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DISTANCE VERSION OF HYPER ZAGREB INDEX OF SOME GRAPH OPERATIONS

M. Aruvi, V. Piramanantham and R. Muruganandam

ABSTRACT. The distance version of Hyper-Zagreb index of a graph is defined and exact values of this index have been found for some graph operations.

1. Introduction

A topological index of a graph is a numerical quantity which is structural invariant, i.e. it is fixed under graph automorphism. The simplest topological indices are the number of vertices and edges of a graph. In chemistry, biochemistry and nanotechnology different topological indices are found to be useful in isomer discrimination, structure-property relationship, structure-activity relationship and pharmaceutical drug design.

All graphs considered are simple and connected graphs. We denote the vertex and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. $d_G(v)$ denotes the degree of a vertex v in G. The number of elements in the vertex set of a graph G is called the order of G and is denoted by $v(G)$. The number of elements in the edge set of a graph G is called the size of G and is denoted by $e(G)$. A graph with order n and size m is called a (n, m)-graph. For any $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The edge connecting the vertices u and v will be denoted by uv. The complement of a graph G is denoted by $\overline{G}$.

The join of graphs G_1 and G_2 is denoted by $G_1 + G_2$, and it is the graph with vertex set $V(G_1) \cup V(G_2)$ and the edge set

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{u_1u_2 | u_1 \in V(G_1), u_2 \in V(G_2)\}.$$

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The composition of graphs G_1 and G_2 is denoted by $G_1[G_2]$, and it is the graph with vertex set $V(G_1) \times V(G_2)$, and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if (u_1 is adjacent to v_1) or ($u_1 = v_1$ and u_2 and v_2 are adjacent). The disjunction of graphs G_1 and G_2 is denoted by $G_1 \vee G_2$, and it is the graph with vertex set $V(G_1) \times V(G_2)$ and $E(G_1 \vee G_2) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)\}$. The symmetric difference of graphs G_1 and G_2 is denoted by $G_1 \oplus G_2$, and it is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$ but not both $\}$. The Cartesian product of G_1 and G_2, denoted by $G_1 \Box G_2$, is the graph vertex set $V(G_1) \times V(G_2)$ and any two vertices (u_p, v_s) and (u_q, v_s) are adjacent if and only if $[u_p = u_q$ and $v_rv_s \in E(G_2)]$ or $[v_r = v_s$ and $u_pu_q \in E(G_1)]$.

Let G be a connected graph. The Wiener index $W(G)$ of a graph G is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u, v).$$

Dobrynin and Kochetova [**7**] and Gutman [**9**] independently proposed a vertex degree-Weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)[d_G(u) + d_G(v)] = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u, v)[d_G(u) + d_G(v)].$$

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trianjestic [**10**]. The first Zagreb index $M_1(G)$ of a graph G is defined as

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] = \sum_{v \in V(G)} d_G^2(v).$$

The second Zagreb index $M_2(G)$ of a graph G is defined as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

The first Zagreb coindex $\overline{M}_1(G)$ of a graph G is defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)].$$

The second Zagreb coindex $\overline{M}_2(G)$ of a graph G is defined as

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v).$$

The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [**6**].

These indices were introduced in a paper 1972 [**2**], within a study of the structure dependency of total Π-electron energy (ε). It was shown that ε depends on the sum of squares of the vertex degrees of the molecular graph (later named first zagreb index), and thus provides a measure of the branching of the carbon-atom skeleton. In the same paper, also the sum of cubes of degrees of vertices of the molecular graph was found to influence ε, but this topological index was never again investigated and was left to oblivion. However this index was not further studied till then, except in a recent article by Furtula and Gutman [**8**] where they reinvestigated this index and studied some basic properties of this index. They showed that the predictive ability of this index is almost similar to that of first Zagreb index and for the entropy and acentric factor, both of them yield correlation coefficients greater than 0.95. In [**5**], this index as "forgotten topological index" or "F-index", which is defined for a graph G as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2].$$

As we know that some chemically interesting graphs can be found by different graph operations on some general or particular graphs, it is important to study such graph operations in order to understand how it is related to the corresponding topological indices of the original graphs. In [**11**], Khalifeh et. al, derived some exact formulae for computing first and second Zagreb indices under some graph operations. In [**4**], Das et. al. derived some upper bounds for multiplicative Zagreb indices for different graph operations. There are several other results regarding various topological indices under different graph operations are available in the literature. In [**3**], Azari and Iranmanesh presented explicit formulae for computing the eccentric-distance sum of different graph operations.

In [**13**], Shirdel et.al. found Hyper-Zagreb index $HM(G)$ which is defined as

$$HM(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2.$$

In this paper, we define a new index distance version of Hyper- Zagreb index and denoted by $DHM(G)$, so that

$$\begin{aligned} DHM(G) &= \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)]^2 \\ &= \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u) + d_G(v)]^2. \end{aligned}$$

In this paper, we present some exact expressions for the distance version of Hyper-Zagreb index of different graph operations such as join, composition, disjunction and symmetric difference of two graphs.

2. Main Results and Discussions

2.1. Basic Lemmas.

Lemma 2.1 ([**1**]). *Let G_1 and G_2 be two simple connected graphs. The number of vertices and edges of graph G_i are denoted by n_i and e_i respectively for $i = 1, 2$. Then we have*

(i) $d_{G_1+G_2}(u,v) = \begin{cases} 1, & uv \in E(G_1) \textit{ or } uv \in E(G_2) \\ & \textit{or } (u \in V(G_1) \textit{ and } v \in V(G_2)) \\ 2, & \textit{otherwise.} \end{cases}$

For a vertex u of G_1,

$$d_{G_1+G_2}(u) = d_{G_1}(u) + n_2,$$

and for a vertex v of G_2,

$$d_{G_1+G_2}(v) = d_{G_2}(v) + n_1.$$

(ii) $d_{G_1[G_2]}((u_1,v_1),(u_2,v_2)) = \begin{cases} d_{G_1}(u_1,u_2), & u_1 \neq u_2 \\ 1, & u_1 = u_2, v_1v_2 \in E(G_2) \\ 2, & \textit{otherwise} \end{cases}$

$$d_{G_1[G_2]}(u,v) = n_2 d_{G_1}(u) + d_{G_2}(v).$$

(iii) $d_{G_1 \vee G_2}((u_1,v_1),(u_2,v_2)) = \begin{cases} 1, & u_1u_2 \in E(G_1) \textit{ or } v_1v_2 \in E(G_2) \\ 2, & \textit{otherwise} \end{cases}$

$$d_{G_1 \vee G_2}((u,v)) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - d_{G_1}(u) d_{G_2}(v).$$

(iv) $d_{G_1 \oplus G_2}((u_1,v_1),(u_2,v_2)) = \begin{cases} 1, & u_1u_2 \in E(G_1) \\ & \textit{or } v_1v_2 \in E(G_2) \textit{ but not both} \\ 2, & \textit{otherwise} \end{cases}$

$$d_{G_1 \oplus G_2}((u,v)) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - 2 d_{G_1}(u) d_{G_2}(v).$$

REMARK 2.1. ([**12**]) For a graph G, let $A(G) = \{(x,y) \in V(G) \times V(G) \mid x$ and y are adjacent in $G\}$ and let $B(G) = \{(x,y) \in V(G) \times V(G) \mid x$ and y are not adjacent in $G\}$. For each $x \in V(G)$, $(x,x) \in B(G)$. Clearly, $A(G) \cup B(G) = V(G) \times V(G)$. Let $C(G) = \{(x,x) \mid x \in V(G)\}$ and $D(G) = B(G) - C(G)$. Clearly $B(G) = C(G) \cup D(G)$, $C(G) \cap D(G) = \emptyset$. The summation $\sum\limits_{(x,y)\in A(G)}$ runs over the ordered pairs of $A(G)$. For simplicity, we write the summation $\sum\limits_{(x,y)\in A(G)}$ as $\sum\limits_{xy\in G}$. Similarly, we write the summation $\sum\limits_{(x,y)\in B(G)}$ as $\sum\limits_{xy\notin G}$. Also the summation $\sum\limits_{xy\in E(G)}$ runs over the edges of G. We denote the summation $\sum\limits_{x,y\in V(G)}$ by $\sum\limits_{x,y\in G}$. The summation $\sum\limits_{(x,y)\in D(G)}$ eqivalent to $\sum\limits_{xy\notin G, x\neq y}$ and similarly the summation $\sum\limits_{(x,y)\in C(G)}$ equivalent to $\sum\limits_{xy\notin G, x=y}$.

LEMMA 2.2 ([**12**]). *Let G be a graph. Then*
(i) $\sum\limits_{xy\in G} 1 = 2e(G)$
(ii) $\sum\limits_{xy\in G} d_G(x) = M_1(G)$
(iii) $\sum\limits_{xy\in G} d_G(x) d_G(y) = 2M_2(G)$

(iv) $\sum_{xy \notin G} 1 = 2e(\overline{G}) + v(G)$

(v) $\sum_{xy \notin G} d_G(x) = 2e(\overline{G})(v(G) - 1) + 2e(G) - M_1(\overline{G})$

(vi) $\sum_{xy \notin G} d_G(x)d_G(y) = 2\overline{M}_2(G) + M_1(G)$

(vii) $\sum_{xy \notin G} [d_G(x) + d_G(y)] = 2\overline{M}_1(G) + 4e(G)$

(viii) $\sum_{xy \notin G, x \neq y} d_G^2(x) = (v(G) - 1)M_1(G) - F(G)$

(ix) $\sum_{xy \notin G, x \neq y} d_G(x) = (v(G) - 1)2e(G) - M_1(G)$

(x) $\sum_{xy \notin G, x \neq y} 1 = (v(G) - 1)v(G) - 2e(G)$ *and*

(xi) $\sum_{xy \notin G} d_G^2(x) = (v(G) - 1)M_1(G) - F(G) + M_1(G)$

THEOREM 2.1. *Let $G_i, i = 1, 2,$ be a (n_i, m_i)- graph, put $\overline{m}_i = e(\overline{G}_i)$. Then*

$$\begin{aligned} 2 \times DHM(G_1 + G_2) = {} & 2HM(G_1) + 8n_2^2 m_1 + 8n_2 M_1(G_1) \\ & + 4[M_1(G_1)(n_1 - 1) - F(G_1)] \\ & + 8\overline{M}_2(G_1) + 8n_2^2[n_1(n_1 - 1) - 2m_1] \\ & + 16n_2[2m_1(n_1 - 1) - M_1(G_1)] + 2n_2 M_1(G_1) + 2n_1 n_2^3 \\ & + 2n_1 M_1(G_2) + 2n_2 n_1^3 + 8m_1 n_2^2 + 16m_1 m_2 + 8n_1 m_1 n_2 \\ & + 8n_1 m_2 n_2 + 4n_1^2 n_2^2 + 8n_1^2 m_2 + 2HM(G_2) + 8n_1^2 m_2 \\ & + 8n_1 M_1(G_2) + 4[M_1(G_2)(n_2 - 1) - F(G_2)] + 8\overline{M}_2(G_2) \\ & + 8n_1^2[n_2(n_2 - 1) - 2m_2] + 16n_1[2m_2(n_2 - 1) - M_1(G_2)] \end{aligned}$$

PROOF. Let $G = G_1 + G_2$. Then,

$$\begin{aligned} 2 \times DHM(G) = {} & \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_1+G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x) + d_{G_1+G_2}(y)\Big]^2 \\ = {} & \sum_{x \in V(G_1+G_2)} \Big\{ \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x) + d_{G_1+G_2}(y)\Big]^2 \\ & + \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x) + d_{G_1+G_2}(y)\Big]^2 \Big\} \end{aligned}$$

$$\begin{aligned} 2 \times DHM(G) = {} & \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x) + d_{G_1+G_2}(y)\Big]^2 \\ & + \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x) + d_{G_1+G_2}(y)\Big]^2 \\ = {} & \sum_{x \in V(G_1)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x) + d_{G_1+G_2}(y)\Big]^2 \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{x\in V(G_2)} \sum_{y\in V(G_1)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
&+ \sum_{x\in V(G_1)} \sum_{y\in V(G_2)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
&+ \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
=& \sum_{x\in V(G_1)} \sum_{y\in V(G_1)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
&+ 2\sum_{x\in V(G_1)} \sum_{y\in V(G_2)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
&+ \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
=& S_1 + 2S_2 + S_3,
\end{aligned}
$$

where S_1, S_2, S_3 are terms of the above sums taken in order. Now we calculate S_1, S_2 and S_3 separately.

