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NUMERICAL SOLUTION OF FRACTIONAL-ORDER SIR EPIDEMIC MODEL VIA JACOBI WAVELETS

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Abstract. The mathematical model of the spread of a non-fatal disease in a population named SIR epidemic model is considered as a system of non-linear fractional differential equations. In this manuscript, Jacobi wavelets are first constructed and then uniform convergence of them together with error analysis is investigated. By using an operational matrix of fractional integration and with the aid of collocation points, a scheme is proposed which transforms the main problem to a system of algebraic equations. Finally, numerical results of applying the presented method are compared with other methods.

1. Introduction

Computational models help biologists to discover the behaviour of diseases and viruses in the human body. They describe the connections of a biological system components with together in the mathematical view. It is worth mentioning that most biological systems have memory and in the models using ordinary differential equations with integer-order, such effects are neglected. Because of the relation of fractional calculus to the systems with memory, the models formulated as fractional-order differential equations reveal more properties of biological systems. However, many of these fractional models are non-linear and so analytical solutions can not be determined easily. Thus, the numerical solution of biological processes is a responsibility for mathematicians.

One of the most important biological systems is SIR epidemic model that measures the changes of susceptible, infected and recovered individuals numbers

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in a population. Here, we consider the following SIR epidemic model involving Caputo fractional derivatives [**1**, **15**]

$$\begin{cases} D_*^{\nu_1} S(t) = -a_1 S(t) I(t), \\ D_*^{\nu_2} I(t) = a_1 S(t) I(t) - a_2 I(t), \\ D_*^{\nu_3} R(t) = a_2 I(t), \end{cases} \tag{1.1}$$

with the initial conditions

$$S(0) = S_0, \;\; I(0) = I_0, \;\; R(0) = R_0, \tag{1.2}$$

in which $D_*^{\nu_i}$ denotes to Caputo derivative with order $0 < \nu_i \leqslant 1, \; i = 1, 2, 3$. The concepts of variables and parameters of the model (1.1) are as follows.

- $S(t)$ is the number of individuals in the susceptible compartment S at the time t.
- $I(t)$ is the number of individuals in the infected compartment I at the time t.
- $R(t)$ is the number of individuals in the recoverd compartment R at the time t.
- a_1 is the rate of change of susceptibles to infective population.
- a_2 is the rate of change of infectives to immune population.

In the non-linear model (1.1), it has been assumed that total population remains constant, N, that is $S(t) + I(t) + R(t) = N$. It is also clear that for $\nu_1 = \nu_2 = \nu_3 = 1$, the model (1.1) reduces to conventional model formulated by A. G. McKendrick and W. O. Kermack [**6**].

Wavelets are a class of functions used to localize a given function in the dilation and translation [**10**]. They have been widely derived in mathematical researches and various fields of applied sciences. In the literature, there is a special consideration on families of the wavelets produced by orthogonal polynomials. These families include Legendre wavelets, first to fourth kinds of Chebyshev wavelets, Gegenbauer wavelets. For instance, some applications of the aforesaid wavelets related to fractional problems can be found in [**2**, **4**, **5**, **7**, **9**, **11**, **13**, **14**, **18**, **20**].

More recently, Jacobi wavelets have been constructed by utilising Jacobi polynomials and general definition of wavelet [**3**, **16**, **19**]. The main advantage of this family of wavelets is that other wavelets generated by orthogonal polynomials are special cases of it. However, there are fewer articles about these wavelets and their usages rather than other types of wavelets. Hence, we try to find some new properties of Jacobi wavelets and develop the applications of this family of wavelets.

In the next section, some preliminaries used further in this work are given. Section 3 is assigned to the structure of Jacobi wavelets. In this section, we also prove uniform convergence of the expansion written by the elements of this wavelet and find an upper bound for the error estimation. In section 4, a numerical method is suggested for solving the problem (1.1) under the initial values

(1.2). This method considers the approximate solutions as the Jacobi wavelets components with unknown coefficients. Then, with the aid of operational matrix of fractional integration and collocation points, the proposed method converts the problem (1.1) to a system of algebraic equations. By solving this algebraic system, the unknown coefficients are determined and thus the approximate solutions can be obtained. In section 5, numerical computations of the method will be offered. At the end, a conclusion is announced in the section 6.

2. Preliminaries

First, some basic definitions of fractional calculus are reviewed [**12**].

Definition 2.1. The Riemann-Liouville's fractional-order integration for the function f on $L^1[a,b]$ is defined as follows

$$I^{\nu} f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau, & \nu > 0, \\ f(t), & \nu = 0. \end{cases}$$

Definition 2.2. The Caputo's type derivative of order $\nu > 0$ is defined as

$$D_*^{\nu} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\nu-1} f^{(n)}(\tau) d\tau, \quad n-1 < \nu \leqslant n,$$

where $t > 0$ and n is an integer.

Remark 2.1. Caputo's integral operator for $f \in L^1[a,b]$ has the following useful property

$$I^{\nu} D_*^{\nu} f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}\left(0^+\right) \frac{t^i}{i!}, \quad n-1 < \nu \leqslant n.$$

3. Wavelets and Jacobi wavelets

In this section, using Jacobi polynomials, we construct Jacobi wavelets and declare some properties of this family of wavelets.

