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M-POLYNOMIAL OF SUBDIVISION AND COMPLEMENTARY GRAPHS OF BANANA TREE GRAPH

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Abstract. The main goal of this paper is to define the closed forms of M-polynomials for subdivision and complementary graphs of Banana tree graph. We will also compute closed forms of various degree-based topological indices of those graphs. It is known that the topological indices will be mentioned in here are numerical tendencies which often depict quantitative structural activity/property/toxicity relationships and correlate certain physico-chemical properties such as boiling point, stability and strain energy. To conclude, we shall plot surfaces associated to M-polynomials and characterize some facts about these graphs.

1. Introduction

Graph theory provides an important tool called molecular graph-based structure descriptor or more commonly topological index to correlated the physico-chemical properties of chemical compounds with their molecular structure. Topological indices are the numerical value associated with chemical constitution for correlation of chemical structure with various physical properties, chemical reactivity or biological activity. The topological index of a molecule is a non-empirical numerical quantity that quantifies the structure and the branching pattern of the molecule. Therefore, the topological analysis of a molecule involves translating its molecular structure into a characteristic unique number (or index) that may be considered a descriptor of the molecule under examination. Such indices based on the distances in graph are widely used for establishing relationships between the structure of molecular graph and their physico-chemical properties.

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Key words and phrases. M-polynomial, degree-based index, Banana tree graph.

We recall that while a graph $G(V, E)$ with vertex set $V(G)$ and edge set $E(G)$ is connected if there is a path between any pair of vertices in G, the degree of a vertex is the number of vertices which are connected to that fixed vertex by the edges. Moreover the distance between any two vertices u and v is denoted by $d(u, v)$ (or $d_G(u, v)$) and is defined as the length of shortest path between u and v in graph G. For details on the basics of graph theory, any standard text such as [**27**] can be of great help.

Several algebraic polynomials have useful applications in chemistry. The Hosoya polynomial ([**11**]) would be the best well-known example, and it plays an important role in determining distance-based topological indices. Among other algebraic polynomials, the M-polynomial ([**8**]) was introduced in 2015 and plays the same role in determining closed forms of many degree-based topological indices. These indices are actually score functions that capture a variety of physico-chemical properties of chemical compounds such as boiling point, heat of evaporation, heat of formation, chromatographic retention times, surface tension, and vapor pressure. For the details, we may refer [**4, 6, 7, 16, 25, 29**].

In the following with in each different paragraph, we will recall the degree based topological indices that will be needed in this paper:

The *M-polynomial* of the graph G is defined as

$$M(G; x, y) = \sum_{\delta \leqslant i \leqslant j \leqslant \Delta} m_{ij} x^i y^j \,, \tag{1.1}$$

where $\delta = min\{d_v : v \in V(G)\}$, $\Delta = max\{d_v; v \in V(G)\}$ and $m_{ij}(G)$ the number of edges $vu \in E(G)$ such that $\{d_v, d_u\} = \{i, j\}$.

The *first* and *second Zagreb indices*

$$M_1(G) = \sum_{u \in V(G)} (d_u)^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} (d_u . d_v) \,,$$

respectively, have been introduced more than thirty years ago by I. Gutman and Trinajstic in [**12**]. In fact these Zagreb indices found many applications in QSPR and QSAR studies. For more details on this important topological indices, we refer to [**13, 14, 15, 22, 24**]. According to the [**21**], both the first and the second Zagreb indices give greater weights to the inner vertices and edges, and smaller weights to outer vertices and edges which oppose intuitive reasoning. On the other hand, there also exists the second modified Zagreb index

$${}^{m}M_2(G) = \sum_{uv \in E(G)} \frac{1}{d_u . d_v} \,,$$

for a simple connected graph G. There also exists a degree based index related to Zagreb indices which is named as the *augmented Zagreb index* of G and proposed by Furtula et al. [**9**]. It is calculated by the formula

$$A(G) = \sum_{uv \in E(G)} \left(\frac{d_u d_v}{d_u + d_v - 2} \right)^3 .$$

Moreover, the tight upper and lower bounds for the augmented Zagreb index of chemical tree, and the tree with minimal augmented Zagreb index were obtained again in the reference [**9**].

The *Randic index* ([**23**])

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{(d_u.d_v)}} ,$$

also known as the connectivity index, of G introduced in 1975 by Milan Randic who has shown that this index is to reflect molecular branching. For the key results on the Randic index, we may also refer [**17, 19**]. On the other hand, the *inverse Randic index* is defined as:

$$RR_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha ,$$

where α is an arbitrary real number.

Among 148 discrete Adriatic indices ([**2, 5**]), one of the important of those is the *symmetric division deg index* which is defined by

$$SDD(G) = \sum_{uv \in E(G)} \left(\frac{min(d_u, d_v)}{max(d_u, d_v)} + \frac{max(d_u, d_v)}{min(d_u, d_v)} \right) .$$

For a collection of recent results on $SDD(G)$, one can see the references [**10, 18**].

Some of the topological indices are based on the vertex-degree of the graph G. One of the vertex-degree based topological index is the *Harmonic index* $H(G)$ that is defined by

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v} .$$

For more results on Harmonic index, we refer citations [**20, 26, 28**].

The *inverse sum index* is the descriptor that was selected in [**3**] as a significant predictor of total surface area of octane isomers and for which the extremal graphs obtained with the help of mathematical chemistry have a particular simple and elegant structure. Actually the inverse sum index is given by

$$I(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v} .$$

In [**1**], above well known degree based topological indices with M-polynomials have been listed with a table (see [**1**, Table 1]). In here, by not re-writing again, we will also use this table in some of our proofs which will be mentioned as just *Table*.

Let us also recall the *subdivision graph* $S(G)$ which is the graph obtained from G by replacing each edge by a path of length 2 or by inserting a vertex in every edge of the graph G. Furthermore the *complementary graph* $\bar{G}$ of G is actually a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are connected by an edge uv if and only if they are not adjacent in G.

We finally remind that the *Banana tree graph* $B_{n,k}$ is obtained by connecting one leaf of each n copies of k-star graph with a single root vertex that has order $nk+1$ and size nk. As an example of this graph, we may give Figure 1 in below.

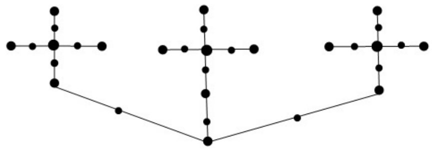


Figure 1: The subdivision graph of Banana tree graph $B_{3,5}$

It has been recently presented some computational aspects for the line graphs of Banana tree graph by Ahmad et. al. (see [**1**]). Motivated from this work, in this paper, we will state and proof the closed forms of M-polynomials of subdivision and complementary graphs of Banana tree graph, and then will compute many topological indices for those graphs obtained from Banana tree graph.

2. M-polynomial of the subdivision graph of Banana tree graph

In the following first main result, we will focus on M-polynomial of the subdivision graph of Banana tree graph.

Theorem 2.1. *Let F be the subdivision graph of Banana tree graph. Then the M-polynomial of F is presented by*

$$M(F;x,y)=k(n-2)xy^2+k(n-1)x^2y^{n-1}+2kx^2y^2+kx^2y^k\,.$$

Proof. In the proof, by considering the subdivision graph F of Banana tree graph for $n=3$ and $k=5$ as shown in Figure 1, we will make a generalization for arbitrary values n and k. Actually we will follow a similar way as in the proof of [**1**, Theorem 3.1].

First note that the graph F contains $2kn+1$ vertices and $2kn$ edges. There are four types of edges in F based on degrees of end vertices of each edge. The first edge partitions $E_1(F)$ contains $k(n-2)$ edges uv, where $d_u=1$, $d_v=2$. The second edge partitions $E_2(F)$ contains $k(n-1)$ edges uv, where $d_u=2$, $d_v=(n-1)$. The third edge partitions $E_3(F)$ contains $2k$ edges uv, where $d_u=2$, $d_v=2$. The fourth edge partitions $E_4(F)$ contains k edges uv, where $d_u=2$, $d_v=k$. Replacing G by F in Equation (1.1), we have

$$\begin{aligned}
M(F;x,y) &= \sum_{i\leqslant j} m_{ij}x^iy^j \\
&= \sum_{1\leqslant 2} m_{(1)(2)}xy^2+\sum_{2\leqslant (n-1)} m_{(2)(n-1)}x^2y^{(n-1)}+\sum_{2\leqslant 2} m_{(2)(2)j}x^2y^2+ \\
&\quad +\sum_{2\leqslant k} m_{(2)(k)}x^2y^k \\
&= \sum_{uv\in E_1(G)} m_{(1)(2)}xy^2+\sum_{uv\in E_2(G)} m_{(2)(n-1)}x^2y^{(n-1)}+
\end{aligned}$$

$$+\sum_{uv\in E_3(G)} m_{(2)(2)j}x^2y^2 + \sum_{uv\in E_4(G)} m_{(2)(k)}x^2y^k$$
$$= |E_1(G)|xy^2 + |E_2(G)|x^2y^{(n-1)} + |E_3(G)|x^2y^2 + |E_4(G)|x^2y^k$$
$$= k(n-2)xy^2 + k(n-1)x^2y^{n-1} + 2kx^2y^2 + kx^2y^k\,,$$

as required. Hence the result. □

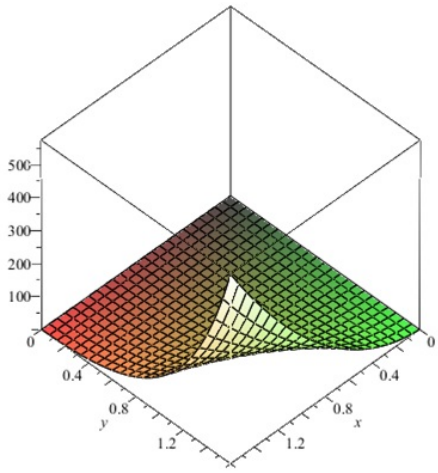


Figure 2: Plot of M-polynomial for subdivision graph of Banana tree graph.