$$
\begin{aligned}
S_1 =& \sum_{x\in V(G_1)} \sum_{y\in V(G_1)} d_{G_1+G_2}(x,y)[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)]^2 \\
=& \sum_{x,y\in V(G_1)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
=& \sum_{xy\in G_1} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
&+ \sum_{xy\notin G_1, x\neq y} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
&+ \sum_{xy\notin G_1, x=y} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
=& 1.\sum_{xy\in G_1} \Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 + 2.\sum_{xy\notin G_1, x\neq y} \Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
&+ 0.\sum_{xy\notin G_1, x=y} \Big[d_{G_1+G_2}(x)+d_{G_1+G_2}(y)\Big]^2 \\
S_1 =& S_{1,1} + 2S_{1,2}
\end{aligned}
$$

where $S_{1,1}$ and $S_{1,2}$ are terms of the above sums taken in order which are computed as follows:

$$
S_{1,1} = \sum_{xy\in G_1} [d_{G_1+G_2}(x)+d_{G_1+G_2}(y)]^2
$$

$$= \sum_{xy \in G_1} [(d_{G_1}(x) + n_2) + (d_{G_1}(y) + n_2)]^2$$
$$= \sum_{xy \in G_1} [d_{G_1}(x) + d_{G_1}(y) + 2n_2]^2$$
$$= \sum_{xy \in G_1} \Big[d^2_{G_1}(x) + d^2_{G_1}(y) + 4n_2^2 + 2d_{G_1}(x)d_{G_1}(y) + 4n_2[d_{G_1}(x) + d_{G_1}(y)]\Big]$$
$$= \sum_{xy \in G_1} [d_{G_1}(x) + d_{G_1}(y)]^2 + 4n_2^2 \sum_{xy \in G_1} 1 + 4n_2 \sum_{xy \in G_1} [d_{G_1}(x) + d_{G_1}(y)]$$
$$= 2HM(G_1) + 8n_2^2 m_1 + 8n_2 M_1(G_1)$$

$$S_{1,2} = \sum_{xy \notin G_1,\ x \neq y} [d_{G_1+G_2}(x) + d_{G_1+G_2}(y)]^2$$
$$= \sum_{xy \notin G_1,\ x \neq y} [(d_{G_1}(x) + n_2) + (d_{G_1}(y) + n_2)]^2$$
$$= \sum_{xy \notin G_1,\ x \neq y} [d_{G_1}(x) + d_{G_1}(y) + 2n_2]^2$$
$$= \sum_{xy \notin G_1,\ x \neq y} \Big[d^2_{G_1}(x) + d^2_{G_1}(y) + 4n_2^2 + 2d_{G_1}(x)d_{G_1}(y)$$
$$+ 4n_2[d_{G_1}(x) + d_{G_1}(y)]\Big]$$
$$= \sum_{xy \notin G_1,\ x \neq y} [d_{G_1}(x) + d_{G_1}(y)]^2 + 4n_2^2 \sum_{xy \notin G_1,\ x \neq y} 1$$
$$+ 4n_2 \sum_{xy \notin G_1,\ x \neq y} [d_{G_1}(x) + d_{G_1}(y)]$$
$$= 2[M_1(G_1)(n_1 - 1) - F(G_1)] + 4\overline{M}_2(G_1) + 4n_2^2[n_1(n_1 - 1) - 2m_1]$$
$$+ 8n_2[2m_1(n_1 - 1) - M_1(G_1)]$$
$$2S_{1,2} = 4[M_1(G_1)(n_1 - 1) - F(G_1)] + 8\overline{M}_2(G_1) + 8n_2^2[n_1(n_1 - 1) - 2m_1]$$
$$+ 16n_2[2m_1(n_1 - 1) - M_1(G_1)]$$

$$S_2 = \sum_{x \in V(G_1)} \sum_{y \in V(G_2)} [(d_{G_1}(x) + n_2) + (d_{G_2}(y) + n_1)]^2$$
$$= \sum_{x \in V(G_1)} \sum_{y \in V(G_2)} \Big[d^2_{G_1}(x) + d^2_{G_2}(y) + n_2^2 + n_1^2 + 2n_2 d_{G_1}(x)$$
$$+ 2d_{G_1}(x)d_{G_2}(y) + 2n_1 d_{G_1}(x) + 2n_2 d_{G_2}(y) + 2n_1 n_2 + 2n_1 d_{G_2}(y)\Big]$$
$$= \sum_{x \in V(G_1)} d^2_{G_1}(x) \sum_{y \in V(G_2)} 1 + \sum_{x \in V(G_1)} 1 \sum_{y \in V(G_1)} d^2_{G_2)}(y)$$

$$
\begin{aligned}
&+ n_2^2 \sum_{x\in V(G_1)} 1 \sum_{y\in V(G_1)} 1 + n_1^2 \sum_{x\in V(G_1)} 1 \sum_{y\in V(G_2)} 1 \\
&+ 2n_2 \sum_{x\in V(G_1)} d_{G_1}(x) \sum_{y\in V(G_2)} 1 + 2 \sum_{x\in V(G_1)} d_{G_1}(x) \sum_{y\in V(G_2)} d_{G_2}(y) \\
&+ 2n_1 \sum_{x\in V(G_1)} d_{G_1}(x) \sum_{y\in V(G_2)} 1 + 2n_2 \sum_{x\in V(G_1)} 1 \sum_{y\in V(G_2)} d_{G_2}(y) \\
&+ 2n_1n_2 \sum_{x\in V(G_1)} 1 \sum_{y\in V(G_2)} 1 + 2n_1 \sum_{x\in V(G_1)} 1 \sum_{y\in V(G_2)} d_{G_2}(y)\Big] \\
= {}& n_2M_1(G_1) + n_1n_2^3 + n_1M_1(G_2) + n_2n_1^3 + 4m_1n_2^2 + 8m_1m_2 + 4n_1m_1n_2 \\
&+ 4n_1m_2n_2 + 2n_1^2n_2^2 + 4n_1^2m_2 \\
2S_2 = {}& 2n_2M_1(G_1) + 2n_1n_2^3 + 2n_1M_1(G_2) + 2n_2n_1^3 + 8m_1n_2^2 + 16m_1m_2 \\
&+ 8n_1m_1n_2 + 8n_1m_2n_2 + 4n_1^2n_2^2 + 8n_1^2m_2
\end{aligned}
$$

$$
\begin{aligned}
S_{3,1} &= \sum_{xy\in G_2} [d_{G_1+G_2}(x) + d_{G_1+G_2}(y)]^2 \\
&= \sum_{xy\in G_2} [(d_{G_2}(x) + n_1) + (d_{G_2}(y) + n_1)]^2 \\
&= \sum_{xy\in G_2} [d_{G_2}(x) + d_{G_2}(y) + 2n_1]^2 \\
&= \sum_{xy\in G_2} \Big[d_{G_2}^2(x) + d_{G_2}^2(y) + 4n_1^2 + 2d_{G_2}(x)d_{G_2}(y) + 4n_1[d_{G_2}(x) + d_{G_2}(y)]\Big] \\
&= \sum_{xy\in G_2} [d_{G_2}(x) + d_{G_2}(y)]^2 + 4n_1^2 \sum_{xy\in G_2} 1 + 4n_1 \sum_{xy\in G_2} [d_{G_2}(x) + d_{G_2}(y)] \\
&= 2HM(G_2) + 8n_1^2m_2 + 8n_1M_1(G_2)
\end{aligned}
$$

$$
\begin{aligned}
S_{3,2} &= \sum_{xy\notin G_2,\ x\neq y} [d_{G_1+G_2}(x) + d_{G_1+G_2}(y)]^2 \\
&= \sum_{xy\notin G_2,\ x\neq y} [(d_{G_2}(x) + n_1)) + (d_{G_2}(y) + n_1)]^2 \\
&= \sum_{xy\notin G_2,\ x\neq y} [d_{G_2}(x) + d_{G_2}(y) + 2n_1]^2 \\
&= \sum_{xy\notin G_2,\ x\neq y} \Big[d_{G_2}^2(x) + d_{G_2}^2(y) + 4n_1^2 + 2d_{G_2}(x)d_{G_2}(y) \\
&\quad + 4n_1[d_{G_2}(x) + d_{G_2}(y)]\Big] \\
&= \sum_{xy\notin G_2,\ x\neq y} [d_{G_2}(x) + d_{G_2}(y)]^2 + 4n_1^2 \sum_{xy\notin G_2\ x\neq y} 1
\end{aligned}
$$

$$+ 4n_1 \sum_{xy \notin G_2,\ x \neq y} [d_{G_2}(x) + d_{G_2}(y)]$$
$$= 2[M_1(G_2)(n_2-1) - F(G_2)] + 4\overline{M}_2(G_2) + 4n_1^2[n_2(n_2-1) - 2m_2]$$
$$+ 8n_1[2m_2(n_2-1) - M_1(G_2)]$$
$$2S_{3,2} = 4[M_1(G_2)(n_2-1) - F(G_2)] + 8\overline{M}_2(G_2) + 8n_1^2[n_2(n_2-1) - 2m_2]$$
$$+ 16n_1[2m_2(n_2-1) - M_1(G_2)]$$

$2 \times DHM(G_1+G_2) = S_1 2S_2 + S_3 = S_{1,1} + 2S_{1,2} + 2S_2 + S_{3,1} + 2S_{3,2}$. By substituting $S_{1,1}$, $S_{1,2}$, S_2, $S_{3,1}$ and $S_{3,2}$, we get the desired result. □

THEOREM 2.2. *Let $G_i, i = 1, 2$, be a (n_i, m_i)- graph ,put $\overline{m}_i = e(\overline{G}_i)$. Then*

$$2 \times DHM(G_1[G_2]) = 4n_2^2 m_2 DHM(G_1) + 4W(G_1)F(G_2) + 8n_2 DD(G_1)M_1(G_2)$$
$$+ 8W(G_1)M_2(G_2) + 8n_2^2 M_1(G_1)[n_2(n_2-1) - 2m_2]$$
$$+ 4n_1[(n_2-1)M_1(G_2) - F(G_2)]$$
$$+ 32n_2 m_1[2m_2(n_2-1) - M_1(G_2)] + 8n_1\overline{M}_2(G_2)$$
$$+ 8n_2^2 m_2 M_1(G_1) + 2n_1 F(G_2) + 16n_2 m_1 M_1(G_2)$$
$$+ 4n_1 M_2(G_2) + 2n_2^2 DHM(G_1)(2\overline{m}_2 + n_2)$$
$$+ 4W(G_1)\big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\big]$$
$$+ 8n_2 DD(G_1)\big[(n_2-1)2\overline{m}_2 - \overline{M}_1(G_2) + 2m_2\big]$$
$$+ 4W(G_1)[2\overline{M}_2(G_2) + M_1(G_2)]$$

PROOF. Let $G = G_1[G_2]$. Then,

$$2 \times \Big[DHM(G_1[G_2])\Big] = \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} d_{G_1[G_2]}((x,u),(y,v))$$
$$\Big[d_{G_1[G_2]}(x,u) + d_{G_1[G_2]}(y,v)\Big]^2$$
$$= \sum_{x,y \in V(G_1)} \Big\{ \sum_{uv \in G_2} \Big[d_{G_1[G_2]}(x,u) + d_{G_1[G_2]}(y,v)\Big]^2$$
$$+ \sum_{uv \notin G_2} \Big[d_{G_1[G_2]}(x,u) + d_{G_1[G_2]}(y,v)\Big]^2 \Big\}$$
$$= \sum_{x,y \in V(G_1)} \sum_{uv \in G_2} \Big[d_{G_1[G_2]}((x,u) + d_{G_1[G_2]}(y,v)\Big]^2$$
$$+ \sum_{x,y \in V(G_1)} \sum_{uv \notin G_2} \Big[d_{G_1[G_2]}((x,u) + d_{G_1[G_2]}((y,v)\Big]^2$$
$$= \sum_{x,y \in G_1, x=y} \sum_{uv \in G_2} \Big[d_{G_1[G_2]}((x,u) + d_{G_1[G_2]}((y,v)\Big]^2$$

$$+\sum_{x,y\in G_1,x\neq y}\sum_{uv\in G_2}\Big[d_{G_1[G_2]}((x,u)+d_{G_1[G_2]}((y,v)\Big]^2$$

$$+\sum_{x,y\in G_1,x=y}\sum_{uv\notin G_2}\Big[d_{G_1[G_2]}((x,u)+d_{G_1[G_2]}((y,v)\Big]^2$$

$$+\sum_{x,y\in G_1,x\neq y}\sum_{uv\notin G_2}\Big[d_{G_1[G_2]}((x,u)+d_{G_1[G_2]}((y,v)\Big]^2$$

$$=J_3+J_1+J_2+J_4,$$

where $J_3,\ J_1,\ J_2$ and J_4 are terms of the above sums taken in order. Now we calculate $J_1,\ J_2,\ J_3$ and J_4 one by one.

$$\begin{aligned}
J_1 &= \sum_{x,y\in G_1,\ x\neq y}\ \sum_{uv\in G_1} d_{G_1[G_2]}\big((x,u),(y,v)\big)[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)]^2\\
&= \sum_{x,y\in G_1,\ x\neq y}\ \sum_{uv\in G_1} d_{G_1}(x,y)[d_{G_2}(u)+n_2d_{G_1}(x)+d_{G_2}(v)+n_2d_{G_1}(y)]^2\\
&= \sum_{x,y\in G_1,\ x\neq y}\ \sum_{uv\in G_1} d_{G_1}(x,y)\Big[d^2_{G_2}(u)+n_2^2d^2_{G_1}(x)+d^2_{G_2}(v)+n_2^2d^2_{G_1}(y)\\
&\quad +2n_2d_{G_1}(x)d_{G_2}(u)+2d_{G_2}(u)d_{G_2}(v)+2n_2d_{G_1}(y)d_{G_2}(u)+2n_2d_{G_1}(x)d_{G_2}(v)\\
&\quad +2n_2^2d_{G_1}(x)d_{G_1}(y)+2n_2d_{G_1}(y)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y)[d^2_{G_2}(u)+d^2_{G_1}(y)+2d_{G_1}(x)d_{G_1}(y)]\sum_{uv\in G_1}1\\
&\quad +\sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y)\sum_{uv\in G_1}[d^2_{G_2}(u)+d^2_{G_2}(v)]\\
&\quad +2n_2\sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y)[d_{G_1}(x)+d_{G_1}(y)]\sum_{uv\in G_1}d_{G_2}(u)\\
&\quad +2n_2\sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y)[d_{G_1}(x)+d_{G_1}(y)]\sum_{uv\in G_1}d_{G_2}(v)\\
&\quad +2\sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y)\sum_{uv\in G_1}d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2(2m_2)2DHM(G_1)+2W(G_1)2F(G_2)+2n_2(2DD(G_1))\sum_{u\in V(G_2)}d^2_{G_2}(u)\\
&\quad +2n_2(2DD(G_1))\sum_{v\in V(G_2)}d^2_{G_2}(v)+2(2W(G_1))2M_2(G_2)\\
&= 4n_2^2m_2DHM(G_1)+4W(G_1)F(G_2)+8n_2DD(G_1)M_1(G_2)+8W(G_1)M_2(G_2)
\end{aligned}$$

$$J_2=\sum_{x,y\in G_1,\ x=y}\ \sum_{uv\notin G_2,u\neq v} d_{G_1[G_2]}\big((x,u),(y,v)\big)[d_{G_1[G_2]}(x,u)+d_{G_1[G_2]}(y,v)]^2$$

$$= \sum_{x,y\in G_1,\ x=y} \sum_{uv\notin G_2, u\neq v} d_{G_1}(x,y)[d_{G_2}(u) + n_2 d_{G_1}(x) + d_{G_2}(v) + n_2 d_{G_1}(y)]^2$$