3.1. Jacobi polynomials. The Jacobi polynomials, $P_m^{\alpha,\beta}$, of the order $m \in \mathbb{N} \cup \{0\}$ are defined for $\alpha > -1$ and $\beta > -1$ on $[-1,1]$ and can be determined by the following recurrence formula [**17**]

$$P_m^{\alpha,\beta}(t) = \begin{cases} 1, & m = 0, \\ \left(\frac{\alpha+\beta+2}{2}\right) t + \frac{\alpha-\beta}{2}, & m = 1, \\ \left(a_{m-1}^{\alpha,\beta} t - b_{m-1}^{\alpha,\beta}\right) P_{m-1}^{\alpha,\beta}(t) - c_{m-1}^{\alpha,\beta} P_{m-2}^{\alpha,\beta}(t), & m > 1, \end{cases} \tag{3.1}$$

where

$$a_{m-1}^{\alpha,\beta} = \frac{(2m+\alpha+\beta-1)(2m+\alpha+\beta)}{2m(m+\alpha+\beta)},$$

$$b_{m-1}^{\alpha,\beta} = \frac{(\beta^2-\alpha^2)(2m+\alpha+\beta-1)}{2m(m+\alpha+\beta)(2m+\alpha+\beta-2)},$$

$$c_{m-1}^{\alpha,\beta} = \frac{(m+\alpha-1)(m+\beta-1)(2m+\alpha+\beta)}{m(m+\alpha+\beta)(2m+\alpha+\beta-2)}.$$

The Jacobi polynomials form an orthogonal basis of $L^2[-1,1]$ with respect to the weight function $\omega^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ such that

$$\int_{-1}^{1} \omega(t) P_m^{\alpha,\beta}(t) P_n^{\alpha,\beta}(t) dt = \begin{cases} 0, & m \neq n, \\ h_m^{\alpha,\beta}, & m = n, \end{cases} \tag{3.2}$$

in which

$$h_m^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)m!\Gamma(m+\alpha+\beta+1)}, \tag{3.3}$$

and Γ refers to the Gamma function.

3.2. Jacobi wavelets. The dilation parameter a and translation parameter b of a mother wavelet ψ define the continuous wavelets

$$\psi_{a,b}(t) = a^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a \in \mathbb{R}^+, \quad b \in \mathbb{R}. \tag{3.4}$$

Now, suppose $a_0 > 1$ and $b_0 > 0$ are fixed and take $a = a_0^{-k}$ and $b = na_0^{-k}b_0$ such that $k, n \in \mathbb{N}$. Instead of using the family of wavelets (3.4), we use the family of wavelets indexed by $\mathbb{N}$, named the discrete wavelets

$$\psi_{k,n}(t) = a_0^{\frac{k}{2}}\psi\left(a_0^k t - nb_0\right), \quad k, n \in \mathbb{N}. \tag{3.5}$$

The family of (3.5) constitutes an orthogonal basis for $L^2(\mathbb{R})$ and by choosing $a_0 = 2$ and $b_0 = 1$, (3.5) forms an orthonormal basis of $L^2(\mathbb{R})$.

For $\alpha > -1$ and $\beta > -1$, we can define Jacobi wavelets, $\psi_{n,m}^{\alpha,\beta}$, for $n = 1, \ldots, 2^{k-1}$, $(k \in \mathbb{N})$ and $m = 0, 1, \ldots$ on [0,1] in the following

$$\psi_{n,m}^{\alpha,\beta}(t) = \begin{cases} \frac{1}{\sqrt{h_m^{\alpha,\beta}}} 2^{\frac{k}{2}} P_m^{\alpha,\beta}\left(2^k t - 2n + 1\right), & \frac{n-1}{2^{k-1}} \leqslant t < \frac{n}{2^{k-1}}, \\ 0, & \textit{otherwise} \end{cases} \tag{3.6}$$

where $h_m^{\alpha,\beta}$ is defined in (3.3) and the coefficient $\frac{1}{\sqrt{h_m^{\alpha,\beta}}}$ is for normality. In (3.6), the dilation parameter is $a = 2^{-k}$, the translation parameter is $b = (2n-1)2^{-k}$ and $k \in \mathbb{N}$ is named the level of resolution. The Jacobi wavelets expressed in (3.6) constitute an orthonormal basis of $L^2[0,1]$ with respect to the weight function $\omega_n(t) = \omega\left(2^k t - 2n + 1\right)$.

3.3. Function approximation. Let $f : [0,1] \longrightarrow \mathbb{R}$ is a measurable function. For $\alpha > -1,\ \beta > -1$ and the weight function $\omega(t)$ the space $L^2_\omega[0,1]$ is defined in the following

$$L^2_\omega[0,1] = \left\{ f : \int_0^1 \omega(t)\, |f(x)|^2 dt < \infty \right\}.$$

We also define the inner product $\langle .,. \rangle_\omega$ as

$$\langle f, g \rangle_\omega = \int_0^1 \omega(t) f(t) g(t) dt.$$

The function $f(t)$ can be expanded at the level $k \in \mathbb{N}$ as

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} f_{n,m} \psi^{\alpha,\beta}_{n,m}(t), \tag{3.7}$$

where

$$f_{n,m} = \left\langle f(t), \psi^{\alpha,\beta}_{n,m}(t) \right\rangle_\omega = \int_0^1 f(t) \psi^{\alpha,\beta}_{n,m}(t) \omega_n(t) dt.$$

Usually, the infinite series in (3.7) is truncated and written in the following

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} f_{n,m} \psi^{\alpha,\beta}_{n,m}(t) = \mathbf{F}^T \mathbf{\Psi}^{\alpha,\beta}(t), \tag{3.8}$$

which approximates $f(t)$ as a finite linear combination of Jacobi wavelets. In (3.8), $\mathbf{F}$ and $\mathbf{\Psi}^{\alpha,\beta}(t)$ are column vectors with $2^{k-1}(M+1)$ entries given by

$$\mathbf{F} = \left[f_{1,0}, \dots, f_{1,M}, f_{2,0}, \dots, f_{2,M}, \dots, f_{2^{k-1},0}, \dots, f_{2^{k-1},M} \right]^T,$$
$$\mathbf{\Psi}^{\alpha,\beta}(t) = \left[\psi^{\alpha,\beta}_{1,0}(t), \dots, \psi^{\alpha,\beta}_{1,M}(t), \dots, \psi^{\alpha,\beta}_{2^{k-1},0}(t), \dots, \psi^{\alpha,\beta}_{2^{k-1},M}(t) \right]^T.$$

4. Uniform convergence and error estimation

In what follows, uniform convergence and error estimation of Jacobi wavelets are assessed.