As a next step of the above theorem, let us compute some degree-based topological indices depicted in the first section of this paper for the subdivision graph of Banana tree graph in terms of M-polynomial.

Corollary 2.1. *Let F be a subdivision graph of the Banana tree graph. Then*

- $M_1(F) = 3k(n-2) + k(n-1)(n+1) + 8k + k(k+2)$.
- $M_2(F) = 2k(n-2) + 2k(n-1)^2 + 8k + 2k^2$.
- ${}^mM_2(F) = \frac{k}{2}[(n-2) + 1 + \frac{1}{k} + 1]$.
- $R_\alpha(F) = 2^{\alpha+1}k(n-2) + 2^{\alpha+1}(n-1)^{\alpha+2}k + 2^{2\alpha+3}k + 2^{\alpha+1}k^{\alpha+2}$.
- $RR_\alpha(F) = \frac{k(n-2)}{2^\alpha} + \frac{k(n-1)^{1-\alpha}}{2^\alpha} + 2^{1-2\alpha}k + \frac{k^{1-\alpha}}{2^\alpha}$.
- $SDD(F) = \frac{5k(n-2)}{2} + \frac{k(n^2-2n+5)}{2} + 4k + \frac{(m^2+4)}{2}$.
- $H(F) = \frac{2k(n-2)}{3} + \frac{2k(n-1)}{n+1} + \frac{2k}{k+2} + k$.
- $I(F) = \frac{2k(n-2)}{3} + \frac{2k(n-1)^2}{n+1} + 2k + \frac{2k^2}{k+2}$.
- $A(F) = 8k(n-2) + 8k(n-1) + 16k + 8k$.

PROOF. By Theorem 2.1, we know that

$$M(F;x,y) = f(x,y) = k(n-2)xy^2 + k(n-1)x^2y^{n-1} + 2kx^2y^2 + kx^2y^k.$$

Therefore, we obtain

$$D_x f(x,y) = k(n-2)xy^2 + 2k(n-1)x^2y^{n-1} + 4kx^2y^2 + 2kx^2y^k,$$

$$D_y f(x,y) = 2k(n-2)xy^2 + k(n-1)^2x^2y^{n-1} + 4kx^2y^2 + k^2x^2y^k,$$

$$D_yD_x f(x,y) = 2k(n-2)xy^2 + 2k(n-1)^2x^2y^{n-1} + 8kx^2y^2 + 2k^2x^2y^k,$$

$$S_y(f(x,y)) = \frac{k(n-2)}{2}xy^2 + kx^2y^{n-1} + kx^2y^2 + x^2y^k,$$

$$S_xS_y(f(x,y)) = \frac{k}{2}\left[(n-2)xy^2 + x^2y^{n-1} + x^2y^2 + \frac{x^2y^k}{k}\right],$$

$$D_y^\alpha(f(x,y)) = 2^{\alpha+1}k(n-2)xy^2 + k(n-1)^{\alpha+2}x^2y^{n-1} + 2^{\alpha+2}kx^2y^2 + k^{\alpha+2}x^2y^k,$$

$$D_x^\alpha D_y^\alpha(f(x,y)) = 2^{\alpha+1}k(n-2)xy^2 + 2^{\alpha+1}(n-1)^{\alpha+2}kx^2y^{n-1} + 2^{2\alpha+3}kx^2y^2 +$$
$$2^{\alpha+1}k^{\alpha+2}x^2y^k,$$

$$S_y^\alpha(f(x,y)) = \frac{k(n-2)}{2^\alpha}xy^2 + k(n-1)^{1-\alpha}x^2y^{n-1} + 2^{1-\alpha}kx^2y^2 + k^{1-\alpha}x^2y^k,$$

$$S_x^\alpha S_y^\alpha(f(x,y)) = \frac{k(n-2)}{2^\alpha}xy^2 + \frac{k(n-1)^{1-\alpha}}{2^\alpha}x^2y^{n-1} + 2^{1-2\alpha}kx^2y^2 + \frac{k^{1-\alpha}}{2^\alpha}x^2y^k,$$

$$S_yD_x(f(x,y)) = \frac{k(n-2)}{2}xy^2 + 2kx^2y^{n-1} + 2kx^2y^2 + 2x^2y^k,$$

$$S_xD_y(f(x,y)) = 2k(n-2)xy^2 + \frac{(n-1)^2}{2}kx^2y^{n-1} + 2kx^2y^2 + \frac{k^2}{2}x^2y^k,$$

$$Jf(x,y) = k(n-2)x^3 + k(n-1)x^{n+1} + 2kx^4 + kx^{k+2},$$

$$S_xJf(x,y) = \frac{k(n-2)}{3}x^3 + \frac{k(n-1)}{n+1}x^{n+1} + \frac{k}{2}x^4 + \frac{k}{k+2}x^{k+2},$$

$$JD_xD_yf(x,y) = 2k(n-2)x^3 + 2k(n-1)^2x^{n+1} + 8kx^4 + 2k^2x^{k+2},$$

$$S_xJD_xD_yf(x,y) = \frac{2k(n-2)}{3}x^3 + \frac{2k(n-1)^2}{n+1}x^{n+1} + 2kx^4 + \frac{2k^2}{k+2}x^{k+2},$$

$$D_y^3f(x,y) = 8k(n-2)xy^2 + k(n-1)^4x^2y^{n-1} + 16kx^2y^2 + k^4x^2y^k,$$

$$D_x^3D_y^3f(x,y) = 8k(n-2)xy^2 + 8k(n-1)^4x^2y^{n-1} + 128kx^2y^2 + 8k^4x^2y^k,$$

$$JD_x^3D_y^3f(x,y) = 8k(n-2)x^3 + 8k(n-1)^4x^{n+1} + 128kx^4 + 8k^4x^{k+2},$$

$$Q_{-2}JD_x^3D_y^3f(x,y) = 8k(n-2)x + 8k(n-1)^4x^{n-1} + 128kx^2 + 8k^4x^k,$$
$$S_x^3Q_{-2}JD_x^3D_y^3f(x,y) = 8k(n-2)x + 8k(n-1)x^{n-1} + 16kx^2 + 8kx^k.$$

After all, by using the *Table*, we have the following graphs of different indices.

$$\begin{aligned} M_1(F) &= (D_x + D_y)f(x,y)|_{x=y=1} \\ &= 3k(n-2) + k(n-1)(n+1) + 8k + k(k+2). \end{aligned}$$

$$\begin{aligned} R_\alpha(F) &= D_x^\alpha D_y^\alpha f(x,y)|_{x=y=1} \\ &= 2^{\alpha+1}k(n-2) + 2^{\alpha+1}(n-1)^{\alpha+2}k + 2^{2\alpha+3}k + 2^{\alpha+1}k^{\alpha+2}. \end{aligned}$$

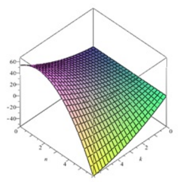
$$\begin{aligned} H(F) &= 2S_xJf(x,y)|_{x=y=1} \\ &= \frac{2k(n-2)}{3} + \frac{2k(n-1)}{n+1} + \frac{2k}{k+2} + k. \end{aligned}$$

$$\begin{aligned} I(F) &= (S_xJD_xD_y)f(x,y)|_{x=y=1} \\ &= \frac{2k(n-2)}{3} + \frac{2k(n-1)^2}{n+1} + 2k + \frac{2k^2}{k+2}. \end{aligned}$$

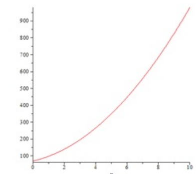
□

2.1. Surfaces representing M-polynomials of subdivision graph of Banana tree graph.

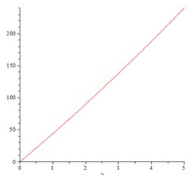
We use Maple 15 to represent graphs of M-polynomials of the subdivision graphs of Banana tree graph given in the proof of Corollary 2.1. From these graphs, it can seen that the behavior of the polynomials differ along different parameters.