$$= 2 \sum_{x,y\in G_1,\ x=y} \sum_{uv\notin G_2} \Big[d^2_{G_2}(u) + n_2^2 d^2_{G_1}(x) + d^2_{G_2}(v) n_2^2 d^2_{G_1}(y)$$

$$+ 2n_2 d_{G_1}(x) d_{G_2}(u) + 2d_{G_2}(u) d_{G_2}(v) + 2n_2 d_{G_1}(y) + d_{G_2}(u)$$

$$+ 2n_2 d_{G_1}(x) d_{G_2}(v) + 2n_2^2 d_{G_1}(x) d_{G_1}(y) + 2n_2 d_{G_1}(y) d_{G_2}(v)\Big]$$

$$= 2\Big\{ \sum_{x,y\in G_1,\ x=y} [d^2_{G_2}(u) + d^2_{G_1}(y) + 2d_{G_1}(x) d_{G_1}(y)] \sum_{uv\notin G_2, u\neq v} 1$$

$$+ \sum_{x,y\in G_1,\ x=y} 1 \sum_{uv\notin G_2, u\neq v} [d^2_{G_2}(u) + d^2_{G_2}(v)]$$

$$+ 2n_2 \sum_{x,y\in G_1,\ x=y} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\notin G_2, u\neq v} d_{G_2}(u)$$

$$+ 2n_2 \sum_{x,y\in G_1,\ x=y} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\notin G_2, u\neq v} d_{G_2}(v)$$

$$+ 2 \sum_{x,y\in G_1,\ x=y} 1 \sum_{uv\notin G_2, u\neq v} d_{G_2}(u) d_{G_2}(v)\Big\}$$

$$= 2\Big[n_2^2 4M_1(G_1)[n_2(n_2-1) - 2m_2] + 2n_1[(n_2-1)M_1(G_2) - F(G_2)]$$

$$+ 2n_2(4m_1)2[2m_2(n_2-1) - M_1(G_2)] + 2n_1 2\overline{M}_2(G_2)\Big]$$

$$= 8n_2^2 M_1(G_1)[n_2(n_2-1) - 2m_2] + 4n_1[(n_2-1)M_1(G_2) - F(G_2)]$$

$$+ 32n_2 m_1[2m_2(n_2-1) - M_1(G_2)] + 8n_1\overline{M}_2(G_2)$$

$$J_3 = \sum_{x,y\in G_1,\ x=y} \sum_{uv\in G_2} d_{G_1[G_2]}\big((x,u),(y,v)\big)[d_{G_1[G_2]}(x,u) + d_{G_1[G_2]}(y,v)]^2$$

$$= \sum_{x,y\in G_1,\ x=y} \sum_{uv\in G_2} d_{G_1}(x,y)[d_{G_2}(u) + n_2 d_{G_1}(x) + d_{G_2}(v) + n_2 d_{G_1}(y)]^2$$

$$= \sum_{x,y\in G_1,\ x=y} \sum_{uv\in G_2} \Big[d^2_{G_2}(u) + n_2^2 d^2_{G_1}(x) + d^2_{G_2}(v) + n_2^2 d^2_{G_1}(y)$$

$$+ 2n_2 d_{G_1}(x) d_{G_2}(u) + 2d_{G_2}(u) d_{G_2}(v) + 2n_2 d_{G_1}(y) d_{G_2}(u)$$

$$+ 2n_2 d_{G_1}(x) d_{G_2}(v) + 2n_2^2 d_{G_1}(x) d_{G_1}(y) + 2n_2 d_{G_1}(y) d_{G_2}(v)\Big]$$

$$= n_2^2 \sum_{x,y\in G_1,\ x=y} [d^2_{G_2}(u) + d^2_{G_1}(y) + 2d_{G_1}(x) d_{G_1}(y)] \sum_{uv\in G_1} 1$$

$$+ \sum_{x,y\in G_1,\ x=y} 1 \sum_{uv\in G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)]$$

$$
\begin{aligned}
&+ 2n_2 \sum_{x,y\in G_1,\ x=y} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(u) \\
&+ 2n_2 \sum_{x,y\in G_1,\ x=y} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(v) \\
&+ 2 \sum_{x,y\in G_1,\ x=y} 1 \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v) \\
&= n_2^2 4M_1(G_1)2m_2 + 2n_1F(G_2) + 2n_2 4m_1M_1(G_2) + 2n_2(4m_1)M_1(G_2) \\
&\quad + 2n_1 2M_2(G_2) \\
&= 8n_2^2 m_2 M_1(G_1) + 2n_1F(G_2) + 16n_2m_1M_1(G_2) + 4n_1M_2(G_2)
\end{aligned}
$$

$$
\begin{aligned}
J_4 &= \sum_{x,y\in G_1,\ x\neq y} \sum_{uv\notin G_2} d_{G_1[G_2]}\big((x,u),(y,v)\big)[d_{G_1[G_2]}(x,u) + d_{G_1[G_2]}(y,v)]^2 \\
&= \sum_{x,y\in G_1,\ x\neq y} \sum_{uv\notin G_2} d_{G_1}(x,y)[d_{G_2}(u) + n_2d_{G_1}(x) + d_{G_2}(v) + n_2d_{G_1}(y)]^2 \\
&= \sum_{x,y\in G_1,\ x\neq y} \sum_{uv\notin G_2} d_{G_1}(x,y)\Big[d^2_{G_2}(u) + n_2^2d^2_{G_1}(x) + d^2_{G_2}(v) + n_2^2d^2_{G_1}(y) \\
&\quad + n_2^2d^2_{G_1}(y) + 2n_2d_{G_1}(x)d_{G_2}(u) + 2d_{G_2}(u)d_{G_2}(v) + 2n_2d_{G_1}(y)d_{G_2}(u) \\
&\quad + 2n_2d_{G_1}(x)d_{G_2}(v) + 2n_2^2d_{G_1}(x)d_{G_1}(y) + 2n_2d_{G_1}(y)d_{G_2}(v)\Big] \\
&= n_2^2 \sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y)[d^2_{G_2}(u) + d^2_{G_1}(y) + 2d_{G_1}(x)d_{G_1}(y)] \sum_{uv\notin G_2} 1 \\
&\quad + \sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y) \sum_{uv\in G_1} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&\quad + 2n_2 \sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y)[d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\notin G_2} d_{G_2}(u) \\
&\quad + 2n_2 \sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y)[d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\notin G_2} d_{G_2}(v) \\
&\quad + 2 \sum_{x,y\in G_1,\ x\neq y} d_{G_1}(x,y) \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) \\
&= n_2^2 2DHM(G_1)(2\overline{m}_2 + n_2) + 2W(G_1)2\big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\big] \\
&\quad + 4n_2 2DD(G_1)\big[(n_2-1)2\overline{m}_2 - \overline{M}_1(G_2) + 2m_2\big] \\
&\quad + 2(2W(G_1))[2\overline{M}_2(G_2) + M_1(G_2)] \\
&= 2n_2^2DHM(G_1)(2\overline{m}_2 + n_2) + 4W(G_1)\big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\big] \\
&\quad + 8n_2DD(G_1)\big[(n_2-1)2\overline{m}_2 - \overline{M}_1(G_2) + 2m_2\big] \\
&\quad + 4W(G_1)[2\overline{M}_2(G_2) + M_1(G_2)]
\end{aligned}
$$

By adding J_1, J_2, J_3 and J_4 we get the required result. □

THEOREM 2.3. *Let $G_i, i = 1, 2$, be a (n_i, m_i)- graph, put $\overline{m}_i = e(\overline{G}_i)$. Then* $2 \times DHM(G_1 \vee G_2)$

$$
\begin{aligned}
&= [2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)] \\
&\qquad [4m_2n_2^2 + 2F(G_2) - 4n_2M_1(G_2)] \\
&+ [2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)][8n_1n_2M_1(G_2) - 4n_1F(G_2) - 8n_1M_2(G_2)] \\
&+ [2M_2(\overline{G}_1) + M_1(G_1)][4m_2n_2^2 - 4n_2M_1(G_2) + 4M_2(G_2)] \\
&+ (2\overline{m}_1 + n_1)[2n_1^2F(G_2) + 4n_1^2M_2(G_2)] \\
&+ [2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)] \\
&\qquad [4m_1n_1^2 + 2F(G_1) - 4n_1M_1(G_1)] \\
&+ [2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)][8n_2n_2M_1(G_1) - 4n_2F(G_1) - 8n_2M_2(G_1)] \\
&+ [2M_2(\overline{G}_2) + M_1(G_2)][4m_1n_1^2 - 4n_1M_1(G_1) + 4M_2(G_1)] \\
&+ (2\overline{m}_2 + n_2)[2n_2^2F(G_1) + 4n_2^2M_2(G_1)] \\
&+ 4n_2^2F(G_1)m_2 + 4n_1^2F(G_2)m_1 + 2F(G_1)F(G_2) + 4n_1n_2M_1(G_1)M_1(G_2) \\
&- 4n_2F(G_1)M_1(G_2) - 4n_1M_1(G_1)F(G_2) + 8n_2^2M_2(G_1)2m_2 + 8n_1^2M_2(G_2)m_1 \\
&+ 4n_1n_2M_1(G_1)2M_1(G_2) - 8n_2M_2(G_1)M_1(G_2) - 8n_1M_2(G_2)M_1(G_1) \\
&+ 8M_2(G_1)M_2(G_2) \\
&+ 4n_2^2\Big[\Big(2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)\Big)(2\overline{m}_2 + n_2)\Big] \\
&+ 4n_1^2\Big[\Big(2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)\Big)(2\overline{m}_1 + n_1)\Big] \\
&+ 2\Big[2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)\Big] \\
&\qquad \Big[2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2)) + M_1(G_2)\Big] \\
&+ 4n_1n_2\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
&- 4n_2\Big[2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)\Big] \\
&\qquad \Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
&- 4n_1\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big] \\
&\qquad \Big[2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)\Big] \\
&+ 8n_2^2[2\overline{M}_2(G_1) + M_1(G_1)](2\overline{m}_2 + n_2) + 2n_1^2(2\overline{m}_1 + n_1)[2\overline{M}_2(G_2) + M_1(G_2)] \\
&+ 4n_1n_2\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
&- 4n_2[2\overline{M}_2(G_1) + M_1(G_1)]\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
&- 4n_1\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big][2\overline{M}_2(G_1) + M_1(G_1)]
\end{aligned}
$$

$$+ 2[2\overline{M}_2(G_1) + M_1(G_1)][2\overline{M}_2(G_2) + M_1(G_2)] + 4n_2^3 M_1(G_1) + 4n_1^3 M_1(G_2)$$
$$+ 4M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2 - 16m_2n_2M_1(G_1) - 16n_1m_1M_1(G_2).$$

PROOF. Let $G = G_1 \vee G_2$. Then,

$$\begin{aligned}
2\times & DHM(G_1 \vee G_2) \\
&= \sum_{x,y\in V(G_1)} \sum_{u,v\in V(G_2)} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{x,y\in V(G_1)} \Big\{ \sum_{uv\in G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&\quad + \sum_{uv\notin G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \Big\} \\
&= \sum_{x,y\in V(G_1)} \sum_{uv\in G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&\quad + \sum_{x,y\in V(G_1)} \sum_{uv\notin G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&\quad + \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&\quad + \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&\quad + \sum_{xy\notin G_1} \sum_{uv\notin G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= S_3 + S_1 + S_2 + S_4
\end{aligned}$$

$$\begin{aligned}
S_1 &= \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{xy\notin G_1} \sum_{uv\in G_2} [d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{xy\notin G_1} \sum_{uv\in G_2} \Big[d^2_{G_1\vee G_2}(x,u) + d^2_{G_1\vee G_2}(y,v) + 2d_{G_1\vee G_2}(x,u)d_{G_1\vee G_2}(y,v)\Big] \\
&= \sum_{xy\notin G_1} \sum_{uv\in G_2} \Big[\{n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)\}^2 \\
&\quad + \{n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - d_{G_1}(y)d_{G_2}(v)\}^2 \\
&\quad + 2\{n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)\} \\
&\qquad \{n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - d_{G_1}(y)d_{G_2}(v)\}\Big]
\end{aligned}$$

$$= \sum_{xy\notin G_1}\sum_{uv\in G_2}\Big[n_2^2d^2_{G_1}(x)+n_1^2d^2_{G_2}(u)+d^2_{G_1}(x)d^2_{G_2}(u)+2n_1n_2d_{G_1}(x)d_{G_2}(u)$$
$$-2n_2d^2_{G_1}(x)d_{G_2}(u)-2n_1d^2_{G_2}(u)d_{G_1}(x)+n_2^2d^2_{G_1}(y)+n_1^2d^2_{G_2}(v)$$
$$+d^2_{G_1}(y)d^2_{G_2}(v)+2n_1n_2d_{G_1}(y)d_{G_2}(v)-2n_2d^2_{G_1}(y)d_{G_2}(v)-2n_1d^2_{G_2}(v)d_{G_1}(y)$$
$$+2\Big\{n_2^2d_{G_1}(x)d_{G_1}(y)+n_1n_2d_{G_1}(x)d_{G_1}(v)-n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)$$
$$+n_1n_2d_{G_1}(y)d_{G_2}(u)+n_1^2d_{G_2}(u)d_{G_2}(v)-n_1d_{G_2}(u)d_{G_1}(y)d_{G_2}(v)$$
$$-n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)-n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)$$
$$+d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big\}\Big]$$
$$= n_2^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d^2_{G_1}(x)+d^2_{G_1}(y)]+n_1^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d^2_{G_2}(u)+d^2_{G_2}(v)]$$
$$+\sum_{xy\notin G_1}\sum_{uv\in G_2}[d^2_{G_1}(x)d^2_{G_2}(u)+d^2_{G_1}(y)d^2_{G_2}(v)]$$
$$+2n_1n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(u)+d_{G_1}(y)d_{G_2}(v)]$$
$$-2n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d^2_{G_1}(x)d_{G_2}(u)+d^2_{G_1}(y)d_{G_2}(v)]$$
$$-2n_1\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d^2_{G_2}(u)+d_{G_1}(y)d^2_{G_2}(v)]$$
$$+2n_2^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_1}(y)]+2n_1^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_2}(x)d_{G_2}(v)]$$
$$+2n_1n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)]$$
$$-2n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)]$$
$$-2n_1\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)]$$
$$+2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)]$$
$$= n_2^2\sum_{xy\notin G_1}[d^2_{G_1}(x)+d^2_{G_1}(y)]\sum_{uv\in G_2}1+n_1^2\sum_{xy\notin G_1}1\sum_{uv\in G_2}[d^2_{G_2}(u)+d^2_{G_2}(v)]$$
$$+\sum_{xy\notin G_1}[d^2_{G_1}(x)+d^2_{G_1}(y)]\sum_{uv\in G_2}d^2_{G_2}(u)$$
$$+2n_1n_2\sum_{xy\notin G_1}[d_{G_1}(x)+d_{G_1}(y)]\sum_{uv\in G_2}d_{G_2}(u)$$

