended soft substructures for the soft substructures of a soft semigroup and study their lattice theoretic properties.

Let us recall that, a 0-adjoined semigroup of a semigroup is the semigroup with 0 in which the multiplication of any two elements of the semigroup is the same as the old one and the multiplication of any semigroup element or the 0 by the 0 is the 0 itself.

Notice that throughout this section U_0 is denoted by $\overline{U}$.

DEFINITION 4.1. For any soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) A of a soft semigroup E over U, the *extended soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal)* or simply *es-subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal)* for A, denoted by A', is defined by $A' = E$ and for each $e \in E$,

$$\sigma_{A'}e = \begin{cases} \sigma_A e \cup \{0\} & \text{if } \sigma_A e \neq \phi \\ \{0\} & \text{if } \sigma_A e = \phi \text{ or } e \in E - A \end{cases}$$

In particular, the es-semigroup for E is given by E', where $E' = E$ and for each $e \in E$,

$$\sigma_{E'}e = \begin{cases} \sigma_E e \cup \{0\} & \text{if } \sigma_E e \neq \phi \\ \{0\} & \text{if } \sigma_E e = \phi \end{cases}$$

Notice that (1) for any soft semigroup E over U, the es-semigroup E' for E is always a soft semigroup over $\overline{U}$

(2) E is a soft subsemigroup of E'.

Whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the set of *all es-subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) for all soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of* E is denoted by $\mathcal{S}_s(\mathsf{E})'$ $(\mathcal{S}_*(\mathsf{E})')$. In other words, $\mathcal{S}_s(\mathsf{E})' = \{\mathsf{A}' | \mathsf{A} \in \mathcal{S}_s(\mathsf{E})\}$, $(\mathcal{S}_*(\mathsf{E})' = \{\mathsf{A}' | \mathsf{A} \in \mathcal{S}_*(\mathsf{E})\})$.

REMARK 4.1. Observe that for any soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) A of a soft semigroup E over U, A' is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of the soft semigroup E' over $\overline{U}$. In fact, A' is a d-total and regular soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E'.

Notice that since the empty soft set Φ is a soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E, Φ', where $\phi' = E$ and $\sigma_{\phi'}e = \{0\}$ for all $e \in E$, is the soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) of E'.

LEMMA 4.1. *For any soft semigroup E over U, whenever $* = s\,(l,\, r,\, i,\, q,\, b)$, the set $\mathcal{S}_s(\mathsf{E})'$ $(\mathcal{S}_*(\mathsf{E})')$ of all es-subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) for all soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of E over U is a proper subset of the complete lattice $\mathcal{S}_s(\mathsf{E}')$ $(\mathcal{S}_*(\mathsf{E}'))$ of all soft subsemigroups (left ideals, right ideals, ideals, quasi-ideals, bi-ideals) of E' over $\overline{U}$.*

PROOF. It follows from the Remark 4.1. □

DEFINITION 4.2. For any regular soft subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) A of a soft semigroup E over U, the es-subsemigroup (left ideal, right ideal, ideal, quasi-ideal, bi-ideal) for A is given by A', where $A' = E$ and for each $e \in E$,

$$\sigma_{A'}e = \begin{cases} \sigma_A e \cup \{0\} & \text{if } e \in A \\ \{0\} & \text{if } e \in E - A \end{cases}$$

LEMMA 4.2. *For any soft semigroup* E *over* U*, the following are true:*

(1) *For any* $\mathsf{B} \in \mathcal{S}_s^d(\mathsf{E}')$ *such that* $0 \in \sigma_B b$ *for all* $b \in B$ *there exists unique* A *in* $\mathcal{S}_s^r(\mathsf{E})$ *such that* $\mathsf{A}' = \mathsf{B}$*;*

(2) *Further,* $\mathcal{S}_s(\mathsf{E})' = \{\mathsf{B} \in \mathcal{S}_s^d(\mathsf{E}')/0 \in \sigma_B b \text{ for all } b \in B\}$*;*

(3) *For any* $\mathsf{B} \in \mathcal{S}_s(\mathsf{E})'$ *there exists unique* A *in* $\mathcal{S}_s^r(\mathsf{E})$ *such that* $\mathsf{A}' = \mathsf{B}$*;*

(4) *Consequently,* $\mathcal{S}_s(\mathsf{E})' = \mathcal{S}_s^r(\mathsf{E})'$*.*

PROOF. (1): Define A by $A = \{e \in B/\sigma_B e - \{0\} \neq \phi\}$ and $\sigma_A e = \sigma_B e - \{0\}$ for all $e \in A$. Clearly, $\mathsf{A} \in \mathcal{S}_s^r(\mathsf{E}) \subseteq \mathcal{S}_s(\mathsf{E})$.

Now A' is given by $A' = E$ and for each $e \in E$,

$$\sigma_{A'} e = \begin{cases} \sigma_A e \cup \{0\} & \text{if } e \in A \\ \{0\} & \text{if } e \in E - A \end{cases}$$

We show that $\mathsf{A}' = \mathsf{B}$ or $\sigma_{A'} e = \sigma_B e$ for all $e \in E$. Let $e \in E$ be fixed. If $e \in A$ then $\sigma_{A'} e = \sigma_A e \cup \{0\} = (\sigma_B e - \{0\}) \cup \{0\} = \sigma_B e$. If $e \notin A$ then $\sigma_{A'} e = \{0\} = \sigma_B e$ or $\mathsf{A}' = \mathsf{B}$.

Let A_1, $\mathsf{A}_2 \in \mathcal{S}_s^r(\mathsf{E})$ such that $\mathsf{A}_1' = \mathsf{B} = \mathsf{A}_2'$. Then $A_1' = B = E = A_2'$ and $\sigma_{A_1'} e = \sigma_B e = \sigma_{A_2'} e$ for all $e \in E$.

We show that $\mathsf{A}_1 = \mathsf{A}_2$ or (i) $A_1 = A_2$ (ii) $\sigma_{A_1} e = \sigma_{A_2} e$ for all $e \in A_1$.

(i): $e \in A_1 - A_2$ implies $\sigma_{A_2} e = \phi$ implies $\{0\} = \sigma_{A_2'} e = \sigma_{A_1'} e = \sigma_{A_1} e \cup \{0\}$ implies $\sigma_{A_1} e = \phi$, which is a contradiction to $\mathsf{A}_1 \in \mathcal{S}_s^r(\mathsf{E})$. Therefore $A_1 \subseteq A_2$. Similarly, $A_2 \subseteq A_1$ and we get that $A_1 = A_2$.

(ii): Let $e \in A_1 = A_2$ be fixed. Since $\sigma_{A_1} e \cup \{0\} = \sigma_{A_1'} e = \sigma_{A_2'} e = \sigma_{A_2} e \cup \{0\}$, $\sigma_{A_1} e = \sigma_{A_2} e$.

Now (i) and (ii) imply $\mathsf{A}_1 = \mathsf{A}_2$.