THEOREM 4.1. *If $f(t)$ is continuous on $[0,1]$ and there exists $L \in \mathbb{R}^+$ such that $|f''(t)| \leqslant L$, then the truncated series* (3.8) *when $M \to \infty$ converges to $f(t)$ uniformly, that is*

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} f_{n,m} \psi^{\alpha,\beta}_{n,m}(t).$$

Moreover, for $m > 1$,

$$|f_{n,m}| < \frac{L \left(\frac{2\alpha+4}{\alpha+\beta+4} \right)^{\frac{\alpha+2}{2}} \left(\frac{2\beta+4}{\alpha+\beta+4} \right)^{\frac{\beta+2}{2}}}{n^{\frac{5}{2}} (m-1)(m+\alpha+\beta+1)}.$$

Proof. For every $\alpha > -1$ and $\beta > -1$, we can write

$$\begin{aligned} f_{n,m} &= \int_0^1 \omega_n(t)\psi_{n,m}^{\alpha,\beta}(t)f(t)dt \\ &= \frac{1}{\sqrt{h_m^{\alpha,\beta}}}2^{\frac{k}{2}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(1-\left(2^k t-2n+1\right)\right)^{\alpha} \left(1+\left(2^k t-2n+1\right)\right)^{\beta} \times \\ & P_m^{\alpha,\beta}\left(2^k t-2n+1\right) f(t)dt. \end{aligned}$$

If we put $x = 2^k t - 2n + 1$, it follows

$$f_{n,m} = \frac{1}{\sqrt{h_m^{\alpha,\beta}}}2^{-\frac{k}{2}} \int_{-1}^1 \omega^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x) f\left(\frac{2n-1+x}{2^k}\right) dx. \tag{4.1}$$

The Rodrigues formula of Jacobi polynomials results in

$$\omega^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x) = -\frac{1}{2m}\frac{d}{dx}\left(\omega^{\alpha+1,\beta+1}(x)P_{m-1}^{\alpha+1,\beta+1}(x)\right). \tag{4.2}$$

Applying the integration by parts technique for (4.1) and using (4.2), we have

$$f_{n,m} = -\frac{2^{-\frac{3k}{2}}}{2m\sqrt{h_m^{\alpha,\beta}}} \int_{-1}^1 \omega^{\alpha+1,\beta+1}(x)P_{m-1}^{\alpha+1,\beta+1}(x) f'\left(\frac{2n-1+x}{2^k}\right) dx. \tag{4.3}$$

Integrating by parts again, enables one to achieve

$$f_{n,m} = \frac{2^{-\frac{5k}{2}}}{4m(m-1)\sqrt{h_m^{\alpha,\beta}}} \int_{-1}^1 \omega^{\alpha+2,\beta+2}(x)P_{m-2}^{\alpha+2,\beta+2}(x) f''\left(\frac{2n-1+x}{2^k}\right) dx.$$

Deriving Holder's inequality for this identity, we obtain

$$|f_{n,m}| \leqslant \frac{L}{2m(m-1)\sqrt{h_m^{\alpha,\beta}}}2^{-\frac{5k}{2}} \left(\int_{-1}^1 \left(\omega^{\alpha+2,\beta+2}(x)P_{m-2}^{\alpha+2,\beta+2}(x)\right)^2 dx\right)^{\frac{1}{2}}. \tag{4.4}$$

Now, put

$$r(x) = \omega^{\alpha+2,\beta+2}(x), \quad s(x) = \omega^{\alpha+2,\beta+2}(x)\left(P_{m-2}^{\alpha+2,\beta+2}(x)\right)^2.$$

From (4.4) and the fact $r(x),\ s(x) > 0$ on $[-1,1]$, one may gain

$$|f_{n,m}| \leqslant \frac{L}{m(m-1)\sqrt{h_m^{\alpha,\beta}}}2^{-\frac{5k}{2}} \left(\max_{-1\leqslant x\leqslant 1} r(x)\right)^{\frac{1}{2}} \left(\int_{-1}^1 s(x)dx\right)^{\frac{1}{2}}. \tag{4.5}$$

A simple computation shows that $x^* = \frac{\beta-\alpha}{\alpha+\beta+4}$ maximizes $r(x)$ on $[-1,1]$. This maximum value is

$$r(x^*) = \max_{-1\leqslant x\leqslant 1} r(x) = \left(\frac{2\alpha+4}{\alpha+\beta+4}\right)^{\alpha+2}\left(\frac{2\beta+4}{\alpha+\beta+4}\right)^{\beta+2}. \tag{4.6}$$