$$
\begin{aligned}
&- 2n_2 \sum_{xy\notin G_1} [d^2_{G_1}(x) + d^2_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(u)\\
&- 2n_1 \sum_{xy\notin G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d^2_{G_2}(u)\\
&+ 2n_2^2 \sum_{xy\notin G_1} [d_{G_1}(x)d_{G_1}(y)] \sum_{uv\in G_2} 1 + 2n_1^2 \sum_{xy\notin G_1} 1 \sum_{uv\in G_2} [d_{G_2}(u)d_{G_2}(v)]\\
&+ 2n_1n_2 \sum_{xy\notin G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(u)\\
&- 2n_2 \sum_{xy\notin G_1} [d_{G_1}(x)d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(u)\\
&- 2n_1 \sum_{xy\notin G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(v)d_{G_2}(u)\\
&+ 2 \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
=\ & n_2^2 2[2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)](2m_2)\\
&+ n_1^2(2\overline{m}_1 + n_1)2F(G_2) + 2[2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1)\\
&+ M_1(G_1)]F(G_2) + 2n_1n_2 \times 2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]M_1(G_2)\\
&- 2n_2 \times 2[2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)]M_1(G_2)\\
&- 2n_1 \times 2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]F(G_2) + 2n_2^2[2\overline{M}_2(G_1)\\
&+ M_1(G_1)]2m_2 + 2n_1^2(2\overline{m}_1 + n_1)(2M_2(G_2)) + 2n_1n_2 \times 2[2\overline{m}_1(n_1-1)\\
&+ 2m_1 - M_1(\overline{G}_1)]M_1(G_2) - 2n_2 \times 2[2\overline{M}_2(G_1) + M_1(G_1)]M_1(G_2)\\
&- 2n_1 \times 2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)](2M_2(G_2)) + 2 \times 2[2\overline{M}_2(G_1)\\
&+ M_1(G_1)]2M_2(G_2)\\
=\ & [2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)][4m_2n_2^2 + 2F(G_2)\\
&- 4n_2M_1(G_2)] + [2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)][8n_1n_2M_1(G_2) - 4n_1F(G_2)\\
&- 8n_1M_2(G_2)] + [2M_2(\overline{G}_1) + M_1(G_1)][4m_2n_2^2 - 4n_2M_1(G_2) + 4M_2(G_2)]\\
&+ (2\overline{m}_1 + n_1)[2n_1^2F(G_2) + 4n_1^2M_2(G_2)]
\end{aligned}
$$

$$
\begin{aligned}
S_2 &= \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2\\
&= \sum_{xy\in G_1} \sum_{uv\notin G_2} [d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2\\
&= \sum_{xy\in G_1} \sum_{uv\notin G_2} \Big[\{n_2d_{G_1}(x) + n_1d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)\}^2\\
&\quad + \{n_2d_{G_1}(y) + n_1d_{G_2}(v) - d_{G_1}(y)d_{G_2}(v)\}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\{n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x) d_{G_2}(u)\} \\
&\qquad \{n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - d_{G_1}(y) d_{G_2}(v)\}\Big] \\
&= \sum_{xy \in G_1} \sum_{uv \notin G_2} \Big[n_2^2 d_{G_1}^2(x) + n_1^2 d_{G_2}^2(u) + d_{G_1}^2(x) d_{G_2}^2(u) + 2n_1 n_2 d_{G_1}(x) d_{G_2}(u) \\
&- 2n_2 d_{G_1}^2(x) d_{G_2}(u) - 2n_1 d_{G_2}^2(u) d_{G_2}(u) + n_2^2 d_{G_1}^2(y) + n_1^2 d_{G_2}^2(v) \\
&+ d_{G_1}^2(y) d_{G_2}^2(v) + 2n_1 n_2 d_{G_1}(y) d_{G_2}(v) - 2n_2 d_{G_1}^2(y) d_{G_2}(v) - 2n_1 d_{G_2}^2(v) d_{G_1}(y) \\
&+ 2\Big\{ n_2^2 d_{G_1}(x) d_{G_1}(y) + n_1 n_2 d_{G_1}(x) d_{G_1}(v) - n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) \\
&+ n_1 n_2 d_{G_1}(y) d_{G_2}(u) + n_1^2 d_{G_2}(u) d_{G_2}(v) - n_1 d_{G_2}(u) d_{G_1}(y) d_{G_2}(v) \\
&- n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) - n_1 d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) \\
&+ d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v) \Big\}\Big] \\
&= n_2^2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}^2(x) + d_{G_1}^2(y)] + n_1^2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&+ 2n_1 n_2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}(u) + d_{G_1}(y) d_{G_2}(v)] \\
&- 2n_2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}^2(x) d_{G_2}(u) + d_{G_1}^2(y) d_{G_2}(v)] \\
&- 2n_1 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}^2(u) + d_{G_1}(y) d_{G_2}^2(v)] \\
&+ 2n_2^2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_1}(y)] + 2n_1^2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_2}(x) d_{G_2}(v)] \\
&+ 2n_1 n_2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u)] \\
&- 2n_2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) + d_{G_1}(x) d_{G_1}(y) d_{G_2}(u)] \\
&- 2n_1 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)] \\
&+ 2 \sum_{xy \in G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)] \\
&= n_2^2 \sum_{uv \notin G_2} [d_{G_2}^2(u) + d_{G_2}^2(v)] \sum_{xy \in G_1} 1 + n_1^2 \sum_{uv \notin G_2} 1 \sum_{xy \in G_1} [d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&+ \sum_{uv \notin G_2} [d_{G_2}^2(u) + d_{G_2}^2(v)] \sum_{xy \in G_1} d_{G_1}^2(x) \\
&+ 2n_1 n_2 \sum_{uv \notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy \in G_1} d_{G_1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&- 2n_2 \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 2n_1 \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy\in G_1} d^2_{G_1}(x) \\
&+ 2n_2^2 \sum_{uv\notin G_2} [d_{G_2}(u) d_{G_2}(v)] \sum_{xy\in G_1} 1 + 2n_1^2 \sum_{uv\notin G_2} 1 \sum_{xy\in G_1} [d_{G_2}(u) d_{G_2}(v)] \\
&+ 2n_1 n_2 \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 2n_2 \sum_{uv\notin G_2} [d_{G_2}(u) d_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 2n_1 \sum_{uv\notin G_2} d_{G_2}(u) d_{G_2}(v) \sum_{xy\in G_1} [d_{G_1}(x) + d_{G_1}(y)] \\
&+ 2 \sum_{uv\notin G_2} d_{G_2}(u) d_{G_2}(v) \sum_{xy\in G_1} d_{G_1}(x) d_{G_1}(y) \\
= {}& n_2^2 2[2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)](2m_1) \\
&+ n_2^2(2\overline{m}_2 + n_2)2F(G_1) + 2[2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) \\
&+ M_1(G_2)]F(G_1) + 2n_2n_2 \times 2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]M_1(G_1) \\
&- 2n_1 \times 2[2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)]M_1(G_1) \\
&- 2n_2 \times 2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]F(G_1) + 2n_1^2[2\overline{M}_2(G_2) \\
&+ M_1(G_2)]2m_1 + 2n_2^2(2\overline{m}_2 + n_2)(2M_2(G_1)) + 2n_2n_2 \times 2[2\overline{m}_2(n_2-1) \\
&+ 2m_2 - M_1(\overline{G}_2)]M_1(G_1) - 2n_1 \times 2[2\overline{M}_2(G_2) + M_1(G_2)]M_1(G_1) \\
&- 2n_2 \times 2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)](2M_2(G_1)) \\
&+ 2 \times 2[2\overline{M}_2(G_2) + M_1(G_2)]2M_2(G_1) \\
= {}& [2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)][4m_1n_1^2 + 2F(G_1) \\
&- 4n_1M_1(G_1)] + [2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)][8n_2n_2M_1(G_1) - 4n_2F(G_1) \\
&- 8n_2M_2(G_1)] + [2M_2(\overline{G}_2) + M_1(G_2)][4m_1n_1^2 - 4n_1M_1(G_1) + 4M_2(G_1)] \\
&+ (2\overline{m}_2 + n_2)[2n_2^2F(G_1) + 4n_2^2M_2(G_1)]
\end{aligned}
$$

$$
\begin{aligned}
S_3 &= \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} \Big[\{n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x) d_{G_2}(u)\}^2 \\
&\quad + \{n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - d_{G_1}(y) d_{G_2}(v)\}^2
\end{aligned}
$$

$$+ 2\{n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x) d_{G_2}(u)\}$$
$$\{n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - d_{G_1}(y) d_{G_2}(v)\}\Big]$$
$$= \sum_{xy\in G_1} \sum_{uv\in G_2} \Big[n_2^2 d_{G_1}^2(x) + n_1^2 d_{G_2}^2(u) + d_{G_1}^2(x) d_{G_2}^2(u) + 2n_1 n_2 d_{G_1}(x) d_{G_2}(u)$$
$$- 2n_2 d_{G_1}^2(x) d_{G_2}(u) - 2n_1 d_{G_2}^2(u) d_{G_2}(u) + n_2^2 d_{G_1}^2(y) + n_1^2 d_{G_2}^2(v)$$
$$+ d_{G_1}^2(y) d_{G_2}^2(v) + 2n_1 n_2 d_{G_1}(y) d_{G_2}(v) - 2n_2 d_{G_1}^2(y) d_{G_2}(v) - 2n_1 d_{G_2}^2(v) d_{G_1}(y)$$
$$+ 2\Big\{n_2^2 d_{G_1}(x) d_{G_1}(y) + n_1 n_2 d_{G_1}(x) d_{G_1}(v) - n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(v)$$
$$+ n_1 n_2 d_{G_1}(y) d_{G_2}(u) + n_1^2 d_{G_2}(u) d_{G_2}(v) - n_1 d_{G_2}(u) d_{G_1}(y) d_{G_2}(v)$$
$$- n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) - n_1 d_{G_1}(x) d_{G_2}(u) d_{G_2}(v)$$
$$+ d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)\Big\}\Big]$$
$$= n_2^2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}^2(x) + d_{G_1}^2(y)] + n_1^2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_2}^2(u) + d_{G_2}^2(v)]$$
$$+ \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}^2(x) d_{G_2}^2(u) + d_{G_1}^2(y) d_{G_2}^2(v)]$$
$$+ 2n_1 n_2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}(x) d_{G_2}(u) + d_{G_1}(y) d_{G_2}(v)]$$
$$- 2n_2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}^2(x) d_{G_2}(u) + d_{G_1}^2(y) d_{G_2}(v)]$$
$$- 2n_1 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}(x) d_{G_2}^2(u) + d_{G_1}(y) d_{G_2}^2(v)]$$
$$+ 2n_2^2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}(x) d_{G_1}(y)] + 2n_1^2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_2}(x) d_{G_2}(v)]$$
$$+ 2n_1 n_2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}(x) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u)]$$
$$- 2n_2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) + d_{G_1}(x) d_{G_1}(y) d_{G_2}(u)]$$
$$- 2n_1 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)]$$
$$+ 2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)]$$
$$= n_2^2 \sum_{uv\in G_2} [d_{G_2}^2(u) + d_{G_2}^2(v)] \sum_{xy\in G_1} 1 + n_1^2 \sum_{uv\in G_2} 1 \sum_{xy\in G_1} [d_{G_2}^2(u) + d_{G_2}^2(v)]$$
$$+ \sum_{uv\in G_2} [d_{G_2}^2(u) + d_{G_2}^2(v)] \sum_{xy\in G_1} d_{G_1}^2(x)$$

$$
\begin{aligned}
&+ 2n_1n_2 \sum_{uv\in G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 2n_2 \sum_{uv\in G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 2n_1 \sum_{uv\in G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy\in G_1} d^2_{G_1}(x) \\
&+ 2n_2^2 \sum_{uv\in G_2} [d_{G_2}(u)d_{G_2}(v)] \sum_{xy\in G_1} 1 + 2n_1^2 \sum_{uv\in G_2} 1 \sum_{xy\in G_1} [d_{G_2}(u)d_{G_2}(v)] \\
&+ 2n_1n_2 \sum_{uv\in G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 2n_2 \sum_{uv\in G_2} [d_{G_2}(u)d_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 2n_1 \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v) \sum_{xy\in G_1} [d_{G_1}(x) + d_{G_1}(y)] \\
&+ 2 \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v) \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y) \\
&= n_2^2 2F(G_1)(2m_2) + n_1^2 2F(G_2)(2m_1) + 2F(G_1)F(G_2) + 2n_1n_2M_1(G_1)2M_1(G_2) \\
&- 2n_2F(G_1)2M_1(G_2) - 2n_1M_1(G_1)2F(G_2) + 2n_2^2(2M_2(G_1))(2m_2) \\
&+ 2n_1^2(2M_2(G_2))(2m_1) + 2n_1n_2M_1(G_1)2M_1(G_2) - 2n_2(2M_2(G_1))2M_1(G_2) \\
&- 2n_1(2M_2(G_2))2M_1(G_1) + 2(2M_2(G_1))2M_2(G_2) \\
&= 4n_2^2F(G_1)m_2 + 4n_1^2F(G_2)m_1 + 2F(G_1)F(G_2) + 4n_1n_2M_1(G_1)M_1(G_2) \\
&- 4n_2F(G_1)M_1(G_2) - 4n_1M_1(G_1)F(G_2) + 8n_2^2M_2(G_1)2m_2 + 8n_1^2M_2(G_2)m_1 \\
&+ 4n_1n_2M_1(G_1)2M_1(G_2) - 8n_2M_2(G_1)M_1(G_2) - 8n_1M_2(G_2)M_1(G_1) \\
&+ 8M_2(G_1)M_2(G_2)
\end{aligned}
$$