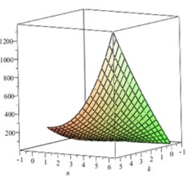
Plot for the first Zagreb index for the subdivision graph of Banana tree graph.



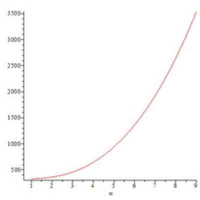
Plot for the first Zagreb index for the subdivision graph of Banana tree graph for k=7.



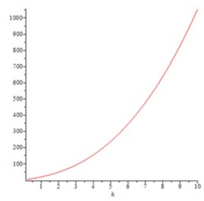
Plot for the first Zagreb index for the subdivision graph of Banana tree graph for n=5.



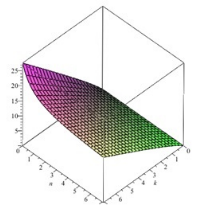
Plot for the Randic index for the subdivision graph of Banana tree graph for α=1/2.



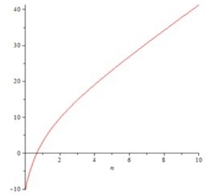
Plot for the Randic index for the subdivision graph of Banana tree graph for k=6 and α=1/2.



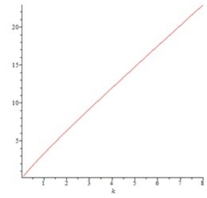
Plot for the Randic index for the subdivision graph of Banana tree graph for n=6 and α=1/2.



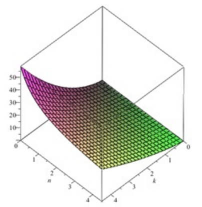
Plot for the Harmonic index for the subdivision graph of Banana tree graph.



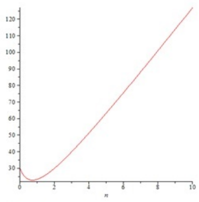
Plot for the Harmonic index for the subdivision graph of Banana tree graph for k=5.



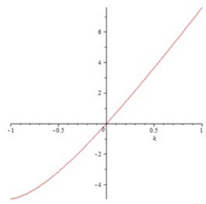
Plot for the Harmonic index for the subdivision graph of Banana tree graph for n=3.



Plot for the inverse sum index for the subdivision graph of Banana tree graph.



Plot for the inverse sum index index for the subdivision graph of Banana tree graph for k=5.



Plot for the inverse sum index for the subdivision graph of Banana tree graph for n=4.

3. M-polynomial of the complementary graph of Banana tree graph

In this second main section of this paper, we will demonstrate the M-polynomial of the complementary graph of Banana tree graph. Hence the other main result is the following.

THEOREM 3.1. *Let H be the complementary graph of Banana graph. Then the M-polynomial of the graph H is*

$$\begin{aligned} M(H;x,y) &= f(x,y) \\ &= \frac{k(k-1)}{2}x^{n(k-1)+1}y^{n(k-1)+1} + k(k-1)x^{n(k-1)+1}y^{kn-2} + \\ &\quad + kx^{n(k-1)+1}y^{k(n-1)} + (k^2n-kn-2k^2+2k)x^{n(k-1)+1}y^{kn-1} + \\ &\quad + \frac{k(k-1)}{2}x^{kn-2}y^{kn-2} + k^2(n-2)x^{kn-2}y^{kn-1} + \\ &\quad k(n-2)x^{k(n-1)}y^{kn-1} + \\ &\quad + \frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}x^{kn-1}y^{kn-1}. \end{aligned}$$

PROOF. In this proof, we will follow quite similar way as in the proof of Theorem 2.1. Hence, by considering the complementary graph H of Banana tree graph for arbitrary values n and k, we will make a generalization.

We note that the graph H contains $\frac{kn(kn-1)}{2}$ edges. There are eight types of edges in H as in the following:

The first edge partitions $E_1(G)$ contains $\frac{k(k-1)}{2}$ edges uv, where $d_u = n(k-1)+1 = d_v$.

The second edge partitions $E_2(G)$ contains $k(k-1)$ edges uv, where $d_u = n(k-1)+1$, $d_v = kn-2$.

The third edge partitions $E_3(G)$ contains k edges uv, where $d_u = n(k-1)+1$, $d_v = k(n-1)$.

The fourth edge partitions $E_4(G)$ contains $(k^2n - kn - 2k^2 + 2k)$ edges uv, where $d_u = n(k-1)+1$, $d_v = kn-1$.

The fifth edge partitions $E_5(G)$ contains $\frac{k(k-1)}{2}$ edges uv, where $d_u = kn-2$, $d_v = kn-2$.

The sixth edge partitions $E_6(G)$ contains $k^2(n-2)$ edges uv, where $d_u = kn-2$, $d_v = kn-1$.

The seventh edge partitions $E_7(G)$ contains $k(n-2)$ edges uv, where $d_u = k(n-1)$, $d_v = kn-1$.

Finally the eighth edge partitions $E_8(G)$ contains $\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}$ edges uv, where $d_u = kn-1$, $d_v = kn-1$.

Replacing G by H in Equation (1.1), we have

$$\begin{aligned}
M(H;x,y) \quad = \quad & \sum_{[n(k-1)+1]\leqslant[n(k-1)+1]} m_{[n(k-1)+1][n(k-1)+1]} x^{n(k-1)+1} y^{n(k-1)+1} + \\
& + \sum_{[n(k-1)+1]\leqslant(kn-2)} m_{[n(k-1)+1](kn-2)} x^{n(k-1)+1} y^{(kn-2)} + \\
& + \sum_{[n(k-1)+1]\leqslant k(n-1)} m_{[n(k-1)+1][k(n-1)]} x^{n(k-1)+1} y^{k(n-1)} + \\
& + \sum_{[n(k-1)+1]\leqslant(kn-1)} m_{[n(k-1)+1](kn-1)} x^{n(k-1)+1} y^{kn-1} + \\
& + \sum_{(kn-2)\leqslant(kn-2)} m_{(kn-2)(kn-2)} x^{kn-2} y^{kn-2} + \\
& + \sum_{(kn-2)\leqslant(mn-1)} m_{(kn-2)(kn-1)} x^{kn-2} y^{kn-1} + \\
& + \sum_{[k(n-1)]\leqslant(kn-1)} m_{[k(n-1)](kn-1)} x^{k(n-1)} y^{kn-1} + \\
& + \sum_{(kn-1)\leqslant(kn-1)} m_{(kn-1)(kn-1)} x^{kn-1} y^{kn-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned} M(H;x,y) &= \sum_{uv\in E_1(G)} m_{[n(k-1)+1][n(k-1)+1]} x^{n(k-1)+1} y^{n(k-1)+1} + \\ &+ \sum_{uv\in E_2(G)} m_{[n(k-1)+1](kn-2)} x^{n(k-1)+1} y^{kn-2} + \\ &+ \sum_{uv\in E_3(G)} m_{[n(k-1)+1][k(n-1)]} x^{n(k-1)+1} y^{k(n-1)} + \\ &+ \sum_{uv\in E_4(G)} m_{[n(k-1)+1](kn-1)} x^{n(k-1)+1} y^{kn-1} + \\ &+ \sum_{uv\in E_5(G)} m_{(kn-2)(kn-2)} x^{kn-2} y^{kn-2} + \\ &+ \sum_{uv\in E_6(G)} m_{(kn-2)(kn-1)} x^{kn-2} y^{kn-1} + \\ &+ \sum_{uv\in E_7(G)} m_{[k(n-1)](kn-1)} x^{k(n-1)} y^{kn-1} + \\ &+ \sum_{uv\in E_8(G)} m_{(kn-1)(kn-1)} x^{kn-1} y^{kn-1}. \end{aligned}$$

In fact, from this last equality we obtain

$$\begin{aligned} M(H;x,y) &= |E_1(G)|x^{n(k-1)+1}y^{n(k-1)+1} + |E_2(G)|x^{n(k-1)+1}y^{kn-2} + \\ &+|E_3(G)|x^{n(k-1)+1}y^{k(n-1)} + |E_4(G)|x^{n(k-1)+1}y^{kn-1} + \\ &+|E_5(G)|x^{kn-2}y^{kn-2} + |E_6(G)|x^{kn-2}y^{kn-1} + \\ &+|E_7(G)|x^{k(n-1)}y^{kn-1} + |E_8(G)|x^{kn-1}y^{kn-1} \end{aligned}$$

which is equal to the

$$\begin{aligned} &= \frac{k(k-1)}{2}x^{n(k-1)+1}y^{n(k-1)+1} + k(k-1)x^{n(k-1)+1}y^{kn-2} + \\ &\quad +kx^{n(k-1)+1}y^{k(n-1)} + (k^2n-kn-2k^2+2k)x^{n(k-1)+1}y^{kn-1} + \\ &\quad +\frac{k(k-1)}{2}x^{kn-2}y^{kn-2} + k^2(n-2)x^{kn-2}y^{kn-1} + \\ &\quad +k(n-2)x^{k(n-1)}y^{kn-1} + \\ &\quad +\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}x^{kn-1}y^{kn-1}. \end{aligned}$$