(2): It follows from the Definition 4.1, the Remark 4.1 and (1) above.

(3): It follows from (1) and (2) above.

(4): By (3) above, $\mathcal{S}_s(\mathsf{E})' \subseteq \mathcal{S}_s^r(\mathsf{E})'$.

On the other hand, $\mathsf{D}' \in \mathcal{S}_s^r(\mathsf{E})'$ implies $\mathsf{D} \in \mathcal{S}_s^r(\mathsf{E}) \subseteq \mathcal{S}_s(\mathsf{E})$ implying $\mathsf{D}' \in \mathcal{S}_s(\mathsf{E})'$ or $\mathcal{S}_s^r(\mathsf{E})' \subseteq \mathcal{S}_s(\mathsf{E})'$. □

LEMMA 4.3. *For any soft semigroup* E *over* U*, whenever* $* = l, r, i, q, b$*, the following are true:*

(1) *For any* $\mathsf{B} \in \mathcal{S}_*^{d,r}(\mathsf{E}')$ *there exists unique* A *in* $\mathcal{S}_*^r(\mathsf{E})$ *such that* $\mathsf{A}' = \mathsf{B}$*;*

(2) *Further,* $\mathcal{S}_*(\mathsf{E})' = \mathcal{S}_*^{d,r}(\mathsf{E}')$*;*

(3) *For any* $\mathsf{B} \in \mathcal{S}_*(\mathsf{E})'$ *there exists unique* A *in* $\mathcal{S}_*^r(\mathsf{E})$ *such that* $\mathsf{A}' = \mathsf{B}$*;*

(4) *Consequently,* $\mathcal{S}_*(\mathsf{E})' = \mathcal{S}_*^r(\mathsf{E})'$*.*

PROOF. It follows in a similar way as the Lemma 4.2. □

THEOREM 4.1. *For any soft semigroup* E *over* U*, whenever* $* = s, q, b, l, r, i$*, the set* $\mathcal{S}_*(\mathsf{E})'$ *is a complete sublattice of the complete lattice* $\mathcal{S}_*(\mathsf{E}')$ *with*

(1) *the induced partial ordering from the super poset* $\mathcal{S}_*(\mathsf{E}')$*;*

(2) *the largest and the least elements in* $\mathcal{S}_*(\mathsf{E})'$ *are* E' *and* Φ' *respectively;*

(3) *for any family* $(\mathsf{A}_i')_{i \in I}$ *in* $\mathcal{S}_*(\mathsf{E})'$*,* $\wedge_{i \in I} \mathsf{A}_i' = \cap_{i \in I} \mathsf{A}_i'$*;*

(4) *for any family* $(\mathsf{A}_i')_{i \in I}$ *in* $\mathcal{S}_*(\mathsf{E})'$*, however;*

(i) *for* $* = s$, q, b, $\vee_{i\in I}A'_i = \bar{\vee}_{i\in I}A'_i$, *where* $\bar{\vee}$ *is the meet induced join in* $\mathcal{S}_*(E')$ *and* $\bar{\vee}_{i\in I}A'_i = A'$, *where* $A' = E$ *and* $\sigma_{A'}e = (\cup_{i\in I}\sigma_{A'_i}e)_{*,\sigma_{E'}e}$ *for all* $e \in E$

(ii) *for* $* = l$, r, i, $\vee_{i\in I}A'_i = \cup_{i\in I}A'_i$.

(5) *Further,* $\mathcal{S}_*(E)'$ *is a complete filter of* $\mathcal{S}_*(E')$.

Proof. (1): It follows from the Definition 2.1.

(2): It follows in a similar way as in the Theorem 3.1(2).

(3): Let $(A'_i)_{i\in I}$ be a subset of $\mathcal{S}_s(E)'$. $B = \wedge_{i\in I}A'_i$ in $\mathcal{S}_s(E')$ implies $B = \cap_{i\in I}A'_i$ in $\mathcal{S}_s(E')$ implies $B = \cap_{i\in I}A'_i = E$ and $\sigma_B e = \cap_{i\in I}\sigma_{A'_i}e$ for all $e \in E$. Since $B = E$ and $0 \in \sigma_B e$ for all $e \in E$ as $0 \in \sigma_{A'_i}e$ for all $e \in A'_i$ and for all $i \in I$, by the Lemma 4.2(2), $B \in \mathcal{S}_s(E)'$.

(4): Let $(A'_i)_{i\in I}$ be a subset of $\mathcal{S}_s(E)'$. $B = \vee_{i\in I}A'_i$ in $\mathcal{S}_s(E')$ implies $B = \bar{\vee}_{i\in I}A'_i$ in $\mathcal{S}_s(E')$ implies $B = \cup_{i\in I}A'_i = E$ and $\sigma_B e = (\cup_{i\in I}\sigma_{A'_i}e)_{s,\sigma_{E'}e}$ for all $e \in E$. Since $B = E$ and $0 \in \sigma_B e$ for all $e \in E$ as $0 \in \sigma_{A'_i}e$ for all $e \in A'_i$ and for all $i \in I$, by the Lemma 4.2(2), $B \in \mathcal{S}_s(E)'$. Now (3) and (4) imply $\mathcal{S}_s(E)'$ is a complete sublattice of $\mathcal{S}_s(E')$.

(5): $A' \in \mathcal{S}_s(E)'$ and $B \in \mathcal{S}_s(E')$ such that $A' \leqslant B$ in $\mathcal{S}_s(E')$ implies $A' \subseteq B$ in $\mathcal{S}_s(E')$ implies $E = A' \subseteq B$ and $\sigma_{A'}e \subseteq \sigma_B e$ for all $e \in A' = E$ implies $B = E$ and $0 \in \sigma_B e$ for all $e \in E$ as $0 \in \sigma_{A'}e$ for all $e \in E$ implying by the Lemma 4.2(2), $B \in \mathcal{S}_s(E)'$ or $\mathcal{S}_s(E)'$ is a complete filter of $\mathcal{S}_s(E')$.

For $* = q$, b, l, r, i, the proofs follow in a similar way as above. □

Theorem 4.2. *For any soft semigroup* E *over* U, *whenever* $* = s\,(q, b, l, r, i)$, *the map* $\varepsilon_* : \mathcal{S}_*(E) \to \mathcal{S}_*(E)'$ *defined by for any* $A \in \mathcal{S}_*(E)$, $\varepsilon_* A = A'$ *being the es-subsemigroup (quasi-ideal, bi-ideal, left ideal, right ideal, ideal) for* A, *satisfies the following properties:*

(1) *The map* ε_* *is onto;*

(2) *For any* $A, B \in \mathcal{S}_*(E)$, $A \leqslant B$ *implies* $\varepsilon_* A \leqslant \varepsilon_* B$.