$$
\begin{aligned}
S_4 &= \sum_{xy\notin G_1} \sum_{uv\notin G_2} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{xy\notin G_1} \Big\{ \sum_{uv\notin G_2\ u\neq v} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&\quad + \sum_{uv\notin G_2\ u=v} d_{G_1\vee G_2]}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \Big\} \\
&= \sum_{xy\notin G_1} \sum_{uv\notin G_2\ u\neq v} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&\quad + \sum_{xy\notin G_1} \sum_{uv\notin G_2\ u=v} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u\neq v} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u\neq v} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&+ \sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u=v} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&+ \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u=v} d_{G_1\vee G_2}\big((x,u),(y,v)\big)[d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&= 2\sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u\neq v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&+ 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u\neq v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&+ 2\sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u=v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&= 2\sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u\neq v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&+ 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u\neq v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&+ 2\sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u=v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&+ 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u=v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&- 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u=v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&= 2\sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&- 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2,u=v} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2
\end{aligned}
$$

$$S_4 = 2S_5 - 2S_6$$

$$
\begin{aligned}
S_5 &= \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_{G_1\vee G_2}(x,u)+d_{G_1\vee G_2}(y,v)]^2\\
&= \sum_{xy\notin G_1} \sum_{uv\notin G_2} \Big[\{n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\}^2\\
&+ \{n_2d_{G_1}(y)+n_1d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\}^2\\
&+ 2\{n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\}\\
&\ \{n_2d_{G_1}(y)+n_1d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\}\Big]\\
&= \sum_{xy\notin G_1} \sum_{uv\notin G_2} \Big[n_2^2d_{G_1}^2(x)+n_1^2d_{G_2}^2(u)+d_{G_1}^2(x)d_{G_2}^2(u)+2n_1n_2d_{G_1}(x)d_{G_2}(u)
\end{aligned}
$$

$$
\begin{aligned}
&- 2n_2 d^2_{G_1}(x) d_{G_2}(u) - 2n_1 d^2_{G_2}(u) d_{G_2}(u) + n_2^2 d^2_{G_1}(y) + n_1^2 d^2_{G_2}(v) \\
&+ d^2_{G_1}(y) d^2_{G_2}(v) + 2n_1 n_2 d_{G_1}(y) d_{G_2}(v) - 2n_2 d^2_{G_1}(y) d_{G_2}(v) - 2n_1 d^2_{G_2}(v) d_{G_1}(y) \\
&+ 2\Big\{ n_2^2 d_{G_1}(x) d_{G_1}(y) + n_1 n_2 d_{G_1}(x) d_{G_1}(v) - n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) \\
&+ n_1 n_2 d_{G_1}(y) d_{G_2}(u) + n_1^2 d_{G_2}(u) d_{G_2}(v) - n_1 d_{G_2}(u) d_{G_1}(y) d_{G_2}(v) \\
&- n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) - n_1 d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) \\
&+ d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v) \Big\} \Big] \\
= {}& n_2^2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d^2_{G_1}(x) + d^2_{G_1}(y)] + n_1^2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d^2_{G_1}(x) d^2_{G_2}(u) + d^2_{G_1}(y) d^2_{G_2}(v)] \\
&+ 2n_1 n_2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}(u) + d_{G_1}(y) d_{G_2}(v)] \\
&- 2n_2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d^2_{G_1}(x) d_{G_2}(u) + d^2_{G_1}(y) d_{G_2}(v)] \\
&- 2n_1 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d^2_{G_2}(u) + d_{G_1}(y) d^2_{G_2}(v)] \\
&+ 2n_2^2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_1}(y)] + 2n_1^2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_2}(x) d_{G_2}(v)] \\
&+ 2n_1 n_2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u)] \\
&- 2n_2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) + d_{G_1}(x) d_{G_1}(y) d_{G_2}(u)] \\
&- 2n_1 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)] \\
&+ 2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)] \\
= {}& n_2^2 \sum_{xy \notin G_1} [d^2_{G_1}(x) + d^2_{G_1}(y)] \sum_{uv \notin G_2} 1 + n_1^2 \sum_{xy \notin G_1} 1 \sum_{uv \notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ 2 \sum_{xy \notin G_1} d^2_{G_1}(x) \sum_{uv \notin G_2} d^2_{G_2}(u) + 2n_1 n_2 \sum_{xy \notin G_1} d_{G_1}(x) \sum_{uv \notin G_2} d_{G_2}(u) \\
&+ \sum_{xy \notin G_1} d_{G_1}(y) \sum_{uv \notin G_2} d_{G_2}(v) - 2n_2 \Big[\sum_{xy \notin G_1} d^2_{G_1}(x) \sum_{uv \notin G_2} d_{G_2}(u) \\
&+ \sum_{xy \notin G_1} d^2_{G_1}(y) \sum_{uv \notin G_2} d_{G_2}(v) \Big] - 2n_1 \Big[\sum_{xy \notin G_1} d_{G_1}(x) \sum_{uv \notin G_2} d^2_{G_2}(u)
\end{aligned}
$$

$$+\sum_{xy\notin G_1} d_{G_1}(y)\sum_{uv\notin G_2} d^2_{G_2}(v)\Big]+2n_2^2\sum_{xy\notin G_1}[d_{G_1}(x)d_{G_1}(y)]\sum_{uv\notin G_2}1$$
$$+2n_1^2\sum_{xy\notin G_1}1\sum_{uv\notin G_2}[d_{G_2}(u)d_{G_2}(v)]+2n_1n_2\Big[\sum_{xy\notin G_1}d_{G_1}(x)\sum_{uv\notin G_2}d_{G_2}(v)$$
$$+\sum_{xy\notin G_1}d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(u)\Big]-2n_2\Big[\sum_{xy\notin G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(u)$$
$$+\sum_{xy\notin G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(v)\Big]$$
$$-2n_1\Big[\sum_{xy\notin G_1}d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(u)d_{G_2}(v)$$
$$+\sum_{xy\notin G_1}d_{G_1}(x)\sum_{uv\notin G_2}d_{G_2}(u)d_{G_2}(v)\Big]$$
$$+2\sum_{xy\notin G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(v)d_{G_2}(u)$$
$$=n_2^2 2\Big[2\Big(2\overline{m}_1(n_1-1)^2+F(\overline{G}_1)-2(n_1-1)M_1(\overline{G}_1)+M_1(G_1)\Big)(2\overline{m}_2+n_2)\Big]$$
$$+n_1^2 2\Big[2\Big(2\overline{m}_2(n_2-1)^2+F(\overline{G}_2)-2(n_2-1)M_1(\overline{G}_2)+M_1(G_2)\Big)(2\overline{m}_1+n_1)\Big]$$
$$+2\Big[2\overline{m}_1(n_1-1)^2+F(\overline{G}_1)-2(n_1-1)M_1(\overline{G}_1)+M_1(G_1)\Big]$$
$$\Big[2\overline{m}_2(n_2-1)^2+F(\overline{G}_2)-2(n_2-1)M_1(\overline{G}_2))+M_1(G_2)\Big]$$
$$+2n_1n_2\Big\{2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\Big\}$$
$$-2n_2\Big\{2\Big[2\overline{m}_1(n_1-1)^2+F(\overline{G}_1)-2(n_1-1)M_1(\overline{G}_1)+M_1(G_1)\Big]\Big[2\overline{m}_2(n_2-1)$$
$$+2m_2-M_1(\overline{G}_2)\Big]\Big\}-2n_1\Big\{2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]$$
$$\Big[2\overline{m}_2(n_2-1)^2+F(\overline{G}_2)-2(n_2-1)M_1(\overline{G}_2)+M_1(G_2)\Big]\Big\}$$
$$+2n_2^2[2\overline{M}_2(G_1)+M_1(G_1)](2\overline{m}_2+n_2)+2n_1^2(2\overline{m}_1+n_1)[2\overline{M}_2(G_2)+M_1(G_2)]$$
$$+2n_1n_2\Big\{2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\Big\}$$
$$-2n_2 2[2\overline{M}_2(G_1)+M_1(G_1)]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]$$
$$-2n_1 2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big][2\overline{M}_2(G_1)+M_1(G_1)]$$
$$+2[2\overline{M}_2(G_1)+M_1(G_1)][2\overline{M}_2(G_2)+M_1(G_2)]$$
$$=4n_2^2\Big[\Big(2\overline{m}_1(n_1-1)^2+F(\overline{G}_1)-2(n_1-1)M_1(\overline{G}_1)+M_1(G_1)\Big)(2\overline{m}_2+n_2)\Big]$$
$$+4n_1^2\Big[\Big(2\overline{m}_2(n_2-1)^2+F(\overline{G}_2)-2(n_2-1)M_1(\overline{G}_2)+M_1(G_2)\Big)(2\overline{m}_1+n_1)\Big]$$

$$
\begin{aligned}
&+ 2\Big[2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)\Big] \\
&\quad \Big[2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2)) + M_1(G_2)\Big] \\
&+ 4n_1n_2\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
&- 4n_2\Big[2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)\Big] \\
&\quad \Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] - 4n_1\Big[2\overline{m}_1(n_1-1) \\
&+ 2m_1 - M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)\Big] \\
&+ 8n_2^2[2\overline{M}_2(G_1) + M_1(G_1)](2\overline{m}_2 + n_2) \\
&+ 2n_1^2(2\overline{m}_1 + n_1)[2\overline{M}_2(G_2) + M_1(G_2)] + 4n_1n_2\Big[2\overline{m}_1(n_1-1) + 2m_1 \\
&- M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] - 4n_2[2\overline{M}_2(G_1) \\
&+ M_1(G_1)]\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] - 4n_1\Big[2\overline{m}_1(n_1-1) \\
&+ 2m_1 - M_1(\overline{G}_1)\Big][2\overline{M}_2(G_1) + M_1(G_1)] \\
&+ 2[2\overline{M}_2(G_1) + M_1(G_1)][2\overline{M}_2(G_2) + M_1(G_2)]
\end{aligned}
$$

$$
\begin{aligned}
S_6 &= 2\sum_{xy\notin G_1,\ x=y}\ \sum_{uv\notin G_2, u=v} [d_{G_1\vee G_2}(x,u) + d_{G_1\vee G_2}(y,v)]^2 \\
&= \sum_{xy\notin G_1,\ x=y}\ \sum_{uv\notin G_2, u=v} \Big[n_2^2 d_{G_1}^2(x) + n_1^2 d_{G_2}^2(u) + d_{G_1}^2(x) d_{G_2}^2(u) \\
&\quad + 2n_1n_2 d_{G_1}(x)d_{G_2}(u) - 2n_2 d_{G_1}^2(x)d_{G_2}(u) - 2n_1 d_{G_2}^2(u)d_{G_2}(u) + n_2^2 d_{G_1}^2(y) \\
&\quad + n_1^2 d_{G_2}^2(v) + d_{G_1}^2(y)d_{G_2}^2(v) + 2n_1n_2 d_{G_1}(y)d_{G_2}(v) - 2n_2 d_{G_1}^2(y)d_{G_2}(v) \\
&\quad - 2n_1 d_{G_2}^2(v)d_{G_1}(y) + 2\Big\{n_2^2 d_{G_1}(x)d_{G_1}(y) + n_1n_2 d_{G_1}(x)d_{G_1}(v) \\
&\quad - n_2 d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + n_1n_2 d_{G_1}(y)d_{G_2}(u) + n_1^2 d_{G_2}(u)d_{G_2}(v) \\
&\quad - n_1 d_{G_2}(u)d_{G_1}(y)d_{G_2}(v) - n_2 d_{G_1}(x)d_{G_1}(y)d_{G_2}(u) - n_1 d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) \\
&\quad + d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big\}\Big] \\
&= n_2^2 \sum_{xy\notin G_1,\ x=y}\ \sum_{uv\notin G_2, u=v} [d_{G_1}^2(x) + d_{G_1}^2(y)] \\
&\quad + n_1^2 \sum_{xy\notin G_1,\ x=y}\ \sum_{uv\notin G_2, u=v} [d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&\quad + \sum_{xy\notin G_1,\ x=y}\ \sum_{uv\notin G_2, u=v} [d_{G_1}^2(x)d_{G_2}^2(u) + d_{G_1}^2(y)d_{G_2}^2(v)]
\end{aligned}
$$