On the other hand, according to (3.2), it is clear that

$$\int_{-1}^{1} s(x)dx = h_{m-2}^{\alpha+2,\beta+2}. \tag{4.7}$$

From (4.5),(4.6), (4.7) and knowing $n \leqslant 2^{k-1}$, for $m > 1$, it follows

$$\begin{aligned}
|f_{n,m}| &\leqslant \frac{L\left(\frac{2\alpha+4}{\alpha+\beta+4}\right)^{\frac{\alpha+2}{2}}\left(\frac{2\beta+4}{\alpha+\beta+4}\right)^{\frac{\beta+2}{2}}}{(2n)^{\frac{5}{2}}m(m-1)}\sqrt{\frac{h_{m-2}^{\alpha+2,\beta+2}}{h_m^{\alpha,\beta}}}\\
&= \frac{L\left(\frac{2\alpha+4}{\alpha+\beta+4}\right)^{\frac{\alpha+2}{2}}\left(\frac{2\beta+4}{\alpha+\beta+4}\right)^{\frac{\beta+2}{2}}}{(2n)^{\frac{5}{2}}m(m-1)}\sqrt{\frac{16m(m-1)}{(m+\alpha+\beta+2)(m+\alpha+\beta+1)}}\\
&= \frac{L\left(\frac{2\alpha+4}{\alpha+\beta+4}\right)^{\frac{\alpha+2}{2}}\left(\frac{2\beta+4}{\alpha+\beta+4}\right)^{\frac{\beta+2}{2}}}{n^{\frac{5}{2}}\sqrt{2m(m-1)(m+\alpha+\beta+2)(m+\alpha+\beta+1)}}\\
&< \frac{L\left(\frac{2\alpha+4}{\alpha+\beta+4}\right)^{\frac{\alpha+2}{2}}\left(\frac{2\beta+4}{\alpha+\beta+4}\right)^{\frac{\beta+2}{2}}}{n^{\frac{5}{2}}(m-1)(m+\alpha+\beta+1)}.
\end{aligned}$$

In a similar way, considering (4.3), for $m = 1$, we get

$$|f_{n,1}| < \frac{G\left(\frac{2\alpha+2}{\alpha+\beta+2}\right)^{\frac{\alpha+2}{2}}\left(\frac{2\beta+2}{\alpha+\beta+2}\right)^{\frac{\beta+2}{2}}}{(\alpha+\beta+3)n^{\frac{3}{2}}},$$

where $G = \sup|f'(t)|$ on $[0,1]$. We mention that G is exist because by the Mean Value Theorem for every $t \in [0,1]$, there exists an $0 < \eta_t < t$ such that $f'(t) - f'(0) = f''(\eta_t)t$ and the assumption $|f''(t)| < L$ on $[0,1]$ entails $|f'(t)| \leqslant |f'(0)| + L$.

Consequently, the series $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} f_{n,m}$ is absolutely convergent. Therefore, $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} f_{n,m}\psi_{n,m}^{\alpha,\beta}(t)$ converges to $f(t)$ uniformly. This completes the proof. □

Theorem 4.2. *Under the assumptions of Theorem (4.1), assume that*

$$f_{k,M}(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} f_{n,m}\psi_{n,m}^{\alpha,\beta}(t),$$

is the Jacobi wavelets approximation of $f(t)$ at the level k. Then, the error estimation on $[0,1]$ is bounded as

$$\|\varepsilon_{k,M}\| < C'\left(\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty}\frac{1}{n^5(m-1)^2(m+\alpha+\beta+1)^2}\right)^{\frac{1}{2}},$$

so that

$$C' = L\left(\frac{2\alpha+4}{\alpha+\beta+4}\right)^{\frac{\alpha+2}{2}}\left(\frac{2\beta+4}{\alpha+\beta+4}\right)^{\frac{\beta+2}{2}}.$$

PROOF.

$$\begin{aligned}
\|\varepsilon_{k,M}\|^2 &= \int_0^1 \left|f(t)-f_{k,M}(t)\right|^2 \omega_n(t)dt \\
&= \int_0^1 \Big|\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} f_{n,m}\psi_{n,m}^{\alpha,\beta}(t) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} f_{n,m}\psi_{n,m}^{\alpha,\beta}(t)\Big|^2 \omega_n(t)dt \\
&= \int_0^1 \Big|\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} f_{n,m}\psi_{n,m}^{\alpha,\beta}(t)\Big|^2 \omega_n(t)dt \\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} |f_{n,m}|^2 \int_0^1 \left|\psi_{n,m}^{\alpha,\beta}(t)\right|^2 \omega_n(t)dt \\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} |f_{n,m}|^2 \\
&< C'^2 \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} \frac{1}{n^5(m-1)^2(m+\alpha+\beta+1)^2},
\end{aligned}$$

where

$$C' = L\left(\frac{2\alpha+4}{\alpha+\beta+4}\right)^{\frac{\alpha+2}{2}}\left(\frac{2\beta+4}{\alpha+\beta+4}\right)^{\frac{\beta+2}{2}}.$$

□

5. Operational matrix and implementation of numerical method

In this section, operational matrix of fractional integration for Jacobi wavelets is first designed and then a numerical method is implemented for solving (1.1) with the initial conditions (1.2).

5.1. Operational matrix of fractional integration. Let k is fixed and M is given. Put $\hat{m} = 2^{k-1}(M+1)$. Taking the collocation points

$$\mathbf{T} = \left\{t_j |\ t_j = \frac{2j-1}{2\hat{m}},\ j = 1,\dots,\hat{m}\right\},$$

we define the $\hat{m}\times\hat{m}$ Jacobi wavelets matrix $\mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta}$ as

$$\mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} = \left[\mathbf{\Psi}^{\alpha,\beta}(t_1), \mathbf{\Psi}^{\alpha,\beta}(t_2), \dots, \mathbf{\Psi}^{\alpha,\beta}(t_{\hat{m}})\right]. \tag{5.1}$$

DEFINITION 5.1. The $\hat{m}$-set of BPFs on $[0,1]$ is defined in the following

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{\hat{m}} \leqslant t < \frac{i}{\hat{m}}, \\ 0, & otherwise, \end{cases}$$

where $i = 1, \ldots, \hat{m}$.