For any family $(A_i)_{i\in I}$ *in* $\mathcal{S}_*(E)$,

(3) $\varepsilon_*(\cap_{i\in I}A_i) = \cap_{i\in I}\varepsilon_* A_i$

(4) (i) *for* $* = s$, q, b, $\varepsilon_*(\bar{\vee}_{i\in I}A_i) = \bar{\vee}_{i\in I}\varepsilon_* A_i$;

(ii) *for* $* = l$, r, i, $\varepsilon_*(\cup_{i\in I}A_i) = \cup_{i\in I}\varepsilon_* A_i$;

(5) *The map* ε_* *is a complete epimorphism.*

Proof. (1): It is straightforward.

(2): A, $B \in \mathcal{S}_s(E)$ such that $A \leqslant B$ implies by the Theorem 3.1(1), $A \subseteq B$ implying $A \subseteq B$ and $\sigma_A e \subseteq \sigma_B e$ for all $e \in A$. Let $\varepsilon_s A = A'$. Then $A' = E$ and for each $e \in E$,

$$\sigma_{A'}e = \begin{cases} \sigma_A e \cup \{0\} & \text{if } \sigma_A e \neq \phi \\ \{0\} & \text{if } \sigma_A e = \phi \text{ or } e \in E - A \end{cases}$$

Let $\varepsilon_s B = B'$. Then $B' = E$ and for each $e \in E$,

$$\sigma_{B'}e = \begin{cases} \sigma_B e \cup \{0\} & \text{if } \sigma_B e \neq \phi \\ \{0\} & \text{if } \sigma_B e = \phi \text{ or } e \in E - B \end{cases}$$

We show that $\mathsf{A}' \subseteq \mathsf{B}'$ or $\sigma_{A'}e \subseteq \sigma_{B'}e$ for all $e \in E$. Let $e \in E$ be fixed. If $e \in A$ and $\sigma_A e \neq \phi$ then $\sigma_{A'}e = \sigma_A e \cup \{0\} \subseteq \sigma_B e \cup \{0\} = \sigma_{B'}e$. If $e \in A$ and $\sigma_A e = \phi$ then $\sigma_{A'}e = \{0\} \subseteq \sigma_{B'}e$. If $e \in B - A$ then $\sigma_{A'}e = \{0\} \subseteq \sigma_{B'}e$. If $e \in E - B$ then $\sigma_{A'}e = \{0\} = \sigma_{B'}e$. Therefore $\mathsf{A}' \subseteq \mathsf{B}'$ or by the Theorem 3.1(1), A' is a soft subsemigroup of B' or $\varepsilon_s\mathsf{A} = \mathsf{A}' \leqslant \mathsf{B}' = \varepsilon_s\mathsf{B}$.

(3): Let $\cap_{i\in I}\mathsf{A}_i = \mathsf{A}$. Then $A = \cap_{i\in I}A_i$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i}e$ for all $e \in A$.

Let $\varepsilon_s\mathsf{A} = \mathsf{A}'$. Then $A' = E$ and for each $e \in E$,

$$\sigma_{A'}e = \begin{cases} \sigma_A e \cup \{0\} & \text{if } \sigma_A e \neq \phi \\ \{0\} & \text{if } \sigma_A e = \phi \text{ or } e \in E - A \end{cases}$$

Let $\varepsilon_s\mathsf{A}_i = \mathsf{B}'_i$. Then $B'_i = E$ and for each $e \in E$,

$$\sigma_{B'_i}e = \begin{cases} \sigma_{A_i} e \cup \{0\} & \text{if } \sigma_{A_i} e \neq \phi \\ \{0\} & \text{if } \sigma_{A_i} e = \phi \text{ or } e \in E - A_i \end{cases}$$

Let $\cap_{i\in I}\mathsf{B}'_i = \mathsf{B}'$. Then $B' = \cap_{i\in I}B'_i = E$ and $\sigma_{B'}e = \cap_{i\in I}\sigma_{B'_i}e$ for all $e \in E$. We show that $\mathsf{A}' = \mathsf{B}'$ or $\sigma_{A'}e = \sigma_{B'}e$ for all $e \in E$.

Let $e \in E$ be fixed.

(i) If $A = \phi$ then $e \in E - A$ implies $\sigma_{A'}e = \{0\}$. $A = \cap_{i\in I}A_i = \phi$ implies there exists $i_0 \in I$ such that $e \notin A_{i_0}$ implies $\sigma_{B'_{i_0}}e = \{0\}$ implying $\sigma_{B'}e = \{0\} = \sigma_{A'}e$.

(ii) If $A \neq \phi$ and $e \notin A$ then $e \in E - A$ implies $\sigma_{A'}e = \{0\} = \sigma_{B'}e$ as in (i) above.

(iii) If $A \neq \phi$, $e \in A$ and $\sigma_A e = \phi$ then $\sigma_{A'}e = \{0\}$. If $\sigma_{A_{i_0}}e = \phi$ for some $i_0 \in I$ then $\sigma_{B'_{i_0}}e = \{0\}$ implies $\sigma_{B'}e = \{0\} = \sigma_{A'}e$. If $\sigma_{A_i}e \neq \phi$ for all $i \in I$ implies $\sigma_{B'_i}e \neq \{0\}$ for all $i \in I$ or $\sigma_{B'_i}e = \sigma_{A_i}e \cup \{0\}$ for all $i \in I$ implying $\sigma_{B'}e = \cap_{i\in I}\sigma_{B'_i}e = \cap_{i\in I}(\sigma_{A_i}e \cup \{0\}) = (\cap_{i\in I}\sigma_{A_i}e) \cup \{0\} = \sigma_A e \cup \{0\} = \{0\} = \sigma_{A'}e$.

(iv) If $A \neq \phi$, $e \in A$ and $\sigma_A e \neq \phi$ then $\sigma_{A'}e = \sigma_A e \cup \{0\}$. $e \in A = \cap_{i\in I}A_i$ and $\sigma_A e \neq \phi$ implies $e \in A_i$ for all $i \in I$ and $\sigma_{A_i}e \neq \phi$ for all $i \in I$ imply $\sigma_{B'_i}e = \sigma_{A_i}e \cup \{0\}$ for all $i \in I$ implying $\sigma_{B'}e = \cap_{i\in I}(\sigma_{A_i}e \cup \{0\}) = (\cap_{i\in I}\sigma_{A_i}e) \cup \{0\} = \sigma_A e \cup \{0\} = \sigma_{A'}e$.