Hence the result. $\square$

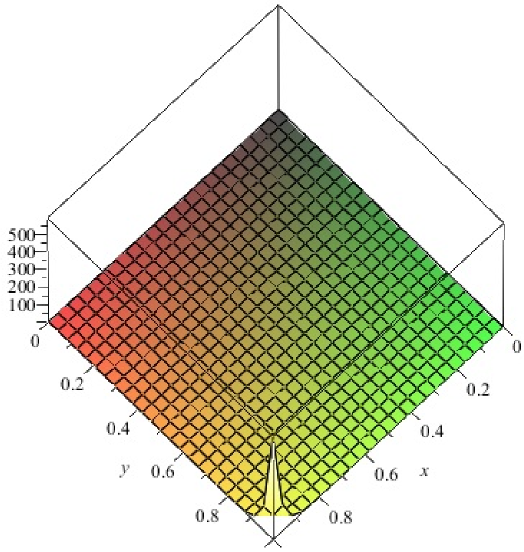


Figure 3: Plot of M-polynomial for the complementary graph of Banana tree graph.

As a next step of Theorem 3.1, let us compute some degree-based topological indices which were depicted in the first section of this paper for the complementary graph of Banana tree graph in terms of M-polynomial.

Corollary 3.1. *Let H be the complementary graph of the Banana tree graph. Then*

- $$M_1(H) = [n(k-1)+1]k(k-1) + (2kn-n-1)k(k-1) + \\ +(2kn-k-n+1)k + (2kn-n) + (k^2n-kn-2k^2+2k) + \\ +(kn-2)k(k-1) + (2kn-3)k^2(n-2) + \\ [k(n-1)+(kn-1)]k(n-2) + (kn-1)(k^2n^2-4k^2n-kn+4k^2+2k).$$

- $$M_2(H) = \frac{[n(k-1)+1]^2[k(k-1)]}{2} + (k-2)[n(k-1)+1][k(k-1)] + \\ +k^2(n-1)[n(k-1)+1] + \\ +(kn-1)[n(k-1)+1](k^2n-kn-2k^2+2k) + \frac{(kn-2)^2[k(k-1)]}{2} + \\ +(kn-1)(kn-2)[k^2(n-2)] + k^2(kn-1)(n-1)(n-2) + (kn-1)^2 \\ \frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}.$$

- $${}^mM_2(H) = \frac{k(k-1)}{2[n(k-1)+1]^2} + \frac{k(k-1)}{(kn-2)[n(k-1)+1]} + \\ +\frac{1}{(n-1)[n(k-1)+1]} +$$

$$\frac{k^2n-kn-2k^2+2k}{(kn-1)[n(k-1)+1]}+\frac{k(k-1)}{2(kn-2)^2}+$$
$$+\frac{k^2(n-2)}{(kn-1)(kn-2)}+\frac{k(n-2)}{(kn-1)[k(n-1)]}$$
$$\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2(kn-1)^2}.$$

- $R_\alpha(H)=\frac{[n(k-1)+1]^{2(\alpha+1)}[k(k-1)]}{2}+(kn-2)^{\alpha+1}[n(k-1)+1]^{\alpha+1}$
$k(k-1)+[k(n-1)]^{\alpha+1}$
$[n(k-1)+1]^{\alpha+1}k+(kn-1)^{\alpha+1}[n(k-1)+1]^{\alpha+1}(k^2n-kn-2k^2+2k)+$
$\frac{(kn-2)^{2(\alpha+1)}[k(k-1)]}{2}+(kn-1)^{\alpha+1}(kn-2)^{\alpha+1}k^2(n-2)+(kn-1)^{\alpha+1}$
$[k(n-1)]^{\alpha+1}k(n-2)+\frac{(kn-1)^{2(\alpha+1)}(k^2n^2-4k^2n-kn+4k^2+2k)}{2}.$

- $$RR_\alpha(H)=\frac{k(k-1)}{2[n(k-1)+1]^{2\alpha}}+\frac{k(k-1)}{(kn-2)^\alpha[n(k-1)+1]^\alpha}+$$
$$+\frac{k}{[k(n-1)]^\alpha[n(k-1)+1]^\alpha}$$
$$+\frac{k^2n-kn-2k^2+2k}{(kn-1)^\alpha[n(k-1)+1]^\alpha}+\frac{k(k-1)}{2(kn-2)^{2\alpha}}+\frac{k^2(n-2)}{(kn-1)^\alpha(kn-2)^\alpha}$$
$$+\frac{k(n-2)}{(kn-1)^\alpha[k(n-1)]^\alpha}+\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2(kn-1)^{2\alpha}}.$$

- $$SDD(H)=k(k-1)+\{\frac{[n(k-1)+1]}{(kn-2)}+\frac{(kn-2)}{[n(k-1)+1]}\}[k(k-1)]+$$
$$+\frac{[n(k-1)+1]}{(n-1)}+\frac{k^2(n-1)}{[n(k-1)+1]}+$$
$$+\{\frac{[n(k-1)+1]}{(kn-1)}+\frac{(kn-1)}{[n(k-1)+1]}\}(k^2n-kn-2k^2+2k)$$
$$+k(k-1)+\{\frac{(kn-2)}{(kn-1)}+\frac{(kn-1)}{kn-2}\}[k^2(n-2)]+$$
$$+\{\frac{k(n-1)}{kn-1}+\frac{(kn-1)}{k(n-1)}\}[k(n-2)]+$$
$$+(k^2n^2-4k^2n-kn+4k^2+2k).$$

- $$H(H)=\frac{k(k-1)}{2[n(k-1)+1]}+\frac{2k(k-1)}{2kn-(n+1)}+\frac{2k}{2kn-(k+n)+1}+$$
$$+\frac{2(k^2n-kn-2k^2+2k)}{2mn-n}$$

$$+\frac{k(k-1)}{2(kn-2)}+\frac{2k^2(n-2)}{2kn-3}+\frac{2k(n-2)}{2kn-(k+1)}+$$
$$+\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2(kn-1)}.$$

- $$I(H)=\frac{[n(k-1)+1][k(k-1)]}{8}+\frac{(k-2)[n(k-1)+1][k(k-1)]}{2kn-(n+1)}+$$
$$+\frac{k^2(n-1)[n(k-1)+1]}{2kn-(k+n)+1}$$
$$+\frac{(kn-1)[n(k-1)+1](k^2n-kn-2k^2+2k)}{2kn-n}+\frac{(kn-2)[k(k-1)]}{4}$$
$$+\frac{(kn-1)(kn-2)[k^2(n-2)]}{2kn-3}+\frac{k^2(kn-1)(n-1)(n-2)}{2kn-(k+1)}$$
$$+\frac{(kn-1)(k^2n^2-4k^2n-kn+4k^2+2k)}{4}.$$

- $$A(H)=\frac{[n(k-1)+1]^6[k(k-1)]}{2[[n(k-1)]^3}+\frac{(kn-2)^3[n(k-1)+1]^3[k(k-1)]}{[2kn-(n+3)]^3}$$
$$+\frac{[k(n-1)]^3[n(k-1)+1]^3k}{[2kn-(k+n+1)]^3}+$$
$$+\frac{(kn-1)^3[n(k-1)+1]^3(k^2n-kn-2k^2+2k)}{[2kn-(n+2)]^3}$$
$$+\frac{(kn-2)^6[k(k-1)]}{2[2(kn-2)]^3}+\frac{(kn-1)^3(kn-2)^3k^2(n-2)}{(2kn-5)^3}+$$
$$+\frac{(kn-1)^3[k(n-1)^3][k(n-2)]}{[2kn-(k+3)]^3}$$
$$+(kn-1)^6\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2[2(kn-2)]^3}.$$