REMARK 5.1. BPFs are disjoint and orthogonal, that is

- $b_i(t)b_j(t) = \begin{cases} 0, & i \neq j, \\ b_i(t), & i = j. \end{cases}$
- $\int_0^1 b_i(\tau)b_j(\tau)d\tau = \begin{cases} 0, & i \neq j, \\ \frac{1}{\hat{m}}, & i = j. \end{cases}$

According to the orthogonality of BPFs, the function $f(t) \in L^2[0,1]$ can be written as

$$f(t) \approx \sum_{i=1}^{\hat{m}} f_i b_i(t) = \mathbf{f}_{\hat{m}}^T \mathbf{B}_{\hat{m}}(t),$$

where

$$\mathbf{f}_{\hat{m}} = [f_1, f_2, \ldots, f_{\hat{m}}]^T, \quad \mathbf{B}_{\hat{m}}(t) = [b_1(t), b_2(t), \ldots, b_{\hat{m}}(t)]^T,$$

in which for $i = 1, \ldots \hat{m}$,

$$f_i = \hat{m} \int_0^1 f(t)b_i(t)dt.$$

DEFINITION 5.2. For two vectors $\mathbf{f}_{\hat{m}} = [f_i]$ and $\mathbf{g}_{\hat{m}} = [g_i]$,

$$\mathbf{f}_{\hat{m}} \otimes \mathbf{g}_{\hat{m}} = (f_i \times g_i)_{\hat{m}}.$$

Similarly, for two matrices $\mathbf{A} = [a_{i,j}]$ and $\mathbf{B} = [b_{i,j}]$ of $\hat{m} \times \hat{m}$

$$\mathbf{A} \otimes \mathbf{B} = (a_{i,j} \times b_{i,j})_{\hat{m} \times \hat{m}}.$$

LEMMA 5.1. *Let the functions $f(t)$, $g(t) \in L^2[0,1]$ are expanded into BPFs, that is $f(t) = \mathbf{f}_{\hat{m}}^T \mathbf{B}_{\hat{m}}(t)$ and $g(t) = \mathbf{g}_{\hat{m}}^T \mathbf{B}_{\hat{m}}(t)$. Then*

$$f(t)g(t) = \left(\mathbf{f}_{\hat{m}}^T \otimes \mathbf{g}_{\hat{m}}^T\right) \mathbf{B}_{\hat{m}}(t).$$

PROOF.

$$\begin{aligned} f(t)g(t) &= \mathbf{f}_{\hat{m}}^T \mathbf{B}_{\hat{m}}(t) \mathbf{B}_{\hat{m}}^T(t) \mathbf{g}_{\hat{m}} = f_1 g_1 b_1(t) + f_2 g_2 b_2(t) + \ldots + f_{\hat{m}} g_{\hat{m}} b_{\hat{m}}(t) \\ &= \left(\mathbf{f}_{\hat{m}}^T \otimes \mathbf{g}_{\hat{m}}^T\right) \mathbf{B}_{\hat{m}}(t). \end{aligned}$$

□

5.1.1. *Operational matrix.* From [**8**], the fractional integration of order ν of the BPFs vector $\mathbf{B}_{\hat{m}}(t)$ is given as

$$I^{\nu}\mathbf{B}_{\hat{m}}(t) \approx \mathbf{F}^{\nu}\mathbf{B}_{\hat{m}}(t), \tag{5.2}$$

where

$$\mathbf{F}^{\nu} = \frac{1}{\hat{m}^{\nu}}\frac{1}{\Gamma(\nu+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \dots & \xi_{\hat{m}-1} \\ 0 & 1 & \xi_1 & \dots & \xi_{\hat{m}-2} \\ 0 & 0 & 1 & \dots & \xi_{\hat{m}-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

in which $\xi_i = (i+1)^{\nu+1} - 2i^{\nu+1} + (i-1)^{\nu+1}$, $i = 1, \dots, \hat{m}-1$.

Let $\mathbf{P}^{\alpha,\beta,\nu}_{\hat{m}\times\hat{m}}$ denotes to the fractional integration operational matrix, that is

$$I^{\nu}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) \approx \mathbf{P}^{\alpha,\beta,\nu}_{\hat{m}\times\hat{m}}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t). \tag{5.3}$$

Now, for $t \in \mathbf{T}$, using (5.1), we have

$$\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) = \mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}\mathbf{B}_{\hat{m}}(t). \tag{5.4}$$

By (5.2) and (5.4), one gets

$$I^{\nu}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) = I^{\nu}\mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}\mathbf{B}_{\hat{m}}(t) = \mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}I^{\nu}\mathbf{B}_{\hat{m}}(t) \approx \mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}\mathbf{F}^{\nu}\mathbf{B}_{\hat{m}}(t). \tag{5.5}$$

Also, with the aid of (5.4), we conclude

$$\mathbf{B}_{\hat{m}}(t) = \left(\mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}\right)^{-1}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t). \tag{5.6}$$

Consequently, from (5.5) and (5.6), it follows

$$I^{\nu}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) \approx \mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}\mathbf{F}^{\nu}\left(\mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}\right)^{-1}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t). \tag{5.7}$$

Considering (5.3) and (5.7) results in

$$\mathbf{P}^{\alpha,\beta,\nu}_{\hat{m}\times\hat{m}} = \mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}\mathbf{F}^{\nu}\left(\mathbf{\Phi}^{\alpha,\beta}_{\hat{m}\times\hat{m}}\right)^{-1}.$$

