JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
IISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
J. Int. Math. Virtual Inst., Vol. **10**(1)(2020), 139-156
DOI: 10.7251/JIMVI2001139O

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

WEAK IMPLICATIVE ALGEBRA TO THE SET-THEORETICAL YANG-BAXTER EQUATION

Tahsin Oner, Tugce Katican, and Necla Kircali Gursoy

Abstract. In this paper, we present a weak implication algebra which is a bounded Hilbert algebra with the specific condition after introducing basic definitions and properties of Hilbert and bounded Hilbert algebras. Then we build some solutions to the set-theoretical Yang-Baxter equation by using properties of weak implication algebra.

1. Introduction

Henkin and Skolem introduced a Hilbert algebra for frameworks in non-classical logics [**6**]. This algebraic structure is an algebraic counterpart of Hilbert' s positive implicative propositional calculus [**17**], that is, a part of the propositional logic involving the implication operator and the constant 1. After Diego analysed the concept of a Hilbert algebra and related notions [**5**], Busneag and Ghiţă studied on some lattice properties of Hilbert algebras [**4**]. During the recent years, A. S. Nasab and A. B. Saeid introduced a weak implication algebra which is a bounded Hilbert algebra $(W, \longrightarrow, 1)$ with the condition $(w_1 \longrightarrow w_2) \longrightarrow w_1 = w_1$ for all $w_1, w_2 \in W$ such that $w_2 \neq 0$, named as (I), (because W is a Boolean algebra when $w_2 = 0$). They showed that every Boolean algebra is a weak implication algebra and its inverse is generally not true. Then they demonstrated that a totally ordered Hilbert algebra and a weak implication algebra are not same [**18**].

Besides, the Yang-Baxter equation was fundamentally used in theoretical physics [**16**] and statical mechanics [**1, 2**], [**19**], and so it can be applied to various workspaces of science, technology and industry. During the recent years, this equation has

2010 *Mathematics Subject Classification.* Primary 16T25; Secondary 03G25; 03G10; 05E40.
Key words and phrases. Yang-Baxter equation, weak implication algebra, hilbert algebra, bounded hilbert algebra.

been widely used in various scientific frameworks such as quantum groups, quantum mechanics, quantum computing, knot theory, braid goups, integrable systems, non-commutative geometry, C*-algebras, etc.(see, for instance, [**7**]-[**12**]). In addition to these, to build set-teoretical solutions to this equation becomes significant for researchers in a wide range of mathematical frameworks. Especially, Oner et al. studied on set-theoretical solutions to the Yang-Baxter equation using some algebras such as Wajsberg Algebras [**13**], BL-algebras [**14**] and MTL-algebras [**15**]. Therefore, we search solutions to the set-theoretical Yang-Baxter equation in a weak implication algebra by using its properties.

It is given a definition of the Yang-Baxter equation which is widely used in various scientific workspaces after recalling basic definitions and concepts related to a weak implication algebra which is a bounded Hilbert algebra with the condition (I). Then we investigate some solutions to the set-teoretical Yang-Baxter equation by using properties of a weak implication algebra. Indeed, we build some solutions that are not usually solutions to the set-theoretical Yang-Baxter equation in a Hilbert or a bounded Hilbert algebra but are solutions in a weak implication algebra. Also, it is illustrated that they are not solutions to the set-theoretical Yang-Baxter equation in a Hilbert or a bounded Hilbert algebra.

2. Preliminaries

In this part, we remind certain definitions and notions about a weak implication algebra and the Yang-Baxter equation.

Definition 2.1. ([**5**]) A Hilbert algebra is an algebra $(W, \longrightarrow, 1)$ of type $(2, 0)$ such that the following axioms are satisfied for all $w_1, w_2, w_3 \in W$:

$(H1)$ $w_1 \longrightarrow (w_2 \longrightarrow w_1) = 1$
$(H2)$ $(w_1 \longrightarrow (w_2 \longrightarrow w_3)) \longrightarrow ((w_1 \longrightarrow w_2) \longrightarrow (w_1 \longrightarrow w_3)) = 1$
$(H3)$ If $w_1 \longrightarrow w_2 = w_2 \longrightarrow w_1 = 1$, then $w_1 = w_2$.