Proof. Considering the equality in Theorem 3.1, we have

$$\begin{aligned}D_xf(x,y) \quad = \quad & \frac{k(k-1)[n(k-1)+1]}{2}x^{n(k-1)+1}y^{n(k-1)+1}+\\ & +[n(k-1)+1]k(k-1)x^{n(k-1)+1}y^{kn-2}\\ & +[n(k-1)+1]kx^{n(k-1)+1}y^{k(n-1)}+\\ & +[n(k-1)+1](k^2n-kn-2k^2+2k)x^{n(k-1)+1}\\ & y^{kn-1}+(kn-2)\frac{k(k-1)}{2}x^{kn-2}y^{kn-2}+\\ & (kn-2)k^2(n-2)x^{kn-2}y^{kn-1}+k^2(n-1)\\ & (n-2)x^{k(n-1)}y^{kn-1}+(kn-1)\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}\\ & x^{kn-1}y^{kn-1},\end{aligned}$$

$$\begin{aligned}
D_y f(x,y) &= [n(k-1)+1]\frac{k(k-1)}{2}x^{n(k-1)+1}y^{n(k-1)+1}+\\
&+(kn-2)k(k-1)x^{n(k-1)+1}y^{kn-2}+\\
&+k^2(n-1)x^{n(k-1)+1}y^{k(n-1)}+(kn-1)(k^2n-kn-2k^2+2k)\\
&x^{n(k-1)+1}y^{kn-1}+(kn-2)\frac{k(k-1)}{2}x^{kn-2}y^{kn-2}+\\
&(kn-1)k^2(n-2)x^{kn-2}y^{kn-1}+\\
&(kn-1)[k(n-2)]x^{k(n-1)}y^{kn-1}+\\
&(kn-1)\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}x^{kn-1}y^{kn-1},
\end{aligned}$$

$$\begin{aligned}
D_yD_x f(x,y) &= [n(k-1)+1]^2\frac{k(k-1)}{2}x^{n(k-1)+1}y^{n(k-1)+1}+\\
&+(k-2)[n(k-1)+1][k(k-1)]x^{n(k-1)+1}y^{kn-2}+\\
&+k^2(n-1)[n(k-1)+1]x^{n(k-1)+1}y^{k(n-1)}\\
&+(kn-1)[n(k-1)+1]\\
&(k^2n-kn-2k^2+2k)x^{n(k-1)+1}y^{kn-1}+(kn-2)^2\\
&\frac{k(k-1)}{2}x^{kn-2}y^{kn-2}+(kn-1)\\
&(kn-2)[k^2(n-2)]x^{kn-2}y^{kn-1}+k^2(kn-1)\\
&(n-1)(n-2)x^{k(n-1)}y^{kn-1}+(kn-1)^2\\
&\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}x^{kn-1}y^{kn-1},
\end{aligned}$$

$$\begin{aligned}
S_y(f(x,y)) &= \frac{k(k-1)}{2[n(k-1)+1]}x^{n(k-1)+1}y^{n(k-1)+1}+\frac{k(k-1)}{(kn-2)}\\
&x^{n(k-1)+1}y^{kn-2}+\frac{1}{(n-1)}x^{n(k-1)+1}y^{k(n-1)}\\
&+\frac{k^2n-kn-2k^2+2k}{(kn-1)}x^{n(k-1)+1}y^{kn-1}+\frac{k(k-1)}{2(kn-2)}\\
&x^{kn-2}y^{kn-2}+\frac{k^2(n-2)}{kn-1}x^{kn-2}y^{kn-1}\\
&+\frac{k(n-2)}{kn-1}x^{k(n-1)}y^{kn-1}+\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2(kn-1)}\\
&x^{kn-1}y^{kn-1},
\end{aligned}$$

$$\begin{aligned}
S_xS_y(f(x,y)) &= \frac{k(k-1)}{2[n(k-1)+1]^2}x^{n(k-1)+1}y^{n(k-1)+1}+\\
&+\frac{k(k-1)}{(kn-2)[n(k-1)+1]}x^{n(k-1)+1}y^{kn-2}
\end{aligned}$$

$$+\frac{1}{(n-1)[n(k-1)+1]}x^{n(k-1)+1}y^{k(n-1)}+$$
$$+\frac{k^2n-kn-2k^2+2k}{(kn-1)[n(k-1)+1]}x^{n(k-1)+1}y^{kn-1}$$
$$+\frac{k(k-1)}{2(kn-2)^2}x^{kn-2}y^{kn-2}+$$
$$+\frac{k^2(n-2)}{(kn-1)(kn-2)}x^{kn-2}y^{kn-1}$$
$$+\frac{k(n-2)}{(kn-1)[k(n-1)]}x^{k(n-1)}y^{kn-1}+$$
$$+\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2(kn-1)^2}$$
$$x^{kn-1}y^{kn-1},$$

$$\begin{aligned} D_y^\alpha(f(x,y)) &= [n(k-1)+1]^{\alpha+1}\frac{k(k-1)}{2}x^{n(k-1)+1}y^{n(k-1)+1}+ \\ &(kn-2)^{\alpha+1}k(k-1)x^{n(k-1)+1}y^{kn-2} \\ &+[k(n-1)]^{\alpha+1}kx^{n(k-1)+1}y^{k(n-1)}+ \\ &(kn-1)^{\alpha+1}(k^2n-kn-2k^2+2k) \\ &x^{n(k-1)+1}y^{kn-1}+(kn-2)^{\alpha+1} \\ &\frac{k(k-1)}{2}x^{kn-2}y^{kn-2}+(kn-1)^{\alpha+1}k^2(n-2)x^{kn-2}y^{kn-1} \\ &+(kn-1)^{\alpha+1}[k(n-2)]x^{k(n-1)}y^{kn-1}+(kn-1)^{\alpha+1} \\ &\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2} \\ &x^{kn-1}y^{kn-1}, \end{aligned}$$

$$\begin{aligned} D_x^\alpha D_y^\alpha(f(x,y)) &= [n(k-1)+1]^{2(\alpha+1)}\frac{k(k-1)}{2}x^{n(k-1)+1}y^{n(k-1)+1}+ \\ &(kn-2)^{\alpha+1}[n(k-1)+1]^{\alpha+1} \\ &k(k-1)x^{n(k-1)+1}y^{kn-2}+[k(n-1)]^{\alpha+1}[n(k-1)+1]^{\alpha+1} \\ &kx^{n(k-1)+1}y^{k(n-1)} \\ &+(kn-1)^{\alpha+1}[n(k-1)+1]^{\alpha+1}(k^2n-kn-2k^2+2k) \\ &x^{n(k-1)+1}y^{kn-1}+ \\ &(kn-2)^{2(\alpha+1)}\frac{k(k-1)}{2}x^{kn-2}y^{kn-2}+ \\ &(kn-1)^{\alpha+1}(kn-2)^{\alpha+1}k^2(n-2)x^{kn-2}y^{kn-1} \\ &+(kn-1)^{\alpha+1}[k(n-1)]^{\alpha+1}k(n-2)x^{k(n-1)}y^{kn-1}+ \\ &(kn-1)^{2(\alpha+1)} \end{aligned}$$

$$\frac{k^2n^2 - 4k^2n - kn + 4k^2 + 2k}{2}$$
$$x^{kn-1}y^{kn-1},$$

$$\begin{aligned}
S_y^\alpha(f(x,y)) &= \frac{k(k-1)}{2[n(k-1)+1]^\alpha}x^{n(k-1)+1}y^{n(k-1)+1} \\
&+\frac{k(k-1)}{(kn-2)^\alpha}x^{n(k-1)+1}y^{kn-2} \\
&+\frac{k}{[k(n-1)]^\alpha}x^{n(k-1)+1}y^{k(n-1)} \\
&+\frac{k^2n - kn - 2k^2 + 2k}{(kn-1)^\alpha}x^{n(k-1)+1}y^{kn-1} \\
&+\frac{k(k-1)}{2(kn-2)^\alpha}x^{kn-2}y^{kn-2} \\
&+\frac{k^2(n-2)}{(kn-1)^\alpha}x^{kn-2}y^{kn-1} + \\
&\frac{k(n-2)}{(kn-1)^\alpha}x^{k(n-1)}y^{kn-1} \\
&+\frac{k^2n^2 - 4k^2n - kn + 4k^2 + 2k}{2(kn-1)^\alpha}x^{kn-1}y^{kn-1},
\end{aligned}$$

$$\begin{aligned}
S_x^\alpha S_y^\alpha(f(x,y)) &= \frac{k(k-1)}{2[n(k-1)+1]^{2\alpha}}x^{n(k-1)+1}y^{n(k-1)+1} \\
&+\frac{k(k-1)}{(kn-2)^\alpha[n(k-1)+1]^\alpha}x^{n(k-1)+1}y^{kn-2} + \\
&\frac{k}{[k(n-1)]^\alpha[n(k-1)+1]^\alpha}x^{n(k-1)+1}y^{k(n-1)} \\
&+\frac{k^2n - kn - 2k^2 + 2k}{(kn-1)^\alpha[n(k-1)+1]^\alpha}x^{n(k-1)+1} \\
&y^{kn-1} + \frac{k(k-1)}{2(kn-2)^{2\alpha}}x^{kn-2}y^{kn-2} + \\
&\frac{k^2(n-2)}{(kn-1)^\alpha(kn-2)^\alpha}x^{kn-2}y^{kn-1} + \\
&\frac{k(n-2)}{(kn-1)^\alpha[k(n-1)]^\alpha}x^{k(n-1)}y^{kn-1} \\
&+\frac{k^2n^2 - 4k^2n - kn + 4k^2 + 2k}{2(kn-1)^{2\alpha}} \\
&x^{kn-1}y^{kn-1},
\end{aligned}$$