5.2. Implementation of the numerical method. Recall the system (1.1) with the initial conditions (1.2). Let

$$\begin{cases} D^{\nu_1}_{*}S(t) \approx \mathbf{C}^{T}_{\hat{m}}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t), \\ D^{\nu_2}_{*}I(t) \approx \mathbf{D}^{T}_{\hat{m}}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t), \\ D^{\nu_3}_{*}R(t) \approx \mathbf{K}^{T}_{\hat{m}}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t), \end{cases} \tag{5.8}$$

in which $\mathbf{C}_{\hat{m}} = [c_1, \dots, c_{\hat{m}}]^T$, $\mathbf{D}_{\hat{m}} = [d_1, \dots, d_{\hat{m}}]^T$, $\mathbf{K}_{\hat{m}} = [k_1, \dots, k_{\hat{m}}]^T$. Integrating of fractional-order and considering the initial conditions (1.2) yield

$$\begin{cases} S(t) = I^{\nu_1}D^{\nu_1}_{*}S(t) + S_0 \approx \mathbf{C}^{T}_{\hat{m}}I^{\nu_1}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) + S_0 \approx \mathbf{C}^{T}_{\hat{m}}\mathbf{P}^{\alpha,\beta,\nu_1}_{\hat{m}\times\hat{m}}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) + S_0, \\ I(t) = I^{\nu_2}D^{\nu_2}_{*}I(t) + I_0 \approx \mathbf{D}^{T}_{\hat{m}}I^{\nu_2}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) + I_0 \approx \mathbf{D}^{T}_{\hat{m}}\mathbf{P}^{\alpha,\beta,\nu_2}_{\hat{m}\times\hat{m}}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) + I_0, \\ R(t) = I^{\nu_3}D^{\nu_3}_{*}R(t) + R_0 \approx \mathbf{K}^{T}_{\hat{m}}I^{\nu_3}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) + R_0 \approx \mathbf{K}^{T}_{\hat{m}}\mathbf{P}^{\alpha,\beta,\nu_3}_{\hat{m}\times\hat{m}}\mathbf{\Psi}^{\alpha,\beta}_{\hat{m}}(t) + R_0. \end{cases}$$

Collocating above equations at the points $\mathbf{T}$ and using (5.4) imply

$$\begin{cases} S(t) \approx \mathbf{C}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_1} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} \mathbf{B}_{\hat{m}}(t) + \mathbf{S}_0 \mathbf{B}_{\hat{m}}(t), \\ I(t) \approx \mathbf{D}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_2} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} \mathbf{B}_{\hat{m}}(t) + \mathbf{I}_0 \mathbf{B}_{\hat{m}}(t), \\ R(t) \approx \mathbf{K}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_3} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} \mathbf{B}_{\hat{m}}(t) + \mathbf{R}_0 \mathbf{B}_{\hat{m}}(t), \end{cases} \tag{5.9}$$

where $\mathbf{S}_0 = [S_0, \ldots, S_0]_{1\times\hat{m}}$, $\mathbf{I}_0 = [I_0, \ldots, I_0]_{1\times\hat{m}}$ and $\mathbf{R}_0 = [R_0, \ldots, R_0]_{1\times\hat{m}}$. Substituting (5.9) and (5.8) into (1.1) and dispersing t at collocation points $\mathbf{T}$, we achieve the following non-linear system of $3\hat{m}$ algebraic equations for $3\hat{m}$ unknowns

$$\begin{cases} \mathbf{C}_{\hat{m}}^T \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} = -a_1 \left(\mathbf{C}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_1} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} + \mathbf{S}_0 \right) \otimes \left(\mathbf{D}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_2} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} + \mathbf{I}_0 \right), \\ \mathbf{D}_{\hat{m}}^T \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} = a_1 \left(\mathbf{C}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_1} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} + \mathbf{S}_0 \right) \otimes \left(\mathbf{D}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_2} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} + \mathbf{I}_0 \right) \\ \qquad\qquad -a_2 \left(\mathbf{D}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_2} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} + \mathbf{I}_0 \right), \\ \mathbf{K}_{\hat{m}}^T \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} = a_2 \left(\mathbf{D}_{\hat{m}}^T \mathbf{P}_{\hat{m}\times\hat{m}}^{\alpha,\beta,\nu_2} \mathbf{\Phi}_{\hat{m}\times\hat{m}}^{\alpha,\beta} + \mathbf{I}_0 \right), \end{cases}$$

which can be solved by the Newton-Raphson procedure with an initial guess or MATLAB's fsolve command and $T(t)$, $I(t)$, $V(t)$ are obtained on $[0, 1]$.

6. Numerical experiments

Throughout this section, we assume $\nu_1 = \nu_2 = \nu_3 = \nu$, for the sake of simplicity. The parameters and initial values of (1.1) are given as

$$a_1 = 0.01, \ a_2 = 0.02, \ S_0 = 20, \ I_0 = 15, \ R_0 = 10.$$

We also consider the Jacobi wavelets basis for $\alpha = 1, \beta = 2$ which is chosen arbitrary. For given k and M, we determine the coefficients c_i, d_i, k_i for $i = 1, \ldots, \hat{m}$ and then $T(t)$, $I(t)$ and $V(t)$ are obtained.

In order to test the efficiency and accuracy of the proposed method in the case $\nu = 1$, we compare the numerical results of applying the method with the well-known Runge-Kutta method of fourth-order, Table 1. For $0 < \nu < 1$, we calculate the residual errors which are defined as

$$\begin{cases} E(S(t)) = \left| D_*^{\nu} S(t) + a_1 S(t) I(t) \right|, \\ E(I(t)) = \left| D_*^{\nu} I(t) - a_1 S(t) I(t) + a_2 I(t) \right|, \\ E(R(t)) = \left| D_*^{\nu} R(t) - a_2 I(t) \right|. \end{cases}$$

Table 2 compares the residual errors of seventh-order approximations of Homotopy analysis method [**1**] with the proposed method for $k = 1$, $M = 7$ in the case $\nu = 0.75$.