$$
\begin{aligned}
&+2n_1n_2 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d_{G_1}(x)d_{G_2}(u)+d_{G_1}(y)d_{G_2}(v)]\\
&-2n_2 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d^2_{G_1}(x)d_{G_2}(u)+d^2_{G_1}(y)d_{G_2}(v)]\\
&-2n_1 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d_{G_1}(x)d^2_{G_2}(u)+d_{G_1}(y)d^2_{G_2}(v)]\\
&+2n_2^2 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d_{G_1}(x)d_{G_1}(y)]\\
&+2n_1^2 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d_{G_2}(x)d_{G_2}(v)]\\
&+2n_1n_2 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d_{G_1}(x)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)]\\
&-2n_2 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)]\\
&-2n_1 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)]\\
&+2 \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2, u=v} [d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)]\\
=\,& n_2^2 \sum_{xy\notin G_1,\ x=y} [d^2_{G_1}(x)+d^2_{G_1}(y)] \sum_{uv\notin G_2, u=v} 1\\
&+n_1^2 \sum_{xy\notin G_1,\ x=y} 1 \sum_{uv\notin G_2, u=v} [d^2_{G_2}(u)+d^2_{G_2}(v)]\\
&+2 \sum_{xy\notin G_1,\ x=y} d^2_{G_1}(x) \sum_{uv\notin G_2, u=v} d^2_{G_2}(u)\\
&+2n_1n_2 \sum_{xy\notin G_1,\ x=y} d_{G_1}(x) \sum_{uv\notin G_2, u=v} d_{G_2}(u)\\
&+\sum_{xy\notin G_1,\ x=y} d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(v) - 2n_2\Big[\sum_{xy\notin G_1,\ x=y} d^2_{G_1}(x) \sum_{uv\notin G_2, u=v} d_{G_2}(u)\\
&+\sum_{xy\notin G_1,\ x=y} d^2_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(v)\Big]\\
&-2n_1\Big[\sum_{xy\notin G_1,\ x=y} d_{G_1}(x) \sum_{uv\notin G_2, u=v} d^2_{G_2}(u)\\
&+\sum_{xy\notin G_1,\ x=y} d_{G_1}(y) \sum_{uv\notin G_2, u=v} d^2_{G_2}(v)\Big]\\
&+2n_2^2 \sum_{xy\notin G_1,\ x=y} [d_{G_1}(x)d_{G_1}(y)] \sum_{uv\notin G_2, u=v} 1
\end{aligned}
$$

$$+ 2n_1^2 \sum_{xy \notin G_1,\ x=y} 1 \sum_{uv \notin G_2, u=v} [d_{G_2}(u) d_{G_2}(v)]$$
$$+ 2n_1 n_2 \Big[\sum_{xy \notin G_1,\ x=y} d_{G_1}(x) \sum_{uv \notin G_2, u=v} d_{G_2}(v)$$
$$+ \sum_{xy \notin G_1,\ x=y} d_{G_1}(y) \sum_{uv \notin G_2, u=v} d_{G_2}(u) \Big]$$
$$- 2n_2 \Big[\sum_{xy \notin G_1,\ x=y} d_{G_1}(x) d_{G_1}(y) \sum_{uv \notin G_2, u=v} d_{G_2}(u)$$
$$+ \sum_{xy \notin G_1,\ x=y} d_{G_1}(x) d_{G_1}(y) \sum_{uv \notin G_2, u=v} d_{G_2}(v) \Big]$$
$$- 2n_1 \Big[\sum_{xy \notin G_1,\ x=y} d_{G_1}(y) \sum_{uv \notin G_2, u=v} d_{G_2}(u) d_{G_2}(v)$$
$$+ \sum_{xy \notin G_1,\ x=y} d_{G_1}(x) \sum_{uv \notin G_2, u=v} d_{G_2}(u) d_{G_2}(v) \Big]$$
$$+ 2 \sum_{xy \notin G_1,\ x=y} d_{G_1}(x) d_{G_1}(y) \sum_{uv \notin G_2, u=v} d_{G_2}(v) d_{G_2}(u)$$
$$= 2n_2^3 M_1(G_1) + 2n_1^3 M_1(G_2) + 4M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2 + 2n_2^3 M_1(G_1)$$
$$+ 2n_1^3 M_1(G_2) - 16m_2n_2M_1(G_1) - 16n_1m_1M_1(G_2)$$
$$= 4n_2^3 M_1(G_1) + 4n_1^3 M_1(G_2) + 4M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2$$
$$- 16m_2n_2M_1(G_1) - 16n_1m_1M_1(G_2).$$

By substituting S_1, S_2 S_3 S_5 and S_6, the desired result follows from the simplication. □

THEOREM 2.4. *Let $G_i, i = 1, 2,$ be a (n_i, m_i)- graph, put $\overline{m}_i = e(\overline{G}_i)$. Then* $2 \times DHM(G_1 \oplus G_2)$

$$= [2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)][4m_2n_2^2 + 8F(G_2)$$
$$- 8n_2M_1(G_2)] + [2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)][8n_1n_2M_1(G_2) - 8n_1F(G_2)$$
$$+ 16n_1M_2(G_2)] + [2M_2(\overline{G}_1) + M_1(G_1)][4m_2n_2^2 - 8n_2M_1(G_2) + 16M_2(G_2)]$$
$$+ (2\overline{m}_1 + n_1)[2n_1^2F(G_2) + 4n_1^2M_2(G_2)] + [2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2)$$
$$- 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)][4m_1n_1^2 + 8F(G_1) - 8n_1M_1(G_1)] + [2\overline{m}_2(n_2-1)$$
$$+ 2m_2 - M_1(\overline{G}_2)][8n_1n_2M_1(G_1) - 8n_2F(G_1) - 16n_2M_2(G_1)] + [2M_2(\overline{G}_2)$$
$$+ M_1(G_2)][4m_1n_1^2 - 8n_1M_1(G_1) + 16M_2(G_1)] + (2\overline{m}_2 + n_2)[2n_2^2F(G_1)$$
$$+ 4n_2^2M_2(G_1)] + 4n_2^2F(G_1)m_2 + 4n_1^2F(G_2)m_1 + 8F(G_1)F(G_2)$$
$$+ 4n_1n_2M_1(G_1)M_1(G_2) - 8n_2F(G_1)M_1(G_2) - 8n_1M_1(G_1)F(G_2)$$
$$+ 8n_2^2M_2(G_1)2m_2 + 8n_1^2M_2(G_2)m_1 + 4n_1n_2M_1(G_1)2M_1(G_2)$$
$$- 16n_2M_2(G_1)M_1(G_2) - 16n_1M_2(G_2)M_1(G_1) + 32M_2(G_1)M_2(G_2)$$

$$+ 4n_2^2\Big[\Big(2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)\Big)(2\overline{m}_2 + n_2)\Big]$$
$$+ 4n_1^2\Big[\Big(2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)\Big)(2\overline{m}_1 + n_1)\Big]$$
$$+ 8\Big[2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)\Big]\Big[2\overline{m}_2(n_2-1)^2$$
$$+ F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2)) + M_1(G_2)\Big] + 4n_1n_2\Big[2\overline{m}_1(n_1-1) + 2m_1$$
$$- M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] - 8n_2\Big[2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1)$$
$$- 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)\Big]\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big]$$
$$- 8n_1\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2)$$
$$- 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)\Big] + 2n_2^2[2\overline{M}_2(G_1) + M_1(G_1)](2\overline{m}_2 + n_2)$$
$$+ 2n_1^2(2\overline{m}_1 + n_1)[2\overline{M}_2(G_2) + M_1(G_2)] + 4n_1n_2\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big]$$
$$\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] - 8n_2[2\overline{M}_2(G_1) + M_1(G_1)]\Big[2\overline{m}_2(n_2-1)$$
$$+ 2m_2 - M_1(\overline{G}_2)\Big] - 8n_1\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big][2\overline{M}_2(G_1) + M_1(G_1)]$$
$$+ 16[2\overline{M}_2(G_1) + M_1(G_1)][2\overline{M}_2(G_2) + M_1(G_2)] + 4n_2^3M_1(G_1) + 4n_1^3M_1(G_2)$$
$$+ 16M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2 - 32m_2n_2M_1(G_1) - 32n_1m_1M_1(G_2)$$

Proof. Let $G = G_1 \oplus G_2$. Then,

$$2\times DHM(G_1 \oplus G_2)$$
$$= \sum_{x,y\in V(G_1)} \sum_{u,v\in V(G_2)} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2$$
$$= \sum_{x,y\in V(G_1)} \Big\{\sum_{uv\in G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2$$
$$+ \sum_{uv\notin G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2\Big\}$$
$$= \sum_{x,y\in V(G_1)} \sum_{uv\in G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2$$
$$+ \sum_{x,y\in V(G_1)} \sum_{uv\notin G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2$$
$$= \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2$$
$$+ \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2$$
$$+ \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2$$

$$+\sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2$$
$$= B_3 + B_1 + B_2 + B_4$$

$$\begin{aligned}
B_1 &= \sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2} \Big[d^2(x,u)+d^2(y,v)+2d(x,u)d(y,v)\Big]\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2} \Big[\{n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\}^2\\
&\quad + \{n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\}^2\\
&\quad + 2\{n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\}\\
&\qquad \{n_2 d_{G_1}(y)+n_1 d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\}\Big]\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2} \Big[n_2^2 d_{G_1}^2(x)+n_1^2 d_{G_2}^2(u)+4d_{G_1}^2(x)d_{G_2}^2(u)+2n_1n_2 d_{G_1}(x)d_{G_2}(u)\\
&\quad - 4n_2 d_{G_1}^2(x)d_{G_2}(u)-4n_1 d_{G_2}^2(u)d_{G_2}(u)+n_2^2 d_{G_1}^2(y)+n_1^2 d_{G_2}^2(v)\\
&\quad + 4d_{G_1}^2(y)d_{G_2}^2(v)+2n_1n_2 d_{G_1}(y)d_{G_2}(v)-4n_2 d_{G_1}^2(y)d_{G_2}(v)\\
&\quad - 4n_1 d_{G_2}^2(v)d_{G_1}(y)+2\Big\{n_2^2 d_{G_1}(x)d_{G_1}(y)+n_1n_2 d_{G_1}(x)d_{G_1}(v)\\
&\quad - 2n_2 d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+n_1n_2 d_{G_1}(y)d_{G_2}(u)+n_1^2 d_{G_2}(u)d_{G_2}(v)\\
&\quad - 2n_1 d_{G_2}(u)d_{G_1}(y)d_{G_2}(v)-2n_2 d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&\quad - 2n_1 d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+4d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big\}\Big]\\
&= n_2^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}^2(x)+d_{G_1}^2(y)]+n_1^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_2}^2(u)+d_{G_2}^2(v)]\\
&\quad + 4\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}^2(x)d_{G_2}^2(u)+d_{G_1}^2(y)d_{G_2}^2(v)]\\
&\quad + 2n_1n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(u)+d_{G_1}(y)d_{G_2}(v)]\\
&\quad - 4n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}^2(x)d_{G_2}(u)+d_{G_1}^2(y)d_{G_2}(v)]\\
&\quad - 4n_1\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}^2(u)+d_{G_1}(y)d_{G_2}^2(v)]\\
&\quad + 2n_2^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_1}(y)]+2n_1^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_2}(x)d_{G_2}(v)]
\end{aligned}$$

$$
\begin{aligned}
&+ 2n_1n_2 \sum_{xy\notin G_1} \sum_{uv\in G_2} [d_{G_1}(x)d_{G_2}(v) + d_{G_1}(y)d_{G_2}(u)] \\
&- 4n_2 \sum_{xy\notin G_1} \sum_{uv\in G_2} [d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)] \\
&- 4n_1 \sum_{xy\notin G_1} \sum_{uv\in G_2} [d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) + d_{G_1}(y)d_{G_2}(v)d_{G_2}(u)] \\
&+ 8 \sum_{xy\notin G_1} \sum_{uv\in G_2} [d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)d_{G_2}(u)] \\
= {} & n_2^2 \sum_{xy\notin G_1} [d^2_{G_1}(x) + d^2_{G_1}(y)] \sum_{uv\in G_2} 1 + n_1^2 \sum_{xy\notin G_1} 1 \sum_{uv\in G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ 4 \sum_{xy\notin G_1} [d^2_{G_1}(x) + d^2_{G_1}(y)] \sum_{uv\in G_2} d^2_{G_2}(u) \\
&+ 2n_1n_2 \sum_{xy\notin G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(u) \\
&- 4n_2 \sum_{xy\notin G_1} [d^2_{G_1}(x) + d^2_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(u) \\
&- 4n_1 \sum_{xy\notin G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d^2_{G_2}(u) \\
&+ 2n_2^2 \sum_{xy\notin G_1} [d_{G_1}(x)d_{G_1}(y)] \sum_{uv\in G_2} 1 + 2n_1^2 \sum_{xy\notin G_1} 1 \sum_{uv\in G_2} [d_{G_2}(u)d_{G_2}(v)] \\
&+ 2n_1n_2 \sum_{xy\notin G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(u) \\
&- 4n_2 \sum_{xy\notin G_1} [d_{G_1}(x)d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(u) \\
&- 4n_1 \sum_{xy\notin G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{uv\in G_2} d_{G_2}(v)d_{G_2}(u) \\
&+ 8 \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(v)d_{G_2}(u) \\
= {} & n_2^2 2[2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)](2m_2) \\
&+ n_1^2(2\overline{m}_1 + n_1)2F(G_2) + 4\times 2[2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) \\
&+ M_1(G_1)]F(G_2) + 2n_1n_2 \times 2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]M_1(G_2) \\
&- 4n_2 \times 2[2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)]M_1(G_2) \\
&- 4n_1 \times 2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]F(G_2) \\
&+ 2n_2^2[2\overline{M}_2(G_1) + M_1(G_1)]2m_2 + 2n_1^2(2\overline{m}_1 + n_1)(2M_2(G_2)) \\
&+ 2n_1n_2 \times 2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]M_1(G_2) - 4n_2 \times 2[2\overline{M}_2(G_1) \\
&+ M_1(G_1)]M_1(G_2) - 4n_1 \times 2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)](2M_2(G_2))
\end{aligned}
$$