$$S_yD_x(f(x,y)) = \frac{k(k-1)}{2}x^{n(k-1)+1}y^{n(k-1)+1} +$$

$$\frac{[n(k-1)+1][k(k-1)]}{(kn-2)}x^{n(k-1)+1}y^{kn-2}$$
$$+\frac{[n(k-1)+1]}{(n-1)}x^{n(k-1)+1}y^{k(n-1)}+$$
$$\frac{[n(k-1)+1](k^2n-kn-2k^2+2k)}{(kn-1)}x^{n(k-1)+1}$$
$$y^{kn-1}+\frac{k(k-1)}{2}x^{kn-2}y^{kn-2}+$$
$$\frac{(kn-2)[k^2(n-2)]}{kn-1}x^{kn-2}y^{kn-1}+$$
$$\frac{[k(n-1)]k(n-2)}{kn-1}$$
$$x^{k(n-1)}y^{kn-1}+$$
$$\frac{k^2n^2-4k^2n-kn+4m^2+2k}{2}x^{kn-1}y^{kn-1},$$

$$\begin{aligned}
S_xD_y(f(x,y)) \quad = \quad & \frac{k(k-1)}{2}x^{n(k-1)+1}y^{n(k-1)+1}+\\
& \frac{k(k-1)(kn-2)}{[n(k-1)+1]}x^{n(k-1)+1}y^{kn-2}+\\
& \frac{k^2(n-1)}{[n(k-1)+1]}x^{n(k-1)+1}y^{k(n-1)}+\\
& \frac{(kn-1)(k^2n-kn-2k^2+2k)}{[n(k-1)+1]}x^{n(k-1)+1}y^{kn-1}\\
& +\frac{k(k-1)}{2}x^{kn-2}y^{kn-2}+\\
& \frac{(k-1)[k^2(n-2)]}{kn-2}x^{kn-2}y^{kn-1}+\\
& \frac{(kn-1)(n-2)}{n-1}x^{k(n-1)}y^{kn-1}\\
& +\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}x^{kn-1}y^{n-1},
\end{aligned}$$

$$\begin{aligned}
Jf(x,y) \quad = \quad & \frac{k(k-1)}{2}x^{2[n(k-1)+1]}+k(k-1)x^{2kn-(n+1)}+\\
& kx^{2kn-(k+n)+1}+(k^2n-kn-2k^2+2k)\\
& x^{2kn-n}+\frac{k(k-1)}{2}x^{2(kn-2)}+k^2(n-2)\\
& x^{2kn-3}+k(n-2)x^{2kn-(k+1)}\\
& +\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}x^{2(kn-1)},
\end{aligned}$$

$$\begin{aligned}
S_xJf(x,y) &= \frac{k(k-1)}{4[n(k-1)+1]}x^{2[n(k-1)+1]}\\
&+\frac{k(k-1)}{2kn-(n+1)}x^{2kn-(n+1)}+\frac{k}{2kn-(k+n)+1}\\
&x^{2kn-(k+n)+1}+\frac{(k^2n-kn-2k^2+2k)}{2mn-n}\\
&x^{2kn-n}+\frac{k(k-1)}{4(kn-2)}x^{2(kn-2)}+\\
&\frac{k^2(n-2)}{2kn-3}x^{2kn-3}+\frac{k(n-2)}{2kn-(k+1)}x^{2kn-(k+1)}+\\
&\frac{k^2n^2-4k^2n-kn+4k^2+2k}{4(kn-1)}x^{2(kn-1)},
\end{aligned}$$

$$\begin{aligned}
JD_xD_yf(x,y) &= [n(k-1)+1]^2\frac{k(k-1)}{2}x^{2[n(k-1)+1]}+\\
&(k-2)[n(k-1)+1][k(k-1)]x^{2kn-(n+1)}\\
&+k^2(n-1)[n(k-1)+1]x^{2kn-(k+n)+1}+\\
&(kn-1)[n(k-1)+1](k^2n-kn-2k^2+2k)x^{2kn-n}\\
&+(kn-2)^2\frac{k(k-1)}{2}x^{2(kn-2)}+(kn-1)\\
&(kn-2)[k^2(n-2)]x^{2kn-3}+\\
&+k^2(kn-1)(n-1)(n-2)x^{2kn-(k+1)}\\
&+\frac{(kn-1)^2(k^2n^2-4k^2n-kn+4k^2+2k)}{2}x^{2(kn-1)},
\end{aligned}$$

$$\begin{aligned}
S_xJD_xD_yf(x,y) &= \frac{[n(k-1)+1][k(k-1)]}{8}x^{2[n(k-1)+1]}+\\
&\frac{(k-2)[n(k-1)+1][k(k-1)]}{2kn-(n+1)}x^{2kn-(n+1)}+\\
&+\frac{k^2(n-1)[n(k-1)+1]}{2kn-(k+n)+1}x^{2kn-(k+n)+1}+\\
&\frac{(kn-1)[n(k-1)+1](k^2n-kn-2k^2+2k)}{2kn-n}x^{2kn-n}\\
&+\frac{(kn-2)[k(k-1)]}{4}x^{2(kn-2)}\\
&+\frac{(kn-1)(kn-2)[k^2(n-2)]}{2kn-3}x^{2kn-3}+\\
&\frac{k^2(kn-1)(n-1)(n-2)}{2kn-(k+1)}x^{2kn-(k+1)}+\\
&+\frac{(kn-1)(k^2n^2-4k^2n-kn+4k^2+2k)}{4}x^{2(kn-1)},
\end{aligned}$$

$$\begin{aligned}
D_y^3 f(x,y) &= [n(k-1)+1]^3 \frac{k(k-1)}{2} x^{n(k-1)+1} y^{n(k-1)+1} + \\
&(kn-2)^3 k(k-1) x^{n(k-1)+1} y^{kn-2} \\
&+[k(n-1)]^3 k x^{n(k-1)+1} y^{k(n-1)} + \\
&(kn-1)^3 (k^2 n - kn - 2k^2 + 2k) x^{n(k-1)+1} y^{kn-1} \\
&+(kn-2)^3 \frac{k(k-1)}{2} x^{kn-2} y^{kn-2} + \\
&(kn-1)^3 k^2 (n-2) x^{kn-2} y^{kn-1} + (kn-1)^3 \\
&[k(n-2)] x^{k(n-1)} y^{kn-1} + (kn-1)^3 \\
&\frac{k^2 n^2 - 4k^2 n - kn + 4k^2 + 2k}{2} x^{kn-1} y^{kn-1},
\end{aligned}$$

$$\begin{aligned}
D_x^3 D_y^3 f(x,y) &= \frac{[n(k-1)+1]^6 [k(k-1)]}{2} x^{n(k-1)+1} y^{n(k-1)+1} + \\
&(kn-2)^3 [n(k-1)+1]^3 [k(k-1)] \\
&x^{n(k-1)+1} y^{kn-2} + [k(n-1)]^3 [n(k-1)+1]^3 \\
&k x^{n(k-1)+1} y^{k(n-1)} + (kn-1)^3 \\
&[n(k-1)+1]^3 (k^2 n - kn - 2k^2 + 2k) x^{n(k-1)+1} y^{kn-1} \\
&+(kn-2)^6 \frac{k(k-1)}{2} x^{kn-2} y^{kn-2} \\
&+(kn-1)^3 (kn-2)^3 k^2 (n-2) x^{kn-2} y^{kn-1} + \\
&(kn-1)^3 [k(n-1)^3][k(n-2)] x^{k(n-1)} \\
&y^{kn-1} + (kn-1)^6 \frac{k^2 n^2 - 4k^2 n - kn + 4k^2 + 2k}{2} \\
&x^{kn-1} y^{kn-1},
\end{aligned}$$

$$\begin{aligned}
J D_x^3 D_y^3 f(x,y) &= \frac{[n(k-1)+1]^6 [k(k-1)]}{2} x^{2[n(k-1)+1]} + \\
&(kn-2)^3 [n(k-1)+1]^3 [k(k-1)] x^{2kn-(n+1)} \\
&+[k(n-1)]^3 [n(k-1)+1]^3 k x^{2kn-(k+n)+1} + \\
&(kn-1)^3 [n(k-1)+1]^3 \\
&(k^2 n - kn - 2k^2 + 2k) x^{2kn-n} + (kn-2)^6 \\
&\frac{k(k-1)}{2} x^{2(kn-2)} + (kn-1)^3 (kn-2)^3 \\
&k^2 (n-2) x^{2kn-3} + (kn-1)^3 [k(n-1)^3][k(n-2)] \\
&x^{2kn-(k+1)} + (kn-1)^6 \\
&\frac{k^2 n^2 - 4k^2 n - kn + 4k^2 + 2k}{2} x^{2(kn-1)},
\end{aligned}$$