(4): Let $\overline{\vee}_{i\in I}\mathsf{A}_i = \mathsf{A}$. Then $A = \cup_{i\in I}A_i$ and $\sigma_A e = (\cup_{i\in I_e}\sigma_{A_i}e)_{s,\sigma_E e}$ for all $e \in A$, where $I_e = \{i \in I / e \in A_i\}$. Let $\varepsilon_s\mathsf{A} = \mathsf{A}'$. Then $A' = E$ and for each $e \in E$,

$$\sigma_{A'}e = \begin{cases} \sigma_A e \cup \{0\} & \text{if } \sigma_A e \neq \phi \\ \{0\} & \text{if } \sigma_A e = \phi \text{ or } e \in E - A \end{cases}$$

Let $\varepsilon_s\mathsf{A}_i = \mathsf{B}'_i$. Then $\mathsf{B}'_i = E$ and for each $e \in E$,

$$\sigma_{B'_i}e = \begin{cases} \sigma_{A_i} e \cup \{0\} & \text{if } \sigma_{A_i} e \neq \phi \\ \{0\} & \text{if } \sigma_{A_i} e = \phi \text{ or } e \in E - A_i \end{cases}$$

Let $\overline{\vee}_{i\in I}\mathsf{B}'_i = \mathsf{B}'$. Then $B' = \cup_{i\in I}B'_i = E$ and $\sigma_{B'}e = (\cup_{i\in I}\sigma_{B'_i}e)_{s,\sigma_{E'}e}$ for all $e \in E$. We show that $\mathsf{A}' = \mathsf{B}'$ or $\sigma_{A'}e = \sigma_{B'}e$ for all $e \in E$.

Let $e \in E$ be fixed.

(i) If $e \notin A$ then $\sigma_{A'}e = \{0\}$. $e \notin A = \cup_{i\in I}A_i$ implies $e \notin A_i$ for all $i \in I$ implies $\sigma_{B'_i}e = \{0\}$ for all $i \in I$ implying $\sigma_{B'}e = \{0\} = \sigma_{A'}e$.

(ii) If $e \in A = \cup_{i\in I} A_i$ then $I_e \neq \phi$. Define I_e as $I_e = J_e \uplus (I_e - J_e)$, where $J_e = \{i \in I_e / \sigma_{A_i} e \neq \phi\}$, and I as $I = I - I_e \uplus I_e = I - I_e \uplus J_e \uplus I_e - J_e$, where $\uplus$ denotes the disjoint union. Therefore

$$\sigma_{B_i'} e = \begin{cases} \{0\} & \text{if } i \in I - I_e \\ \sigma_{A_i} e \cup \{0\} & \text{if } i \in J_e \\ \{0\} & \text{if } i \in I_e - J_e \end{cases}$$

Now $\sigma_{B'} e = ((\cup_{i\in I-I_e} \sigma_{B_i'} e) \cup (\cup_{i\in J_e} \sigma_{B_i'} e) \cup (\cup_{i\in I_e - J_e} \sigma_{B_i'} e))_{s, \sigma_{E'} e} = ((\cup_{i\in J_e} \sigma_{A_i} e) \cup \{0\})_{s, \sigma_{E'} e}$, $\sigma_A e = ((\cup_{i\in J_e} \sigma_{A_i} e) \cup (\cup_{i\in I_e - J_e} \sigma_{A_i} e))_{s, \sigma_E e} = (\cup_{i\in J_e} \sigma_{A_i} e)_{s, \sigma_E e}$ and $\sigma_{A'} e = \sigma_A e \cup \{0\} = ((\cup_{i\in J_e} \sigma_{A_i} e))_{s, \sigma_E e} \cup \{0\} = ((\cup_{i\in J_e} \sigma_{A_i} e) \cup \{0\})_{s, \sigma_{E'} e}$. Clearly, $\sigma_{A'} e = \sigma_{B'} e$.

(5): It follows from (1), (3) and (4)(i) above.

For $* = q, b, l, r, i$, the proofs follow in a similar way as above. □

The following Example shows that in the above Theorem, whenever $* = s, q, b, l, r, i$, the map ε_* is *not* one-one.

EXAMPLE 4.1. Let U be a semigroup and $\mathsf{E} = (\{(e_1, U), (e_2, U)\}, \{e_1, e_2\})$ be a soft semigroup over U. Then $\overline{U} = U \cup \{0\}$ is also a semigroup. Let $\mathsf{A}_1 = (\{(e_1, \phi), (e_2, U)\}, \{e_1, e_2\})$ and $\mathsf{A}_2 = (\{(e_2, U)\}, \{e_2\})$ be in $\mathcal{S}_*(\mathsf{E})$. Then $\varepsilon_* \mathsf{A}_1 = \mathsf{A}_1' = (\{(e_1, \{0\}), (e_2, \overline{U})\}, \{e_1, e_2\})$ and $\varepsilon_* \mathsf{A}_2 = \mathsf{A}_2' = (\{(e_1, \{0\}), (e_2, \overline{U})\}, \{e_1, e_2\})$. Clearly, $\varepsilon_* \mathsf{A}_1 = \varepsilon_* \mathsf{A}_2$ but $\mathsf{A}_1 \neq \mathsf{A}_2$ or ε_* is *not* one-one.

DEFINITION 4.3. Whenever $* = s, q, b, l, r, i$, the complete epimorphism ε_* as in the Theorem 4.2 is called the extension operator.

THEOREM 4.3. *For any soft semigroup E over U, whenever $* = s, q, b, l, r, i$, the restricted map $\varepsilon_* | \mathcal{S}_*^r(\mathsf{E}) : \mathcal{S}_*^r(\mathsf{E}) \to \mathcal{S}_*(\mathsf{E})'$ defined by for any $\mathsf{A} \in \mathcal{S}_*^r(\mathsf{E})$, $(\varepsilon_* | \mathcal{S}_*^r(\mathsf{E}))(\mathsf{A}) = \varepsilon_* \mathsf{A}$ as in the Theorem 4.2, satisfies the following properties:*

(1) *The map $\varepsilon_* | \mathcal{S}_*^r(\mathsf{E})$ is both one-one and onto;*

(2) *For any $\mathsf{A}, \mathsf{B} \in \mathcal{S}_*^r(\mathsf{E})$, $\mathsf{A} \leqslant \mathsf{B}$ implies $\varepsilon_* \mathsf{A} \leqslant \varepsilon_* \mathsf{B}$.*

For any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_*^r(\mathsf{E})$;

(3) (i) *for $* = s, q, b$, $\varepsilon_*(\bigtriangledown_{i\in I} \mathsf{A}_i) = \bigtriangledown_{i\in I} \varepsilon_* \mathsf{A}_i$*

(ii) *for $* = l, r, i$, $\varepsilon_*(\cup_{i\in I} \mathsf{A}_i) = \cup_{i\in I} \varepsilon_* \mathsf{A}_i$;*

(4) (i) *for $* = s, q, b$, $\varepsilon_*(\sqcap_{i\in I} \mathsf{A}_i) = \cap_{i\in I} \varepsilon_* \mathsf{A}_i$;*

(ii) *for $* = l, r, i$, $\varepsilon_*(\overline{\wedge}_{i\in I} \mathsf{A}_i) = \cap_{i\in I} \varepsilon_* \mathsf{A}_i$;*

(5) *The map $\varepsilon_* | \mathcal{S}_*^r(\mathsf{E})$ is a complete isomorphism.*

PROOF. (1): Let $\mathsf{A}, \mathsf{B} \in \mathcal{S}_s^r(\mathsf{E})$ such that $\varepsilon_s \mathsf{A} = \varepsilon_s \mathsf{B}$. Let $\varepsilon_s \mathsf{A} = \mathsf{A}'$ and $\varepsilon_s \mathsf{B} = \mathsf{B}'$. Then $A' = E = B'$ and for each $e \in E$,

$$\sigma_{A'} e = \begin{cases} \sigma_A e \cup \{0\} & \text{if } e \in A \\ \{0\} & \text{if } e \in E - A \end{cases} \quad \text{and} \quad \sigma_{B'} e = \begin{cases} \sigma_B e \cup \{0\} & \text{if } e \in B \\ \{0\} & \text{if } e \in E - B \end{cases}$$

$\mathsf{A}' = \mathsf{B}'$ implies $A' = E = B'$ and $\sigma_{A'} e = \sigma_{B'} e$ for all $e \in E$. We show that $\mathsf{A} = \mathsf{B}$ or (i) $A = B$ (ii) $\sigma_A e = \sigma_B e$ for all $e \in A$.