$$
\begin{aligned}
&+ 8[2\overline{M}_2(G_1) + M_1(G_1)]2M_2(G_2)\\
&= [2(n_1-1)^2\overline{m}_1 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1)][4m_2n_2^2 + 8F(G_2)\\
&- 8n_2M_1(G_2)] + [2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)][8n_1n_2M_1(G_2) - 8n_1F(G_2)\\
&+ 16n_1M_2(G_2)] + [2M_2(\overline{G}_1) + M_1(G_1)][4m_2n_2^2 - 8n_2M_1(G_2) + 16M_2(G_2)]\\
&+ (2\overline{m}_1 + n_1)[2n_1^2F(G_2) + 4n_1^2M_2(G_2)]
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= \sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2} [d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2} \Big[\{n_2d_{G_1}(x) + n_1d_{G_2}(u) - 2d_{G_1}(x)d_{G_2}(u)\}^2\\
&+ \{n_2d_{G_1}(y) + n_1d_{G_2}(v) - 2d_{G_1}(y)d_{G_2}(v)\}^2\\
&+ 2\{n_2d_{G_1}(x) + n_1d_{G_2}(u) - 2d_{G_1}(x)d_{G_2}(u)\}\\
&\quad \{n_2d_{G_1}(y) + n_1d_{G_2}(v) - 2d_{G_1}(y)d_{G_2}(v)\}\Big]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2} \Big[n_2^2d_{G_1}^2(x) + n_1^2d_{G_2}^2(u) + 4d_{G_1}^2(x)d_{G_2}^2(u) + 2n_1n_2d_{G_1}(x)d_{G_2}(u)\\
&- 4n_2d_{G_1}^2(x)d_{G_2}(u) - 4n_1d_{G_2}^2(u)d_{G_2}(u) + n_2^2d_{G_1}^2(y) + n_1^2d_{G_2}^2(v)\\
&+ 4d_{G_1}^2(y)d_{G_2}^2(v) + 2n_1n_2d_{G_1}(y)d_{G_2}(v) - 4n_2d_{G_1}^2(y)d_{G_2}(v)\\
&- 4n_1d_{G_2}^2(v)d_{G_1}(y) + 2\Big\{n_2^2d_{G_1}(x)d_{G_1}(y) + n_1n_2d_{G_1}(x)d_{G_1}(v)\\
&- 2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + n_1n_2d_{G_1}(y)d_{G_2}(u) + n_1^2d_{G_2}(u)d_{G_2}(v)\\
&- 2n_1d_{G_2}(u)d_{G_1}(y)d_{G_2}(v) - 2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- 2n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) + 4d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big\}\Big]\\
&= n_2^2\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}^2(x) + d_{G_1}^2(y)] + n_1^2\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)]\\
&+ 4\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}^2(x)d_{G_2}^2(u) + d_{G_1}^2(y)d_{G_2}^2(v)]\\
&+ 2n_1n_2\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)]\\
&- 4n_2\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}^2(x)d_{G_2}(u) + d_{G_1}^2(y)d_{G_2}(v)]\\
&- 4n_1\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}(x)d_{G_2}^2(u) + d_{G_1}(y)d_{G_2}^2(v)]\\
&+ 2n_2^2\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}(x)d_{G_1}(y)] + 2n_1^2\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_2}(x)d_{G_2}(v)]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2n_1n_2 \sum_{xy\in G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_2}(v) + d_{G_1}(y)d_{G_2}(u)] \\
&- 4n_2 \sum_{xy\in G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)] \\
&- 4n_1 \sum_{xy\in G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) + d_{G_1}(y)d_{G_2}(v)d_{G_2}(u)] \\
&+ 8 \sum_{xy\in G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)] \\
= {}& n_2^2 \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \sum_{xy\in G_1} 1 + n_1^2 \sum_{uv\notin G_2} 1 \sum_{xy\in G_1} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ 4 \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \sum_{xy\in G_1} d^2_{G_1}(x) \\
&+ 2n_1n_2 \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 4n_2 \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 4n_1 \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy\in G_1} d^2_{G_1}(x) \\
&+ 2n_2^2 \sum_{uv\notin G_2} [d_{G_2}(u)d_{G_2}(v)] \sum_{xy\in G_1} 1 + 2n_1^2 \sum_{uv\notin G_2} 1 \sum_{xy\in G_1} [d_{G_2}(u)d_{G_2}(v)] \\
&+ 2n_1n_2 \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 4n_2 \sum_{uv\notin G_2} [d_{G_2}(u)d_{G_2}(v)] \sum_{xy\in G_1} d_{G_1}(x) \\
&- 4n_1 \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) \sum_{xy\in G_1} [d_{G_1}(x) + d_{G_1}(y)] \\
&+ 8 \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y) \\
= {}& n_2^2 2[2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)](2m_1) \\
&+ n_2^2(2\overline{m}_2 + n_2)2F(G_1) + 4\times 2[2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) \\
&+ M_1(G_2)]F(G_1) + 2n_1n_2 \times 2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]M_1(G_1) \\
&- 4n_2 \times 2[2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)]M_1(G_1) \\
&- 24n_1 \times 2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]F(G_1) \\
&+ 2n_2^2[2\overline{M}_2(G_2) + M_1(G_2)]2m_1 + 2n_1^2(2\overline{m}_2 + n_2)(2M_2(G_1)) \\
&+ 2n_1n_2 \times 2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]M_1(G_1) - 4n_2 \times 2[2\overline{M}_2(G_2) \\
&+ M_1(G_2)]M_1(G_1) - 4n_1 \times 2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)](2M_2(G_1))
\end{aligned}
$$

$$
\begin{aligned}
&+ 8[2\overline{M}_2(G_2) + M_1(G_2)]2M_2(G_1)\\
&= [2(n_2-1)^2\overline{m}_2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2)][4m_1n_1^2 + 8F(G_1)\\
&- 8n_1M_1(G_1)] + [2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)][8n_1n_2M_1(G_1) - 8n_2F(G_1)\\
&- 16n_2M_2(G_1) + [2M_2(\overline{G}_2) + M_1(G_2)][4m_1n_1^2 - 8n_1M_1(G_1) + 16M_2(G_1)]\\
&+ (2\overline{m}_2 + n_2)[2n_2^2F(G_1) + 4n_2^2M_2(G_1)]
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= \sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2\\
&= \sum_{xy\in G_1}\sum_{uv\in G_2} [d_{G_1\oplus G_2}(x,u) + d_{G_1\oplus G_2}(y,v)]^2\\
&= \sum_{xy\in G_1}\sum_{uv\in G_2} \Big[\{n_2d_{G_1}(x) + n_1d_{G_2}(u) - 2d_{G_1}(x)d_{G_2}(u)\}^2\\
&+ \{n_2d_{G_1}(y) + n_1d_{G_2}(v) - 2d_{G_1}(y)d_{G_2}(v)\}^2\\
&+ 2\{n_2d_{G_1}(x) + n_1d_{G_2}(u) - 2d_{G_1}(x)d_{G_2}(u)\}\\
&\quad \{n_2d_{G_1}(y) + n_1d_{G_2}(v) - 2d_{G_1}(y)d_{G_2}(v)\}\Big]\\
&= \sum_{xy\in G_1}\sum_{uv\in G_2} \Big[n_2^2d_{G_1}^2(x) + n_1^2d_{G_2}^2(u) + 4d_{G_1}^2(x)d_{G_2}^2(u) + 2n_1n_2d_{G_1}(x)d_{G_2}(u)\\
&- 4n_2d_{G_1}^2(x)d_{G_2}(u) - 4n_1d_{G_2}^2(u)d_{G_2}(u) + n_2^2d_{G_1}^2(y) + n_1^2d_{G_2}^2(v)\\
&+ 4d_{G_1}^2(y)d_{G_2}^2(v) + 2n_1n_2d_{G_1}(y)d_{G_2}(v) - 4n_2d_{G_1}^2(y)d_{G_2}(v)\\
&- 4n_1d_{G_2}^2(v)d_{G_1}(y) + 2\Big\{n_2^2d_{G_1}(x)d_{G_1}(y) + n_1n_2d_{G_1}(x)d_{G_1}(v)\\
&- 2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + n_1n_2d_{G_1}(y)d_{G_2}(u) + n_1^2d_{G_2}(u)d_{G_2}(v)\\
&- 2n_1d_{G_2}(u)d_{G_1}(y)d_{G_2}(v) - 2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- 2n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) + 4d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big\}\Big]\\
&= n_2^2\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}^2(x) + d_{G_1}^2(y)] + n_1^2\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)]\\
&+ 4\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}^2(x)d_{G_2}^2(u) + d_{G_1}^2(y)d_{G_2}^2(v)]\\
&+ 2n_1n_2\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)]\\
&- 4n_2\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}^2(x)d_{G_2}(u) + d_{G_1}^2(y)d_{G_2}(v)]\\
&- 4n_1\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}^2(u) + d_{G_1}(y)d_{G_2}^2(v)]\\
&+ 2n_2^2\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_1}(y)] + 2n_1^2\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_2}(x)d_{G_2}(v)]
\end{aligned}
$$

$$
\begin{aligned}
&+2n_1n_2\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)]\\
&-4n_2\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)]\\
&-4n_1\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)]\\
&+8\sum_{xy\in G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)]\\
=\;&n_2^2\sum_{uv\in G_2}[d^2_{G_2}(u)+d^2_{G_2}(v)]\sum_{xy\in G_1}1+n_1^2\sum_{uv\in G_2}1\sum_{xy\in G_1}[d^2_{G_2}(u)+d^2_{G_2}(v)]\\
&+4\sum_{uv\in G_2}[d^2_{G_2}(u)+d^2_{G_2}(v)]\sum_{xy\in G_1}d^2_{G_1}(x)\\
&+2n_1n_2\sum_{uv\in G_2}[d_{G_2}(u)+d_{G_2}(v)]\sum_{xy\in G_1}d_{G_1}(x)\\
&-4n_2\sum_{uv\in G_2}[d^2_{G_2}(u)+d^2_{G_2}(v)]\sum_{xy\in G_1}d_{G_1}(x)\\
&-4n_1\sum_{uv\in G_2}[d_{G_2}(u)+d_{G_2}(v)]\sum_{xy\in G_1}d^2_{G_1}(x)\\
&+2n_2^2\sum_{uv\in G_2}[d_{G_2}(u)d_{G_2}(v)]\sum_{xy\in G_1}1+2n_1^2\sum_{uv\in G_2}1\sum_{xy\in G_1}[d_{G_2}(u)d_{G_2}(v)]\\
&+2n_1n_2\sum_{uv\in G_2}[d_{G_2}(u)+d_{G_2}(v)]\sum_{xy\in G_1}d_{G_1}(x)\\
&-4n_2\sum_{uv\in G_2}[d_{G_2}(u)d_{G_2}(v)]\sum_{xy\in G_1}d_{G_1}(x)\\
&-4n_1\sum_{uv\in G_2}d_{G_2}(u)d_{G_2}(v)\sum_{xy\in G_1}[d_{G_1}(x)+d_{G_1}(y)]\\
&+8\sum_{uv\in G_2}d_{G_2}(u)d_{G_2}(v)\sum_{xy\in G_1}d_{G_1}(x)d_{G_1}(y)\\
=\;&n_2^2 2F(G_1)(2m_2)+n_1^2 2F(G_2)(2m_1)+8F(G_1)F(G_2)\\
&+2n_1n_2M_1(G_1)2M_1(G_2)-4n_2F(G_1)2M_1(G_2)-4n_1M_1(G_1)2F(G_2)\\
&+2n_2^2(2M_2(G_1))(2m_2)+2n_1^2(2M_2(G_2))(2m_1+2n_1n_2M_1(G_1)2M_1(G_2)\\
&-4n_2(2M_2(G_1))2M_1(G_2)-4n_1(2M_2(G_2))2M_1(G_1)+8(2M_2(G_1))2M_2(G_2)\\
=\;&4n_2^2F(G_1)m_2+4n_1^2F(G_2)m_1+8F(G_1)F(G_2)+4n_1n_2M_1(G_1)M_1(G_2)\\
&-8n_2F(G_1)M_1(G_2)-8n_1M_1(G_1)F(G_2)+8n_2^2M_2(G_1)2m_2+8n_1^2M_2(G_2)m_1\\
&+4n_1n_2M_1(G_1)2M_1(G_2)-16n_2M_2(G_1)M_1(G_2)-16n_1M_2(G_2)M_1(G_1)\\
&+32M_2(G_1)M_2(G_2)
\end{aligned}
$$

$$
\begin{aligned}
B_4 &= \sum_{xy\notin G_1} \sum_{uv\notin G_2} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&= \sum_{xy\notin G_1} \Big\{ \sum_{uv\notin G_2\ u\neq v} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + \sum_{uv\notin G_2\ u=v} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \Big\} \\
&= \sum_{xy\notin G_1} \sum_{uv\notin G_2\ u\neq v} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + \sum_{xy\notin G_1} \sum_{uv\notin G_2\ u=v} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&= \sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u\neq v} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u\neq v} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + \sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u=v} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + \sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u=v} d_{G_1\oplus G_2}\big((x,u),(y,v)\big)[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&= 2\sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u\neq v} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u\neq v} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + 2\sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u=v} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 + 0 \\
&= 2\sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u\neq v} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u\neq v} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + 2\sum_{xy\notin G_1,\ x\neq y} \sum_{uv\notin G_2\ u=v} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad + 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u=v} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&\quad - 2\sum_{xy\notin G_1,\ x=y} \sum_{uv\notin G_2\ u=v} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2 \\
&= 2\sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2
\end{aligned}
$$

$$- 2 \sum_{xy \notin G_1,\ x=y} \sum_{uv \notin G_2, u=v} [d_{G_1 \oplus G_2}(x,u) + d_{G_1 \oplus G_2}(y,v)]^2$$