$$\begin{aligned} Q_{-2}JD_x^3D_y^3f(x,y) &= \frac{[n(k-1)+1]^6[k(k-1)]}{2}x^{2[n(k-1)]}+ \\ & (kn-2)^3[n(k-1)+1]^3[k(k-1)]x^{2kn-(n+3)} \\ & +[k(n-1)]^3[n(k-1)+1]^3kx^{2kn-(k+n+1)}+(kn-1)^3 \\ & [n(k-1)+1]^3(k^2n-kn-2k^2+2k)x^{2kn-(n+2)} \\ & +\frac{(kn-2)^6[k(k-1)]}{2}x^{2(kn-3)} \\ & +(kn-1)^3(kn-2)^3k^2(n-2)x^{2kn-5}+ \\ & (kn-1)^3[k(n-1)^3][k(n-2)] \\ & x^{2kn-(k+3)}+(kn-1)^6 \\ & \frac{k^2n^2-4k^2n-kn+4k^2+2k}{2}x^{2(kn-2)}, \end{aligned}$$

$$\begin{aligned} S_x^3Q_{-2}JD_x^3D_y^3f(x,y) &= \frac{[n(k-1)+1]^6[k(k-1)]}{2[2n(k-1)]^3}x^{2n(k-1)}+ \\ & \frac{(kn-2)^3[n(k-1)+1]^3[k(k-1)]}{[2kn-(n+3)]^3} \\ & x^{2kn-(n+3)} \\ & +\frac{[k(n-1)]^3[n(k-1)+1]^3k}{[2kn-(k+n+1)]^3}x^{2kn-(k+n+1)} \\ & +\frac{(kn-1)^3[n(k-1)+1]^3(k^2n-kn-2k^2+2k)}{[2kn-(n+2)]^3} \\ & x^{2kn-(n+2)} \\ & +\frac{(kn-2)^6[k(k-1)]}{2[2(kn-2)]^3}x^{2(kn-2)}+ \\ & \frac{(kn-1)^3(kn-2)^3k^2(n-2)}{(2kn-5)^3}x^{2kn-5} \\ & +\frac{(kn-1)^3[k(n-1)^3][k(n-2)]}{[2kn-(k+3)]^3} \\ & x^{2kn-(k+3)}+(kn-1)^6 \\ & \frac{k^2n^2-4k^2n-kn+4k^2+2k}{2[2(kn-2)]^3} \\ & x^{2(kn-2)}. \end{aligned}$$

Then by using the *Table*, we have the following graphs of different indices.

$$\begin{aligned} {}^mM_2(H) &= S_xS_yf(x,y)|_{x=y=1} \\ &= \frac{k(k-1)}{2[n(k-1)+1]^2}+\frac{k(k-1)}{(kn-2)[n(k-1)+1]}+\frac{1}{(n-1)[n(k-1)+1]} \end{aligned}$$

$$+\frac{k^2n-kn-2k^2+2k}{(kn-1)[n(k-1)+1]}+\frac{k(k-1)}{2(kn-2)^2}+$$
$$\frac{k^2(n-2)}{(kn-1)(kn-2)}+$$
$$\frac{k(n-2)}{(kn-1)[k(n-1)]}+\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2(kn-1)^2}.$$

$$\begin{aligned}
RR_\alpha(H) &= S_x^\alpha S_y^\alpha f(x,y)|_{x=y=1}\\
&= \frac{k(k-1)}{2[n(k-1)+1]^{2\alpha}}+\frac{k(k-1)}{(kn-2)^\alpha[n(k-1)+1]^\alpha}\\
&\quad+\frac{k}{[k(n-1)]^\alpha[n(k-1)+1]^\alpha}\\
&\quad+\frac{k^2n-kn-2k^2+2k}{(kn-1)^\alpha[n(k-1)+1]^\alpha}+\frac{k(k-1)}{2(kn-2)^{2\alpha}}+\frac{k^2(n-2)}{(kn-1)^\alpha(kn-2)^\alpha}\\
&\quad+\frac{k(n-2)}{(kn-1)^\alpha[k(n-1)]^\alpha}\\
&\quad+\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2(kn-1)^{2\alpha}}.
\end{aligned}$$

$$\begin{aligned}
SDD(H) &= (D_xS_y+S_xD_y)f(x,y)|_{x=y=1}\\
&= k(k-1)+\{\frac{[n(k-1)+1]}{(kn-2)}+\frac{(kn-2)}{[n(k-1)+1]}\}[k(k-1)]+\\
&\quad\frac{[n(k-1)+1]}{(n-1)}\\
&\quad+\frac{k^2(n-1)}{[n(k-1)+1]}+\{\frac{[n(k-1)+1]}{(kn-1)}+\frac{(kn-1)}{[n(k-1)+1]}\}\\
&\quad(k^2n-kn-2k^2+2k)\\
&\quad+k(k-1)+\{\frac{(kn-2)}{(kn-1)}+\frac{(kn-1)}{kn-2}\}[k^2(n-2)]+\\
&\quad\{\frac{k(n-1)}{kn-1}+\frac{(kn-1)}{k(n-1)}\}[k(n-2)]\\
&\quad+(k^2n^2-4k^2n-kn+4k^2+2k).
\end{aligned}$$

$$\begin{aligned}
H(H) &= 2S_xJf(x,y)|_{x=y=1}\\
&= \frac{k(k-1)}{2[n(k-1)+1]}+\frac{2k(k-1)}{2kn-(n+1)}+\frac{2k}{2kn-(k+n)+1}+\\
&\quad\frac{2(k^2n-kn-2k^2+2k)}{2mn-n}\\
&\quad+\frac{k(k-1)}{2(kn-2)}+\frac{2k^2(n-2)}{2kn-3}+\frac{2k(n-2)}{2kn-(k+1)}+
\end{aligned}$$

$$\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2(kn-1)}.$$

$$\begin{aligned}
I(H) &= (S_xJD_xD_y)f(x,y)|_{x=y=1} \\
&= \frac{[n(k-1)+1][k(k-1)]}{8}+\frac{(k-2)[n(k-1)+1][k(k-1)]}{2kn-(n+1)}+ \\
&\quad \frac{k^2(n-1)[n(k-1)+1]}{2kn-(k+n)+1} \\
&\quad +\frac{(kn-1)[n(k-1)+1](k^2n-kn-2k^2+2k)}{2kn-n}+\frac{(kn-2)[k(k-1)]}{4} \\
&\quad +\frac{(kn-1)(kn-2)[k^2(n-2)]}{2kn-3}+\frac{k^2(kn-1)(n-1)(n-2)}{2kn-(k+1)} \\
&\quad +\frac{(kn-1)(k^2n^2-4k^2n-kn+4k^2+2k)}{4}.
\end{aligned}$$

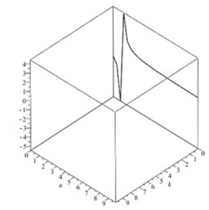
$$\begin{aligned}
A(H) &= (S_x^3Q_{-2}JD_x^3D_y^3)f(x,y)|_{x=y=1} \\
&= \frac{[n(k-1)+1]^6[k(k-1)]}{2[[n(k-1)]^3}+\frac{(kn-2)^3[n(k-1)+1]^3[k(k-1)]}{[2kn-(n+3)]^3} \\
&\quad +\frac{[k(n-1)]^3[n(k-1)+1]^3k}{[2kn-(k+n+1)]^3}+ \\
&\quad \frac{(kn-1)^3[n(k-1)+1]^3(k^2n-kn-2k^2+2k)}{[2kn-(n+2)]^3} \\
&\quad +\frac{(kn-2)^6[k(k-1)]}{2[2(kn-2)]^3}+\frac{(kn-1)^3(kn-2)^3k^2(n-2)}{(2kn-5)^3}+ \\
&\quad \frac{(kn-1)^3[k(n-1)^3][k(n-2)]}{[2kn-(k+3)]^3} \\
&\quad +(kn-1)^6\frac{k^2n^2-4k^2n-kn+4k^2+2k}{2[2(kn-2)]^3}.
\end{aligned}$$

Hence the result.

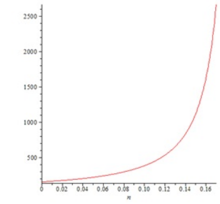
□

3.1. Surfaces representing M-polynomials of complementary graph of Banana tree graph.

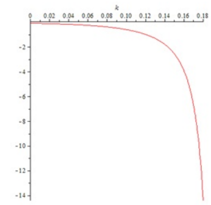
We used Maple 15 to represent graphs of M-polynomials of the complementary graphs of Banana tree graph presented in the proof of Corollary 3.1. From these graphs, it can seen that the behavior of the polynomials differ along different parameters.