(i): $e \in A$ implies $\sigma_{A'}e = \sigma_A e \cup \{0\} \neq \{0\}$ as $\sigma_A e \neq \phi$. If $e \notin B$ then $\sigma_{B'}e = \{0\} \neq \sigma_{A'}e$, which is a contradiction to $\mathsf{A}' = \mathsf{B}'$. Therefore $e \in B$ or $A \subseteq B$. Similarly, $B \subseteq A$ and we get that $A = B$.

(ii): Let $e \in A = B$ be fixed. Since $\sigma_A e \cup \{0\} = \sigma_{A'}e = \sigma_{B'}e = \sigma_B e \cup \{0\}$, we have $\sigma_A e = \sigma_B e$.

Now (i) and (ii) imply $\mathsf{A} = \mathsf{B}$ or the map is one-one. Clearly, by the Lemma 4.2(3), the map is onto.

(2) and (3): Follow from the Theorem 4.2(2) and 4.2(4)(i).

(4): Let $\sqcap_{i\in I}\mathsf{A}_i = \mathsf{A}$. Then $A = \text{Supp}(\cap_{i\in I}\mathsf{A}_i)$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i}e$ for all $e \in A$. Let $\varepsilon_s\mathsf{A} = \mathsf{A}'$. Then $A' = E$ and for each $e \in E$,

$$\sigma_{A'}e = \begin{cases} \sigma_A e \cup \{0\} & \text{if } e \in A \\ \{0\} & \text{if } e \in E - A \end{cases}$$

Let $\varepsilon_s\mathsf{A}_i = \mathsf{B}_i'$. Then $B_i' = E$ and for each $e \in E$,

$$\sigma_{B_i'}e = \begin{cases} \sigma_{A_i} e \cup \{0\} & \text{if } e \in A_i \\ \{0\} & \text{if } e \in E - A_i \end{cases}$$

Let $\cap_{i\in I}\mathsf{B}_i' = \mathsf{B}'$. Then $B' = \cap_{i\in I}B_i' = E$ and $\sigma_{B'}e = \cap_{i\in I}\sigma_{B_i'}e$ for all $e \in E$. We show that $\mathsf{A}' = \mathsf{B}'$ or $\sigma_{A'}e = \sigma_{B'}e$ for all $e \in E$. Let $e \in E$ be fixed.

(i) $A = \phi$ implies $e \in E - A$ implying $\sigma_{A'}e = \{0\}$. $e \notin A$ implies $\cap_{i\in I}\sigma_{A_i}e = \phi$. If $\sigma_{A_{i_0}}e = \phi$ for some $i_0 \in I$ then $e \notin A_{i_0}$ implies $\sigma_{B_{i_0}'}e = \{0\}$ implying $\sigma_{B'}e = \{0\} = \sigma_{A'}e$. If $\sigma_{A_i}e \neq \phi$ for all $i \in I$ then $e \in A_i$ for all $i \in I$ implies $\sigma_{B_i'}e = \sigma_{A_i}e \cup \{0\}$ for all $i \in I$ implies $\sigma_{B'}e = \cap_{i\in I}\sigma_{B_i'}e = \cap_{i\in I}(\sigma_{A_i}e \cup \{0\}) = (\cap_{i\in I}\sigma_{A_i}e) \cup \{0\} = \{0\} = \sigma_{A'}e$.

(ii) $A \neq \phi$ and $e \in A$ implies $\sigma_{A'}e = \sigma_A e \cup \{0\}$. $e \in A = \text{Supp}(\cap_{i\in I}\mathsf{A}_i) \subseteq \cap_{i\in I}A_i$ implies $e \in A_i$ for all $i \in I$ implies $\sigma_{B_i'}e = \sigma_{A_i}e \cup \{0\}$ for all $i \in I$ implying $\sigma_{B'}e = \cap_{i\in I}\sigma_{B_i'}e = \cap_{i\in I}(\sigma_{A_i}e \cup \{0\}) = (\cap_{i\in I}\sigma_{A_i}e) \cup \{0\} = \sigma_A e \cup \{0\} = \sigma_{A'}e$.

(iii) $A \neq \phi$ and $e \in E - A$ implies $\sigma_{A'}e = \{0\}$. $e \in E - A$ implies $\cap_{i\in I}\sigma_A e = \phi$ and as (i) above, $\sigma_{B'}e = \sigma_{A'}e$.

(5): It follows from (1), (3)(i) and (4)(i) above.

For $* = q, b, l, r, i$, the proofs follow in a similar way as above. □

THEOREM 4.4. *For any soft semigroup E over U, whenever $* = s, q, b, l, r, i$, the map $\rho_* : \mathcal{S}_*(\mathsf{E}) \to \mathcal{S}_*^r(\mathsf{E})$ defined by for any $\mathsf{A} \in \mathcal{S}_*(\mathsf{E})$, $\rho_*\mathsf{A} = \mathsf{B}$, where $B = Supp(\mathsf{A})$ and $\sigma_B e = \sigma_A e$ for all $e \in B$, satisfies the following properties:*

(1) *For any $\mathsf{A} \in \mathcal{S}_*(\mathsf{E})$, $\rho_*\mathsf{A} \leqslant \mathsf{A}$. Equality holds whenever A is regular;*

(2) *The map ρ_* is onto;*

(3) *For any $\mathsf{A}, \mathsf{B} \in \mathcal{S}_*(\mathsf{E})$, $\mathsf{A} \leqslant \mathsf{B}$ implies $\rho_*\mathsf{A} \leqslant \rho_*\mathsf{B}$;*

For any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_*(\mathsf{E})$,

(4) (i) *for $* = s, q, b$, $\rho_*(\overline{\vee}_{i\in I}\mathsf{A}_i) = \overline{\vee}_{i\in I}\rho_*\mathsf{A}_i$;*

(ii) *for $* = l, r, i$, $\rho_*(\cup_{i\in I}\mathsf{A}_i) = \cup_{i\in I}\rho_*\mathsf{A}_i$;*

(5) (i) *for $* = s, q, b$, $\rho_*(\cap_{i\in I}\mathsf{A}_i) = \sqcap_{i\in I}\rho_*\mathsf{A}_i$;*

(ii) *for $* = l, r, i$, $\rho_*(\cap_{i\in I}\mathsf{A}_i) = \overline{\wedge}_{i\in I}\rho_*\mathsf{A}_i$;*

(6) *The map ρ_* is a complete epimorphism.*

PROOF. (1): $\rho_s\mathsf{A} = \mathsf{B}$ implies $B = \text{Supp}(\mathsf{A}) \subseteq A$ and $\sigma_B e = \sigma_A e$ for all $e \in B$ implies B is a soft subsemigroup of A or $\mathsf{B} \leqslant \mathsf{A}$. Let A be a regular soft subsemigroup of E and $\rho_s\mathsf{A} = \mathsf{B}$. Then $B = \text{Supp}(\mathsf{A}) = A$ and $\sigma_B e = \sigma_A e$ for all $e \in B$ implying $\mathsf{B} = \mathsf{A}$.