$$B_4 = 2B_5 - BS_6$$

$$\begin{aligned}
B_5 &= \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1 \oplus G_2}(x,u) + d_{G_1 \oplus G_2}(y,v)]^2 \\
&= \sum_{xy \notin G_1} \sum_{uv \notin G_2} \Big[\{n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - 2d_{G_1}(x) d_{G_2}(u)\}^2 \\
&+ \{n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - 2d_{G_1}(y) d_{G_2}(v)\}^2 \\
&+ 2\{n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - 2d_{G_1}(x) d_{G_2}(u)\} \\
&\{n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - 2d_{G_1}(y) d_{G_2}(v)\}\Big] \\
&= \sum_{xy \notin G_1} \sum_{uv \notin G_2} \Big[n_2^2 d_{G_1}^2(x) + n_1^2 d_{G_2}^2(u) + 4d_{G_1}^2(x) d_{G_2}^2(u) + 2n_1 n_2 d_{G_1}(x) d_{G_2}(u) \\
&- 4n_2 d_{G_1}^2(x) d_{G_2}(u) - 4n_1 d_{G_2}^2(u) d_{G_2}(u) + n_2^2 d_{G_1}^2(y) + n_1^2 d_{G_2}^2(v) \\
&+ 4d_{G_1}^2(y) d_{G_2}^2(v) + 2n_1 n_2 d_{G_1}(y) d_{G_2}(v) - 4n_2 d_{G_1}^2(y) d_{G_2}(v) \\
&- 4n_1 d_{G_2}^2(v) d_{G_1}(y) + 2\Big\{n_2^2 d_{G_1}(x) d_{G_1}(y) + n_1 n_2 d_{G_1}(x) d_{G_1}(v) \\
&- 2n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) + n_1 n_2 d_{G_1}(y) d_{G_2}(u) + n_1^2 d_{G_2}(u) d_{G_2}(v) \\
&- 2n_1 d_{G_2}(u) d_{G_1}(y) d_{G_2}(v) - 2n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) \\
&- 2n_1 d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) + 4d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)\Big\}\Big] \\
&= n_2^2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}^2(x) + d_{G_1}^2(y)] + n_1^2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&+ 4 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}^2(x) d_{G_2}^2(u) + d_{G_1}^2(y) d_{G_2}^2(v)] \\
&+ 2n_1 n_2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}(u) + d_{G_1}(y) d_{G_2}(v)] \\
&- 4n_2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}^2(x) d_{G_2}(u) + d_{G_1}^2(y) d_{G_2}(v)] \\
&- 4n_1 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}^2(u) + d_{G_1}(y) d_{G_2}^2(v)] \\
&+ 2n_2^2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_1}(y)] + 2n_1^2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_2}(x) d_{G_2}(v)] \\
&+ 2n_1 n_2 \sum_{xy \notin G_1} \sum_{uv \notin G_2} [d_{G_1}(x) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u)]
\end{aligned}$$

$$
\begin{aligned}
&- 4n_2 \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)] \\
&- 4n_1 \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) + d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)] \\
&+ 8 \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)] \\
=\ & n_2^2 \sum_{xy\notin G_1} [d^2_{G_1}(x) + d^2_{G_1}(y)] \sum_{uv\notin G_2} 1 + n_1^2 \sum_{xy\notin G_1} 1 \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ 4 \sum_{xy\notin G_1} d^2_{G_1}(x) \sum_{uv\notin G_2} d^2_{G_2}(u) \\
&+ 2n_1n_2 \sum_{xy\notin G_1} d_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(u) + \sum_{xy\notin G_1} d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(v) \\
&- 4n_2 \Big[\sum_{xy\notin G_1} d^2_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(u) + \sum_{xy\notin G_1} d^2_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(v) \Big] \\
&- 4n_1 \Big[\sum_{xy\notin G_1} d_{G_1}(x) \sum_{uv\notin G_2} d^2_{G_2}(u) + \sum_{xy\notin G_1} d_{G_1}(y) \sum_{uv\notin G_2} d^2_{G_2}(v) \Big] \\
&+ 2n_2^2 \sum_{xy\notin G_1} [d_{G_1}(x)d_{G_1}(y)] \sum_{uv\notin G_2} 1 + 2n_1^2 \sum_{xy\notin G_1} 1 \sum_{uv\notin G_2} [d_{G_2}(u)d_{G_2}(v)] \\
&+ 2n_1n_2 \Big[\sum_{xy\notin G_1} d_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(v) + \sum_{xy\notin G_1} d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(u) \Big] \\
&- 4n_2 \Big[\sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(u) \\
&+ \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(v) \Big] \\
&- 4n_1 \Big[\sum_{xy\notin G_1} d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) \\
&+ \sum_{xy\notin G_1} d_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) \Big] \\
&+ 8 \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(v)d_{G_2}(u) \\
=\ & n_2^2 2\Big[2\Big(2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1) \Big)(2\overline{m}_2 + n_2) \Big] \\
&+ n_1^2 2\Big[2\Big(2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2) + M_1(G_2) \Big)(2\overline{m}_1 + n_1) \Big] \\
&+ 8\Big[2\overline{m}_1(n_1-1)^2 + F(\overline{G}_1) - 2(n_1-1)M_1(\overline{G}_1) + M_1(G_1) \Big] \\
&\Big[2\overline{m}_2(n_2-1)^2 + F(\overline{G}_2) - 2(n_2-1)M_1(\overline{G}_2)) + M_1(G_2) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2n_1n_2\Big\{2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\Big\}\\
&- 4n_2\Big\{2\Big[2\overline{m}_1(n_1-1)^2+F(\overline{G}_1)-2(n_1-1)M_1(\overline{G}_1)+M_1(G_1)\Big]\\
&\quad\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\Big\} - 4n_1\Big\{2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]\\
&\quad\Big[2\overline{m}_2(n_2-1)^2+F(\overline{G}_2)-2(n_2-1)M_1(\overline{G}_2)+M_1(G_2)\Big]\Big\}\\
&+ 2n_2^2[2\overline{M}_2(G_1)+M_1(G_1)](2\overline{m}_2+n_2)+2n_1^2(2\overline{m}_1+n_1)[2\overline{M}_2(G_2)+M_1(G_2)]\\
&+ 2n_1n_2\Big\{2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\Big\}\\
&- 4n_2 2[2\overline{M}_2(G_1)+M_1(G_1)]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\\
&- 4n_1 2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big][2\overline{M}_2(G_1)+M_1(G_1)]\\
&+ 16[2\overline{M}_2(G_1)+M_1(G_1)][2\overline{M}_2(G_2)+M_1(G_2)]\\
=\ & 4n_2^2\Big[\Big(2\overline{m}_1(n_1-1)^2+F(\overline{G}_1)-2(n_1-1)M_1(\overline{G}_1)+M_1(G_1)\Big)(2\overline{m}_2+n_2)\Big]\\
&+ 4n_1^2\Big[\Big(2\overline{m}_2(n_2-1)^2+F(\overline{G}_2)-2(n_2-1)M_1(\overline{G}_2)+M_1(G_2)\Big)(2\overline{m}_1+n_1)\Big]\\
&+ 8\Big[2\overline{m}_1(n_1-1)^2+F(\overline{G}_1)-2(n_1-1)M_1(\overline{G}_1)+M_1(G_1)\Big]\\
&\quad\Big[2\overline{m}_2(n_2-1)^2+F(\overline{G}_2)-2(n_2-1)M_1(\overline{G}_2))+M_1(G_2)\Big]\\
&+ 4n_1n_2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\\
&- 8n_2\Big[2\overline{m}_1(n_1-1)^2+F(\overline{G}_1)-2(n_1-1)M_1(\overline{G}_1)+M_1(G_1)\Big]\\
&\quad\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\\
&- 8n_1\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]\\
&\Big[2\overline{m}_2(n_2-1)^2+F(\overline{G}_2)-2(n_2-1)M_1(\overline{G}_2)+M_1(G_2)\Big]\\
&+ 2n_2^2[2\overline{M}_2(G_1)+M_1(G_1)](2\overline{m}_2+n_2)+2n_1^2(2\overline{m}_1+n_1)[2\overline{M}_2(G_2)+M_1(G_2)]\\
&+ 4n_1n_2\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\\
&- 8n_2[2\overline{M}_2(G_1)+M_1(G_1)]\Big[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)\Big]\\
&- 8n_1\Big[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)\Big][2\overline{M}_2(G_1)+M_1(G_1)]\\
&+ 16[2\overline{M}_2(G_1)+M_1(G_1)][2\overline{M}_2(G_2)+M_1(G_2)]
\end{aligned}
$$

$$
B_6 = 2\sum_{xy\notin G_1,\ x=y}\ \sum_{uv\notin G_2, u=v}[d_{G_1\oplus G_2}(x,u)+d_{G_1\oplus G_2}(y,v)]^2
$$

$$
\begin{aligned}
&= \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} \Big[n_2^2 d_{G_1}^2(x) + n_1^2 d_{G_2}^2(u) + 4 d_{G_1}^2(x) d_{G_2}^2(u) \\
&\quad + 2n_1 n_2 d_{G_1}(x) d_{G_2}(u) - 4n_2 d_{G_1}^2(x) d_{G_2}(u) - 4n_1 d_{G_2}^2(u) d_{G_2}(u) + n_2^2 d_{G_1}^2(y) \\
&\quad + n_1^2 d_{G_2}^2(v) + 4 d_{G_1}^2(y) d_{G_2}^2(v) + 2n_1 n_2 d_{G_1}(y) d_{G_2}(v) - 4n_2 d_{G_1}^2(y) d_{G_2}(v) \\
&\quad - 4n_1 d_{G_2}^2(v) d_{G_1}(y) + 2 \Big\{ n_2^2 d_{G_1}(x) d_{G_1}(y) + n_1 n_2 d_{G_1}(x) d_{G_1}(v) \\
&\quad - 2n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) + n_1 n_2 d_{G_1}(y) d_{G_2}(u) + n_1^2 d_{G_2}(u) d_{G_2}(v) \\
&\quad - 2n_1 d_{G_2}(u) d_{G_1}(y) d_{G_2}(v) - 2n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) \\
&\quad - 2n_1 d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) + 4 d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v) \Big\} \Big] \\
&= n_2^2 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}^2(x) + d_{G_1}^2(y)] \\
&\quad + n_1^2 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&\quad + 4 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}^2(x) d_{G_2}^2(u) + d_{G_1}^2(y) d_{G_2}^2(v)] \\
&\quad + 2n_1 n_2 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}(x) d_{G_2}(u) + d_{G_1}(y) d_{G_2}(v)] \\
&\quad - 4n_2 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}^2(x) d_{G_2}(u) + d_{G_1}^2(y) d_{G_2}(v)] \\
&\quad - 4n_1 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}(x) d_{G_2}^2(u) + d_{G_1}(y) d_{G_2}^2(v)] \\
&\quad + 2n_2^2 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}(x) d_{G_1}(y)] \\
&\quad + 2n_1^2 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_2}(x) d_{G_2}(v)] \\
&\quad + 2n_1 n_2 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}(x) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u)] \\
&\quad - 4n_2 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) + d_{G_1}(x) d_{G_1}(y) d_{G_2}(u)] \\
&\quad - 4n_1 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) + d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)] \\
&\quad + 8 \sum_{xy \notin G_1,\, x=y} \; \sum_{uv \notin G_2, u=v} [d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) d_{G_2}(u)] \\
&= n_2^2 \sum_{xy \notin G_1,\, x=y} [d_{G_1}^2(x) + d_{G_1}^2(y)] \sum_{uv \notin G_2, u=v} 1
\end{aligned}
$$

$$
\begin{aligned}
&+ n_1^2 \sum_{xy\notin G_1,\ x=y} 1 \sum_{uv\notin G_2, u=v} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ 4 \sum_{xy\notin G_1,\ x=y} d^2_{G_1}(x) \sum_{uv\notin G_2, u=v} d^2_{G_2}(u) \\
&+ 2n_1n_2 \sum_{xy\notin G_1,\ x=y} d_{G_1}(x) \sum_{uv\notin G_2, u=v} d_{G_2}(u) \\
&+ \sum_{xy\notin G_1,\ x=y} d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(v) \\
&- 4n_2 \Big[\sum_{xy\notin G_1,\ x=y} d^2_{G_1}(x) \sum_{uv\notin G_2, u=v} d_{G_2}(u) \\
&+ \sum_{xy\notin G_1,\ x=y} d^2_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(v) \Big] \\
&- 4n_1 \Big[\sum_{xy\notin G_1,\ x=y} d_{G_1}(x) \sum_{uv\notin G_2, u=v} d^2_{G_2}(u) \\
&+ \sum_{xy\notin G_1,\ x=y} d_{G_1}(y) \sum_{uv\notin G_2, u=v} d^2_{G_2}(v) \Big] \\
&+ 2n_2^2 \sum_{xy\notin G_1,\ x=y} [d_{G_1}(x) d_{G_1}(y)] \sum_{uv\notin G_2, u=v} 1 \\
&+ 2n_1^2 \sum_{xy\notin G_1,\ x=y} 1 \sum_{uv\notin G_2, u=v} [d_{G_2}(u) d_{G_2}(v)] \\
&+ 2n_1n_2 \Big[\sum_{xy\notin G_1,\ x=y} d_{G_1}(x) \sum_{uv\notin G_2, u=v} d_{G_2}(v) \\
&+ \sum_{xy\notin G_1,\ x=y} d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(u) \Big] \\
&- 4n_2 \Big[\sum_{xy\notin G_1,\ x=y} d_{G_1}(x) d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(u) \\
&+ \sum_{xy\notin G_1,\ x=y} d_{G_1}(x) d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(v) \Big] \\
&- 4n_1 \Big[\sum_{xy\notin G_1,\ x=y} d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(u) d_{G_2}(v) \\
&+ \sum_{xy\notin G_1,\ x=y} d_{G_1}(x) \sum_{uv\notin G_2, u=v} d_{G_2}(u) d_{G_2}(v) \Big] \\
&+ 8 \sum_{xy\notin G_1,\ x=y} d_{G_1}(x) d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(v) d_{G_2}(u) \\
&= 2n_2^3 M_1(G_1) + 2n_1^3 M_1(G_2) + 8M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2 + 2n_2^3 M_1(G_1) \\
&+ 2n_1^3 M_1(G_2) - 32m_2n_2M_1(G_1) - 32n_1m_1M_1(G_2) + 8M_1(G_1)M_1(G_2)
\end{aligned}
$$

$$= 4n_2^3M_1(G_1) + 4n_1^3M_1(G_2) + 16M_1(G_1)M_1(G_2) + 32n_1n_2m_1m_2$$
$$- 32m_2n_2M_1(G_1) - 32n_1m_1M_1(G_2).$$

$2 \times DHM(G_1 \oplus G_2) = B_1 + B_2 + B_3 + B_4 = B_1 + B_2 + B_3 + 2B_5 - 2B_6$. By substituting B_1, B_2, B_3, B_5 and B_6, we get the desired result. □

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M. Aruvi: Department of Mathematics, Anna University, Tiruchirappalli, India
E-mail address: aruvi.aut@gmail.com

V. Piramanantham: Department of Mathematics, Bharathidasan University, Tiruchirappalli, India
E-mail address: piramanantham@gmail.com

R. Muruganandam: Department of Mathematics, Government Arts College, Tiruchirappalli, India
E-mail address: muruganraju1980@gmail.com