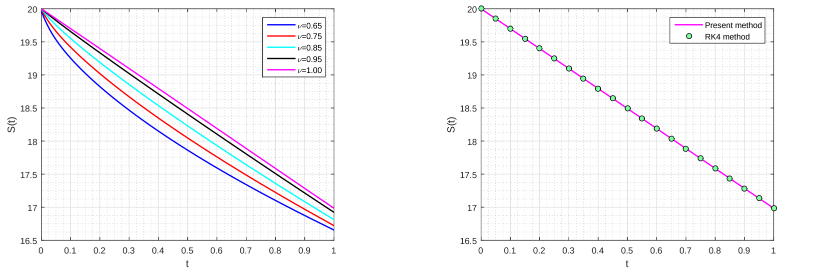
The behaviour of the model (1.1) for various ν and the graphical comparison of our method with fourth-order Runge-Kutta method in the case $\nu = 1$ have been shown in Figures 1, 2 and 3 for $k = 3$, $M = 3$. It is understandable that when ν closes to 1, the solution of fractional-order model $(0 < \nu < 1)$ closes to the solution of integer-order model $(\nu = 1)$.

TABLE 1. Comparison between fourth-order Runge-Kutta solution and our method for $k = 1, M = 7$ in the case $\nu = 1$

t	S(t)		I(t)		R(t)	
	RK4	Ours	RK4	Ours	RK4	Ours
0.0	20.000000	19.999824	15.000000	15.000070	10.000000	10.000105
0.1	19.699578	19.699426	15.270152	15.270199	10.030270	10.030376
0.2	19.398426	19.398297	15.540494	15.540517	10.061081	10.061186
0.3	19.096713	19.096608	15.810855	15.810854	10.092432	10.092538
0.4	18.794612	18.794532	16.081064	16.081039	10.124324	10.124430
0.5	18.492296	18.492240	16.350948	16.350899	10.156756	10.156862
0.6	18.189937	18.189905	16.620336	16.620262	10.189728	10.189833
0.7	17.887708	17.887700	16.889055	16.888958	10.223237	10.223342
0.8	17.585781	17.585798	17.156936	17.156815	10.257283	10.257388
0.9	17.284329	17.284369	17.423807	17.423663	10.291864	10.291968
1.0	16.983520	16.983585	17.689502	17.689335	10.326978	10.327081

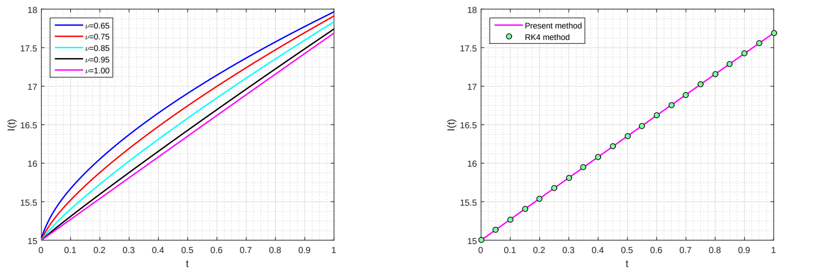
TABLE 2. Residual errors of Homotopy analysis method (seven iteration) and our method for $k = 1, M = 7$ with $\nu = 0.75$

t	E(S(t))		E(I(t))		E(R(t))	
	HAM [**1**]	Ours	HAM [**1**]	Ours	HAM [**1**]	Ours
0.1	8.58906×10^{-4}	3.28307×10^{-5}	3.24403×10^{-2}	3.26663×10^{-5}	3.49976×10^{-3}	2.52537×10^{-8}
0.2	2.21032×10^{-3}	2.69744×10^{-6}	1.01885×10^{-2}	2.83890×10^{-6}	1.81610×10^{-3}	1.33356×10^{-8}
0.3	4.10320×10^{-3}	9.55477×10^{-7}	5.04414×10^{-3}	1.19021×10^{-6}	7.42771×10^{-4}	1.64643×10^{-8}
0.4	2.21434×10^{-4}	1.81059×10^{-6}	1.66542×10^{-4}	1.53201×10^{-6}	1.85557×10^{-4}	1.91991×10^{-8}
0.5	3.19900×10^{-3}	1.52851×10^{-6}	2.43165×10^{-3}	1.76676×10^{-6}	5.13441×10^{-5}	9.21540×10^{-9}
0.6	3.52253×10^{-3}	1.66418×10^{-6}	2.85768×10^{-3}	1.53477×10^{-6}	1.16874×10^{-4}	7.10228×10^{-9}
0.7	9.25863×10^{-4}	7.92870×10^{-7}	3.12027×10^{-3}	8.45278×10^{-7}	3.91056×10^{-5}	2.24806×10^{-8}
0.8	8.75244×10^{-4}	2.04555×10^{-6}	3.14582×10^{-3}	2.20438×10^{-6}	1.86796×10^{-4}	2.18876×10^{-8}
0.9	4.14473×10^{-4}	2.17891×10^{-5}	2.74835×10^{-3}	2.14215×10^{-5}	3.70990×10^{-5}	6.12492×10^{-9}
1.0	1.45279×10^{-3}	2.62004×10^{-4}	2.37497×10^{-3}	2.61652×10^{-4}	2.21411×10^{-4}	1.96767×10^{-7}



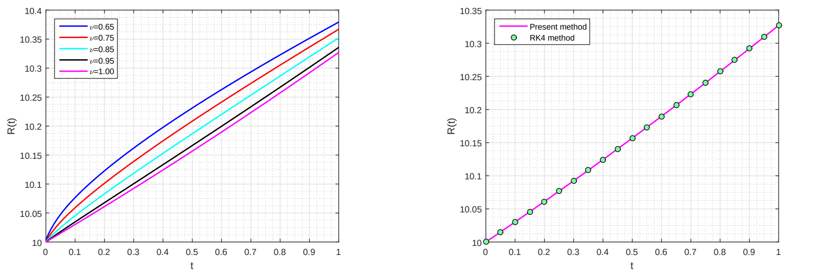
Approximate solutions of $S(t)$ Comparison of $S(t)$ solutions for $\nu = 1$