(2): It is straightforward.

(3): $\mathsf{A}, \mathsf{B} \in \mathcal{S}_s(\mathsf{E})$ such that $\mathsf{A} \leqslant \mathsf{B}$ implies by the Theorem 3.1(1), $\mathsf{A} \subseteq \mathsf{B}$ implying $A \subseteq B$ and $\sigma_A e \subseteq \sigma_B e$ for all $e \in A$. $\rho_s\mathsf{A} = \mathsf{C}$ implies $C = \text{Supp}(\mathsf{A})$ and $\sigma_C e = \sigma_A e$ for all $e \in C$. $\rho_s\mathsf{B} = \mathsf{D}$ implies $D = \text{Supp}(\mathsf{B})$ and $\sigma_D e = \sigma_B e$ for all $e \in D$. We show that $\mathsf{C} \subseteq \mathsf{D}$ or (i) $C \subseteq D$ (ii) $\sigma_C e \subseteq \sigma_D e$ for all $e \in C$.

(i): $e \in C = \text{Supp}(\mathsf{A}) \subseteq A \subseteq B$ implies $\phi \neq \sigma_A e \subseteq \sigma_B e$ implies $\sigma_B e \neq \phi$ implying $e \in \text{Supp}(\mathsf{B}) = D$ or $C \subseteq D$.

(ii): Let $e \in C$ be fixed. Then $\sigma_C e = \sigma_A e \subseteq \sigma_B e = \sigma_D e$.

Now (i) and (ii) imply $\mathsf{C} \subseteq \mathsf{D}$ implying by the Theorem 3.1(1), $\mathsf{C} \leqslant \mathsf{D}$.

(4): $\triangledown_{i\in I}\mathsf{A}_i = \mathsf{A}$ implies $A = \cup_{i\in I} A_i$ and $\sigma_A e = (\cup_{i\in I_e}\sigma_{A_i} e)_{s,\,\sigma_E e}$ for all $e \in A$, where $I_e = \{i \in I / e \in A_i\}$. $\rho_s\mathsf{A} = \mathsf{B}$ implies $B = \text{Supp}(\mathsf{A})$ and $\sigma_B e = \sigma_A e$ for all $e \in B$. $\rho_s\mathsf{A}_i = \mathsf{C}_i$ implies $C_i = \text{Supp}(\mathsf{A}_i)$ and $\sigma_{C_i} e = \sigma_{A_i} e$ for all $e \in C_i$. $\triangledown_{i\in I}\mathsf{C}_i = \mathsf{C}$ implies $C = \cup_{i\in I} C_i$ and $\sigma_C e = (\cup_{i\in I_e}\sigma_{C_i} e)_{s,\,\sigma_E e}$ for all $e \in C$, where $I_e = \{i \in I / e \in C_i\}$.

We show that $\mathsf{B} = \mathsf{C}$ or (i) $B = C$ (ii) $\sigma_B e = \sigma_C e$ for all $e \in B$.

(i): $B = \text{Supp}(\mathsf{A}) = \text{Supp}(\cup_{i\in I}\mathsf{A}_i) = \cup_{i\in I}\text{Supp}(\mathsf{A}_i) = \cup_{i\in I} C_i = C$.

(ii): Let $e \in B = C$ be fixed. Then $\sigma_C e = (\cup_{i\in I_e}\sigma_{C_i} e)_{s,\,\sigma_E e} = (\cup_{i\in I_e}\sigma_{A_i} e)_{s,\,\sigma_E e} = \sigma_A e = \sigma_B e$.

Now (i) and (ii) imply $\mathsf{B} = \mathsf{C}$.

(5): $\cap_{i\in I}\mathsf{A}_i = \mathsf{A}$ implies $A = \cap_{i\in I} A_i$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i} e$ for all $e \in A$. $\rho_s\mathsf{A} = \mathsf{B}$ implies $B = \text{Supp}(\mathsf{A})$ and $\sigma_B e = \sigma_A e$ for all $e \in B$. $\rho_s\mathsf{A}_i = \mathsf{C}_i$ implies $C_i = \text{Supp}(\mathsf{A}_i)$ and $\sigma_{C_i} e = \sigma_{A_i} e$ for all $e \in C_i$. $\sqcap_{i\in I}\mathsf{C}_i = \mathsf{C}$ implies $C = \text{Supp}(\cap_{i\in I}\mathsf{C}_i)$ and $\sigma_C e = \cap_{i\in I}\sigma_{C_i} e$ for all $e \in C$.

We show that $\mathsf{B} = \mathsf{C}$ or (i) $B = C$ (ii) $\sigma_B e = \sigma_C e$ for all $e \in B$.

(i): $e \in B = \text{Supp}(\mathsf{A}) = \text{Supp}(\cap_{i\in I}\mathsf{A}_i) \subseteq \cap_{i\in I}\text{Supp}(\mathsf{A}_i) = \cap_{i\in I} C_i$ implies $\phi \neq \sigma_A e = \cap_{i\in I}\sigma_{A_i} e = \cap_{i\in I}\sigma_{C_i} e$ implies $e \in \text{Supp}(\cap_{i\in I}\mathsf{C}_i) = C$ or $B \subseteq C$. Similarly, $C \subseteq B$ and we get that $B = C$.

(ii): Let $e \in B = C$ be fixed. Then $\sigma_B e = \sigma_A e = \cap_{i\in I}\sigma_{A_i} e = \cap_{i\in I}\sigma_{C_i} e = \sigma_C e$.

Now (i) and (ii) imply $\mathsf{B} = \mathsf{C}$.

(6): It follows from (2), (4)(i) and (5)(i) above.

For $* = q$, b, l, r, i, the proofs follow in a similar way as above. □

The following Example shows that in the above Theorem, whenever $* = s, q, b, l, r, i$, the map ρ_* is *not* one-one.