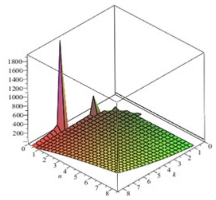
Plot for the modified Zagreb index for the complementary graph of Banana tree graph.



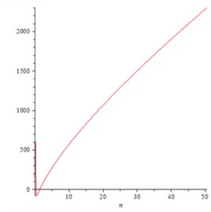
Plot for the modified Zagreb index for the complementary graph of Banana tree graph for k=5.



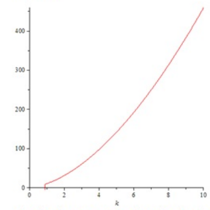
Plot for the modified Zagreb index for the complementary graph of Banana tree graph for n=5.



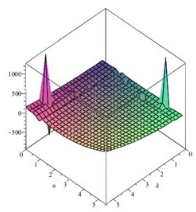
Plot for the inverse Randic index for the complementary graph of Banana tree graph for α=1/2.



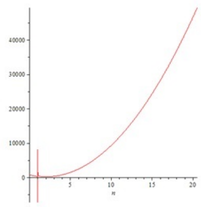
Plot for the inverse Randic index for the complementary graph of Banana tree graph for k=10 and α=1/2.



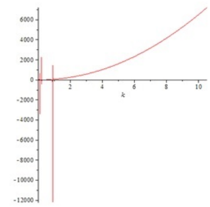
Plot for the inverse Randic index for the complementary graph of Banana tree graph for n=8 and α=1/2.



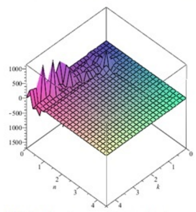
Plot for the SDD index for the complementary graph of Banana tree graph.



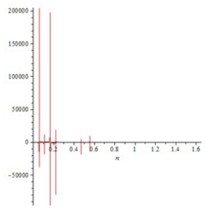
Plot for the SDD index for the complementary graph of Banana tree graph for k=12.



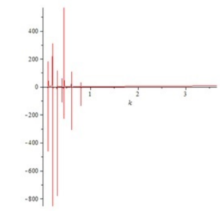
Plot for the SDD index for the complementary graph of Banana tree graph for n=10.



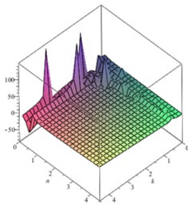
Plot for the Harmonic index for the complementary graph of Banana tree graph.



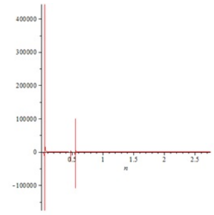
Plot for the Harmonic index for the complementary graph of Banana tree graph for k=9.



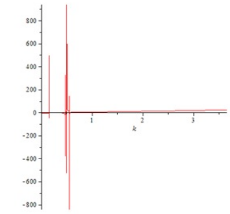
Plot for the Harmonic index for the complementary graph of Banana tree graph for n=5.



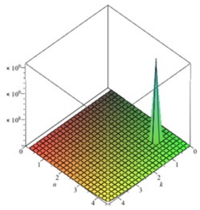
Plot for the inverse sum index for the complementary graph of Banana tree graph.



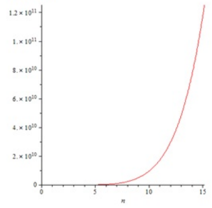
Plot for the inverse sum index for the complementary graph of Banana tree graph for k=10.



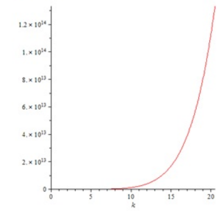
Plot for the inverse sum index for the complementary graph of Banana tree graph for n=10.



Plot for the Augmented Zagreb index for the complementary graph of Banana tree graph.



Plot for Augmented Zagreb index for the complementary graph of Banana tree graph for k=5.



Plot for Augmented Zagreb index for the complementary graph of Banana tree graph for n=11.

References

[1] M. S. Ahmad, W. Nazeer, S. M. Kaug and C. Y. Jung. Some computational aspects for the line graph of Banana tree graph. *Global J. Pure Appl. Math.*, **13**(6)(2017), 2601–2627.

[2] V. Alexander. Upper and lower bounds of Symmetric division deg index. *Iranian J. Math. Chem.*, **5**(2)(2014), 91–98.

[3] A. T. Balaban. Highly discriminating distance based numerical descriptor. *Chem. Phys. Lett.*, **89**(1982), 399–404.

[4] F. M. Bruckler, T. Doslic, A. Graovac and I. Gutman. On a class of distance-based molecular structure descriptors. *Chem. Phys. Lett.*, **503**(2011), 336–338.

[5] D. Vukićević, M. Gasperov. Bond additive modeling 1. Adriatic indices. *Croat. Chem. Acta*, **83**(3)(2010), 243–260.

[6] H. Deng, G. Huang and X. Jiang. A unified linear-programming modeling of some topological indices. *J. Comb. Optim.*, **30**(2015), 826–837.

[7] H. Deng, J. Yang and F. Xia. A general modeling of some vertex-degree based topological indices in benzenoid systems and phenylenes. *Comp. Math. Appl.*, **61**(2011), 3017–3023.

[8] E. Deutsch and S. Klavzar. M-Polynomial and degree-based topological indices. *Iran. J. Math. Chem.*, **6**(2015), 93–102.

[9] B. Furtula, A. Graovac and D. Vukićević. Augmented Zagreb index. *J. Math. Chem.*, **48**(2010), 370–380.

[10] C. K. Gupta, V. Lokesha, B. S. Shetty and P. S. Ranjini. On the Symmetric division deg index of graph. *South. Asian Bull. Math.*, **41**(1)(2016), 1–23.

[11] I. Gutman. Some properties of the Wiener polynomials. *Graph Theory Notes, New York*, **125**(1993), 13–18.

[12] I. Gutman and K. Ch. Das. The first Zagreb indices 30 years after. *MATCH Commun. Math. Comput. Chem.*, **50**(2004), 83–92.

[13] I. Gutman and N. Trinajstić. Graph theory and molecular orbitals. Total φ-electron energy of alternate hydrocarbons. *Chem. Phys. Lett.*, **17**(1972), 535–538.

[14] A. Ilić and D. Stevanović. On comparing Zagreb indices. *MATCH Commun. Math. Comput. Chem.*, **62**(2009), 681–687.

[15] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi. The first and second Zagreb indices of some graph operations. *Disc. Appl. Math.*, **157**(2009), 804–811.

[16] S. Klavzar and I. Gutman. A Comparison of the Schultz molecular topological index with the Wiener index. *J. Chem. Inf. Comput. Sci.*, **36**(1996), 1001–1003.
[17] X. Li and Y. Shi. A survey on the Randic index. *MATCH Commun. Math. Comput. Chem.*, **59**(1)(2008), 127–156.
[18] V. Lokesha, T. Deepika, P. S. Ranjini and I. N. Cangul. Operations of nanostructures via SDD, ABC_4, GA_5 indices. *Appl. Math. Non-linear Science*, **2**(1)(2017), 173–180.
[19] V. Lokesha, B. S. Shetty, P. S. Ranjini, I. N. Cangul and A. S. Cevik. New bounded for Randić and GA indices. *JIA J. Ineq. Appl.*, **180**(2013), 1–7.
[20] V. Lokesha, A. Usha, P. S. Ranjini amd T. Deepika. Harmonic index of cubic polyhedral graphs using bridge graphs. *App. Math. Sci.*, **9**(2015), 4245–4253.
[21] A. Milićević, S. Nikolić and N. Trinajstić. On reformulated Zagreb indices. *Mol. Divers.*, **8**(2004), 393–399.
[22] S. Nikolić, G. Kovačević, A. Milićević and N. Trinajstić. The Zagreb indices 30 years after. *Croat. Chem. Acta*, **76**(2003), 113–124.
[23] M. Randić. Characterization of molecular branching. *J. Amer. Chem. Soc.*, **97**(1975), 6609–6615.
[24] P. S. Ranjini, V. Lokesha and M. A. Rajan. On Zagreb indices of the subdivision graphs. *Int. J. Math. Sc. Eng. Appl.*, **4**(2010), 221–228.
[25] G. Rucker and C. Rucker. On topological indices, boiling points, and cycloalkanes. *J. Chem. Inf. Comput. Sci.*, **39**(1999), 788–802.
[26] B. S. Shetty, V. Lokesha and P. S. Ranjini. On the Harmonic index of graph operations. *Trans. Combin.*, **4**(4)(2015), 5–14.
[27] D. B. West. *An Introduction to Graph Theory*, Prentice-Hall: Upper Saddle River, NJ, USA, 1996.
[28] X. Xu. Relationships between Harmonic index and other topological indices. *Appl. Math. Sci.*, **6**(41)(2012), 2013–2018.
[29] H. Zhang and F. Zhang. The Clar covering polynomial of hexagonal systems. *Discr. Appl. Math.*, **69**(1996), 147–167.

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