FIGURE 1. The numerical behaviour of $S(t)$



Approximate solutions of $I(t)$ Comparison of $I(t)$ solutions for $\nu = 1$

FIGURE 2. The numerical behaviour of $I(t)$



Approximate solutions of $R(t)$ Comparison of $R(t)$ solutions for $\nu = 1$

FIGURE 3. The numerical behaviour of $R(t)$

7. Conclusion

We considered Jacobi wavelets using recurrence formula of Jacobi polynomials and the concept of wavelet. To the best of our knowledge, it was first time that uniform convergence of these wavelets and error estimation of them were studied.

As an application of Jacobi wavelets in non-linear fractional models, an efficient scheme with high accuracy was proposed for solving fractional-order SIR epidemic model expressed as a non-linear system of fractional differential equations. Along the way, an operational matrix of fractional integration using BPFs was obtained to transform the problem to a non-linear system of algebraic equations. The new system was solvable by any standard iteration method such as Newton-Raphson method or fsolve command of MATLAB software.

Numerical results showed that when ν closes to 1, the solution of fractional-order model ($0 < \nu < 1$) closes to the solution of integer-order model ($\nu = 1$). We displayed all of the computations in the matrix form. This scheme makes the computer programming simple and convenient.

We also performed all of the computations by MATLAB R2015a software on a 64-bit PC with 2.20 GHz processor and 8 GB memory. It is notable that the proposed method can be used and extended for solving other non-linear biological systems arising in mathematical biology.

References

[1] O. A. Arqub and A. El-Ajou. Solution of the fractional epidemic model by homotopy analysis method. *Journal of King Saud University - Science*, **25**(1)(2013), 73–81.

[2] H. D. Azodi and M.R. Yaghouti. A new method based on fourth kind Chebyshev wavelets to a fractional-order model of HIV infection of $CD4^{+}T$ cells. *Comput. Methods Differ. Equ.*, **6**(3)(2018), 353–371.

[3] M. R. Eslahchi and M. Kavoosi. The use of Jacobi wavelets for constrained approximation of rational Bezier curves. *Comput. Appl. Math.*, **37**(3)(2018), 3951–3966.

[4] M. H. Heydari, F. M. Maalek Ghaini and M. R. Hooshmandasl. Legendre wavelets method for numerical solution of time-fractional heat equation. *Wavel. Linear Algebra.*, **1**(1)(2014), 19–31.

[5] M. H. Heydari, M. R. Hooshmandasl, F. Mohammadi and C. Cattani. Wavelets method for solving systems of nonlinear singular fractional Volterra integro-differential equations. *Commun. Nonlinear Sci. Numer. Simul.* , **19**(1)(2014), 37–48.

[6] W. O. Kermack and A. G. McKendrick. Contributions to the mathematical theory of epidemics–I^*. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **115**(772)(1927), 700–721.

[7] H. Jafari, S. A. Yousefi, M. A. Firoozjaee, S. Momani and C. M. Khalique. Application of Legendre wavelets for solving fractional differential equations. *Comput. Math. Appl.*, **62**(3)(2011), 1038–1045.

[8] A. Kilicman and Z. A. Al-Zhour. Kronecker operational matrices for fractional calculus and some applications. *Appl. Math. Comput.*, **187**(1)(2007), 250–265.

[9] Y. Li. Solving a nonlinear fractional differential equation using Chebyshev wavelets. *Commun. Nonlinear Sci. Numer. Simul.* , **15**(9)(2010), 2284–2292.

[10] S. Mallat. *A wavelet tour of signal processing*, Academic Press, 2000.

[11] F. Mohammadi. Numerical solution of Bagley-Torvik equation using Chebyshev wavelet operational matrix of fractional derivative. *Int. J. Adv. Appl. Math. Mech.*, **2**(1)(2014), 83–91.

[12] I. Podlubny. *Fractional differential equations*, Academic Press, 1999.

[13] M. U. Rehman and U. Saeed. Gegenbauer wavelets operational matrix method for fractional differential equations. *J. Korean Math. Soc.*, **52**(5)(2015), 1069–1096.

[14] M. U. Rehman and R. A. Khan. The Legendre wavelet method for solving fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.*, **16**(11)(2011), 4163–4173.

[15] S. Z. Rida, A. A. M. Arafa and Y. A. Gaber. Solution of the fractional epidemic model by L-ADM. *J. Fract. Calc. Appl.*, **7**(1)(2016), 189–195.

[16] L. J. Rong and P. Chang. Jacobi wavelet operational matrix of fractional integration for solving fractional integro-differential equation. *J. Phys. Conf. Ser.*, **693** (2016), ID: 012002 (pp. 1–14).

[17] G. Szegö. *Orthogonal Polynomials.* American Mathematical Society, 1939.

[18] Y. Wang and Q. Fan. The second kind Chebyshev wavelet method for solving fractional differential equations. *Appl. Math. Comput.*, **218**(17)(2012), 8592–8601.

[19] M. A. Zaky, I. G. Ameen and M. A. Abdelkawy. A new operational matrix based on Jacobi wavelets for a class of variable-order fractional differential equations. *Proc. Rom. Acad., Ser. A, Math. Phys. Tech. Sci. Inf. Sci.* , **18**(4)(2017), 315–322.

[20] F. Zhou and X. Xu. The third kind Chebyshev wavelets collocation method for solving the time-fractional convection diffusion equations with variable coefficients. *Appl. Math. Comput.*, **280** (2016), 11–29.

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