EXAMPLE 4.2. Let U be a semigroup, $\mathsf{E} = (\{(e_1, U), (e_2, U)\}, \{e_1, e_2\})$ be a soft semigroup over U, $\mathsf{A}_1 = (\{(e_1, U), (e_2, \phi)\}, \{e_1, e_2\})$ and $\mathsf{A}_2 = (\{(e_1, U)\}, \{e_1\})$ be in $\mathcal{S}_*(\mathsf{E})$. Then $\rho_*\mathsf{A}_1 = (\{(e_1, U)\}, \{e_1\}) = \rho_*\mathsf{A}_2$ but $\mathsf{A}_1 \neq \mathsf{A}_2$ or ρ_* is *not* one-one.

DEFINITION 4.4. Whenever $* = s$, q, b, l, r, i, the complete epimorphism defined as in the Theorem 4.4 is called the reparametrization or regularization map.

THEOREM 4.5. *For any soft semigroup E over U, whenever $* = s, q, b, l, r, i$, the operator $\nu_* : \mathcal{S}_*(E)' \to \mathcal{S}^r_*(E)$ defined by $\nu_* B = A$, where A in $\mathcal{S}^r_*(E)$ is unique such that $A' = B$, satisfies the following properties:*

(1) *The map ν_* is both one-one and onto;*

(2) *For any $B, D \in \mathcal{S}_*(E)'$, $B \leqslant D$ implies $\nu_* B \leqslant \nu_* D$*

For any family $(B_i)_{i\in I}$ in $\mathcal{S}_*(E)'$,

(3) (i) *for $* = s, q, b$, $\nu_*(\overline{\vee}_{i\in I} B_i) = \overline{\vee}_{i\in I}\nu_* B_i$*

(ii) *for $* = l, r, i$, $\nu_*(\cup_{i\in I} B_i) = \cup_{i\in I}\nu_* B_i$;*

(4) (i) *for $* = s, q, b$, $\nu_*(\cap_{i\in I} B_i) = \sqcap_{i\in I}\nu_* B_i$;*

(ii) *for $* = l, r, i$, $\nu_*(\cap_{i\in I} B_i) = \overline{\wedge}_{i\in I}\nu_* B_i$;*

(5) *The map ν_* is a complete isomorphism.*

PROOF. (1): Let $\mathsf{B}, \mathsf{D} \in \mathcal{S}_s(\mathsf{E})'$ such that $\nu_s\mathsf{B} = \nu_s\mathsf{D}$. Let $\nu_s\mathsf{B} = \mathsf{A}$, where A is unique in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{A}' = \mathsf{B}$, and $\nu_s\mathsf{D} = \mathsf{C}$, where C is unique in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{C}' = \mathsf{D}$. $\nu_s\mathsf{B} = \nu_s\mathsf{D}$ implies $\mathsf{A} = \mathsf{C}$ implying $\mathsf{B} = \mathsf{A}' = \mathsf{C}' = \mathsf{D}$ as $\varepsilon_s|\mathcal{S}^r_s(\mathsf{E})$ is well defined or ν_s is one-one.

$\mathsf{A} \in \mathcal{S}^r_s(\mathsf{E}) \subseteq \mathcal{S}_s(\mathsf{E})$ implies $\mathsf{A}' \in \mathcal{S}_s(\mathsf{E})'$ which implies $\nu_s\mathsf{A}' = \mathsf{C}$, where C in $\mathcal{S}^r_s(\mathsf{E})$ is unique such that $\mathsf{C}' = \mathsf{A}' \in \mathcal{S}(\mathsf{E})'$, which implies by the uniqueness of C in $\mathcal{S}^r_s(\mathsf{E})$, $\mathsf{C} = \mathsf{A}$ implying $\nu_s\mathsf{A}' = \mathsf{C} = \mathsf{A}$ or ν_s is onto.

(2): $\mathsf{B}, \mathsf{D} \in \mathcal{S}_s(\mathsf{E})'$ such that $\mathsf{B} \leqslant \mathsf{D}$ implies $\mathsf{B} \subseteq \mathsf{D}$ implying $B \subseteq D$ and $\sigma_B e \subseteq \sigma_D e$ for all $e \in B$. Let $\nu_s\mathsf{B} = \mathsf{A}$, where A is unique in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{A}' = \mathsf{B}$, and $\nu_s\mathsf{D} = \mathsf{C}$, where C is unique in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{C}' = \mathsf{D}$. Define A and C such that $A = \{e \in B/\sigma_B e - \{0\} \neq \phi\}$, $\sigma_A e = \sigma_B e - \{0\}$ for all $e \in A$ and $C = \{e \in D/\sigma_D e - \{0\} \neq \phi\}$ and $\sigma_C e = \sigma_D e - \{0\}$ for all $e \in C$. We show that $\mathsf{A} \subseteq \mathsf{C}$ or (i) $A \subseteq C$ (ii) $\sigma_A e \subseteq \sigma_C e$ for all $e \in A$.

(i): $e \in A - C$ implies $\sigma_D e = \{0\}$ implies $\sigma_B e = \{0\}$ as $\sigma_B e \subseteq \sigma_D e$ implying $\sigma_A e = \phi$, which is a contradiction to $\mathsf{A} \in \mathcal{S}^r_s(\mathsf{E})$. Therefore $A \subseteq C$.

(ii): Let $e \in A \subseteq C$ be fixed. Then $\sigma_A e = \sigma_B e - \{0\} \subseteq \sigma_D e - \{0\} = \sigma_C e$.

Now (i) and (ii) imply $\mathsf{A} \subseteq \mathsf{C}$ implying by the Theorem 3.1(1), $\mathsf{A} \leqslant \mathsf{C}$.

(3): Let $(\mathsf{B}_i)_{i\in I}$ be a subset of $\mathcal{S}_s(\mathsf{E})'$. $\mathsf{B}_i \in \mathcal{S}_s(\mathsf{E})'$ implies $\nu_s\mathsf{B}_i = \mathsf{A}_i$, where A_i is unique in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{A}'_i = \mathsf{B}_i$. Since $\mathcal{S}_s(\mathsf{E})'$ is a complete lattice, $\mathsf{B} = \overline{\vee}_{i\in I}\mathsf{B}_i = \vee_{i\in I}\mathsf{B}_i \in \mathcal{S}_s(\mathsf{E})'$. Now $\nu_s\mathsf{B} = \mathsf{A}$, where A is unique in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{A}' = \mathsf{B}$. Since $\mathcal{S}^r_s(\mathsf{E})$ is a complete lattice and $(\mathsf{A}_i)_{i\in I}$ is a subset of $\mathcal{S}^r_s(\mathsf{E})$, $\overline{\vee}_{i\in I}\mathsf{A}_i = \vee_{i\in I}\mathsf{A}_i \in \mathcal{S}^r_s(\mathsf{E})$, where $\overline{\vee}$ is the $\sqcap$ induced join in $\mathcal{S}^r_s(\mathsf{E})$. By the Theorem 4.3(3)(i), $(\overline{\vee}_{i\in I}\mathsf{A}_i)' = \overline{\vee}_{i\in I}\mathsf{A}'_i = \overline{\vee}_{i\in I}\mathsf{B}_i = \mathsf{B}$. By the uniqueness of A in $\mathcal{S}^r_s(\mathsf{E})$, we have $\mathsf{A} = \overline{\vee}_{i\in I}\mathsf{A}_i$. Therefore $\nu_s(\overline{\vee}_{i\in I}\mathsf{B}_i) = \nu_s\mathsf{B} = \mathsf{A} = \overline{\vee}_{i\in I}\mathsf{A}_i = \overline{\vee}_{i\in I}\nu_s\mathsf{B}_i$.

(4): Let $(\mathsf{B}_i)_{i\in I}$ be a subset of $\mathcal{S}_s(\mathsf{E})'$. $\mathsf{B}_i \in \mathcal{S}_s(\mathsf{E})'$ implies $\nu_s\mathsf{B}_i = \mathsf{A}_i$, where A_i is unique in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{A}'_i = \mathsf{B}_i$. Since $\mathcal{S}_s(\mathsf{E})'$ is a complete lattice, $\mathsf{B} = \cap_{i\in I}\mathsf{B}_i = \wedge_{i\in I}\mathsf{B}_i \in \mathcal{S}_s(\mathsf{E})'$. Now $\nu_s\mathsf{B} = \mathsf{A}$, where A is unique in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{A}' = \mathsf{B}$. Since $\mathcal{S}^r_s(\mathsf{E})$ is a complete lattice and $(\mathsf{A}_i)_{i\in I}$ is a subset of $\mathcal{S}^r_s(\mathsf{E})$, $\sqcap_{i\in I}\mathsf{A}_i = \wedge_{i\in I}\mathsf{A}_i \in \mathcal{S}^r_s(\mathsf{E})$. By the Theorem 4.3(4)(i), $(\sqcap_{i\in I}\mathsf{A}_i)' = \cap_{i\in I}\mathsf{A}'_i = \cap_{i\in I}\mathsf{B}_i = \mathsf{B}$. By the uniqueness of A in $\mathcal{S}^r_s(\mathsf{E})$, we have $\mathsf{A} = \sqcap_{i\in I}\mathsf{A}_i$. Therefore $\nu_s(\cap_{i\in I}\mathsf{B}_i) = \nu_s\mathsf{B} = \mathsf{A} = \sqcap_{i\in I}\mathsf{A}_i = \sqcap_{i\in I}\nu_s\mathsf{B}_i$ as required.

(5): It follows from (1), (3)(i) and (4)(i) above.

For $* = q, b, l, r, i$, the proofs follow in a similar way as above. □

LEMMA 4.4. *For any soft semigroup* E *over* U, *whenever* $* = s, q, b, l, r, i$ *and for any* $B, D \in \mathcal{S}_*(E)'$, *define* $B \leqslant D$ *in* $\mathcal{S}_*(E)'$ *iff* $A \leqslant C$ *in* $\mathcal{S}^r_*(E)$, *where* $A' = B$ *and* $C' = D$. *Then* $\leqslant$ *defines a partial order on* $\mathcal{S}_*(E)'$.

PROOF. It follows from the Lemmas 4.2(3) and 4.3(3). □

LEMMA 4.5. *For any soft semigroup* E *over* U, *whenever* $* = s, q, b, l, r, i$, *the induced partial order on* $\mathcal{S}_*(E)'$ *from the super poset* $\mathcal{S}_*(E')$ *and the partial ordering on* $\mathcal{S}_*(E)'$ *defined as in the Lemma 4.4 above are the same.*

PROOF. Let R be the partial ordering on $\mathcal{S}_s(\mathsf{E}')$, $R_1 = \{(\mathsf{B}, \mathsf{D})/\mathsf{B}, \mathsf{D} \in \mathcal{S}_s(\mathsf{E})'$ and $(\mathsf{B}, \mathsf{D}) \in R\}$ be the induced partial ordering on $\mathcal{S}_s(\mathsf{E})'$ from $\mathcal{S}_s(\mathsf{E}')$ and $R_2 = \{(\mathsf{B}, \mathsf{D}) \in \mathcal{S}_s(\mathsf{E})' \times \mathcal{S}_s(\mathsf{E})'/\mathsf{A}, \mathsf{C} \in \mathcal{S}^r_s(\mathsf{E})$ and $\mathsf{A} \leqslant \mathsf{C}$ in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{A}' = \mathsf{B}$ and $\mathsf{C}' = \mathsf{D}\}$ be the partial ordering defined on $\mathcal{S}_s(\mathsf{E})'$ from $\mathcal{S}^r_s(\mathsf{E})$.

Now we show that $R_1 = R_2$. $(\mathsf{F}, \mathsf{G}) \in R_1$ implies F, $\mathsf{G} \in \mathcal{S}_s(\mathsf{E})'$ implying by the Lemma 4.2(3), there exist unique H and I respectively in $\mathcal{S}^r_s(\mathsf{E})$ such that $\mathsf{H}' = \mathsf{F}$ and $\mathsf{I}' = \mathsf{G}$.

(i) $(\mathsf{F}, \mathsf{G}) \in R_1$ implies $\mathsf{F} \leqslant \mathsf{G}$ in $\mathcal{S}_s(\mathsf{E})'$ implies $\mathsf{F} \subseteq \mathsf{G}$ in $\mathcal{S}_s(\mathsf{E})'$ implies by the Theorem 4.5(2), $\mathsf{H} \subseteq \mathsf{I}$ in $\mathcal{S}^r_s(\mathsf{E})$ implies $\mathsf{H} \leqslant \mathsf{I}$ in $\mathcal{S}^r_s(\mathsf{E})$ implies $(\mathsf{F}, \mathsf{G}) \in R_2$ or $R_1 \subseteq R_2$.

(ii) $(\mathsf{F}, \mathsf{G}) \in R_2$ implies $(\mathsf{F}, \mathsf{G}) \in \mathcal{S}_s(\mathsf{E})' \times \mathcal{S}_s(\mathsf{E})'$ implies $\mathsf{F}, \mathsf{G} \in \mathcal{S}_s(\mathsf{E})'$. $(\mathsf{F}, \mathsf{G}) \in R_2$ implies $\mathsf{F} \leqslant \mathsf{G}$ in $\mathcal{S}_s(\mathsf{E})'$ implies $\mathsf{F} = \mathsf{F} \wedge \mathsf{G}$ in $\mathcal{S}_s(\mathsf{E})'$ implies $\mathsf{F} = \mathsf{F} \wedge \mathsf{G}$ in $\mathcal{S}_s(\mathsf{E}')$ as $\mathcal{S}_s(\mathsf{E})'$ is a complete sublattice of $\mathcal{S}_s(\mathsf{E}')$ implies $\mathsf{F} \leqslant \mathsf{G}$ in $\mathcal{S}_s(\mathsf{E}')$ implies $(\mathsf{F}, \mathsf{G}) \in R$ implies $(\mathsf{F}, \mathsf{G}) \in R_1$ or $R_2 \subseteq R_1$.

Now (i) and (ii) imply $R_1 = R_2$.

For $* = q, b, l, r, i$, the proofs follow in a similar way as above. □

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