Proposition 2.1 ([**3, 4**]). *In each Hilbert algebra W, the following relations hold for all $w_1, w_2, w_3 \in W$:*

$(h1)$ $1 \longrightarrow w_1 = w_1$, $w_1 \longrightarrow w_1 = 1$, $w_1 \longrightarrow 1 = 1$,
$(h2)$ $w_1 \leqslant w_2 \longrightarrow w_1$, $w_1 \leqslant (w_1 \longrightarrow w_2) \longrightarrow w_2$,
$(h3)$ $w_1 \longrightarrow (w_1 \longrightarrow w_2) = w_1 \longrightarrow w_2$,
$(h4)$ $((w_1 \longrightarrow w_2) \longrightarrow w_2) \longrightarrow w_2 = w_1 \longrightarrow w_2$,
$(h5)$ $w_1 \longrightarrow (w_2 \longrightarrow w_3) = w_2 \longrightarrow (w_1 \longrightarrow w_3)$,
$(h6)$ $w_1 \longrightarrow w_2 \leqslant (w_2 \longrightarrow w_3) \longrightarrow (w_1 \longrightarrow w_3)$,
$(h7)$ $w_1 \longrightarrow w_2 \leqslant (w_3 \longrightarrow w_1) \longrightarrow (w_3 \longrightarrow w_2)$,
$(h8)$ *if* $w_1 \leqslant w_2$, *then* $w_2 \longrightarrow w_3 \leqslant w_1 \longrightarrow w_3$ *and* $w_3 \longrightarrow w_1 \leqslant w_3 \longrightarrow w_2$,
$(h9)$ $w_1 \longrightarrow (w_2 \longrightarrow w_3) = (w_1 \longrightarrow w_2) \longrightarrow (w_1 \longrightarrow w_3)$,
$(h10)$ $(w_1 \longrightarrow w_2) \longrightarrow ((w_2 \longrightarrow w_1) \longrightarrow w_1) = (w_2 \longrightarrow w_1) \longrightarrow ((w_1 \longrightarrow w_2) \longrightarrow w_2)$,
$(h11)$ $(w_1 \longrightarrow w_2) \longrightarrow (w_2 \longrightarrow w_1) = w_2 \longrightarrow w_1$,
$(h12)$ $((w_1 \longrightarrow w_2) \longrightarrow w_1) \longrightarrow w_2 = w_1 \longrightarrow w_2$.

Lemma 2.1 ([**4**]). *The relation* $\leqslant$ *defined by* $w_1 \leqslant w_2 \Leftrightarrow w_1 \longrightarrow w_2 = 1$ *is a partial order on* W *called the naturel ordering on* W*, and* 1 *is the greatest element of* W *with respect to this order.*

Definition 2.2. ([**4**]) If a Hilbert algebra W has a least element 0 according to the natural ordering on W, then it is called a bounded Hilbert algebra.

In a bounded Hilbert algebra W, the unary operation * on W is defined by $w_1^* = w_1 \longrightarrow 0$ for all $w_1 \in W$.

Proposition 2.2 ([**4**]). *] If* W *is a bounded Hilbert algebra and* $w_1, w_2 \in W$*, then*

($bh1$) $0^* = 1,\ 1^* = 0,$
($bh2$) $w_1 \longrightarrow w_2^* = w_2 \longrightarrow w_1^*,$
($bh3$) $w_1 \longrightarrow w_1^* = w_1^*,\ w_1^* \longrightarrow w_1 = w_1^{**},\ w_1 \leqslant w_1^{**},\ w_1 \leqslant w_1^* \longrightarrow w_2,$
($bh4$) $w_1 \longrightarrow w_2 \leqslant w_2^* \longrightarrow w_1^*,$
($bh5$) *if* $w_1 \leqslant w_2$*, then* $w_2^* \leqslant w_1^*,$
($bh6$) $w_1^{***} = w_1^*,$
($bh7$) $(w_1 \longrightarrow w_2)^{**} = w_1 \longrightarrow w_2^{**} = w_1^{**} \longrightarrow w_2^{**},$
($bh8$) $(w_2 \longrightarrow w_1)^* \leqslant w_1 \longrightarrow w_2.$

Theorem 2.1 ([**4**]). *For a bounded Hilbert algebra* W*, the following conditions are equivalent:*

(a) $w_1^{**} = w_1$ *for every* $w_1 \in W,$
(b) W *is a Boolean algebra according to the natural ordering on* W*, in which* $w_1 \wedge w_2 = (w_1 \longrightarrow w_2^*)^*,\ w_1 \vee w_2 = w_1^* \longrightarrow w_2.$

Corollary 2.1 ([**4**]). *A bounded Hilbert algebra* W *is a Boolean algebra (according to the natural ordering on* W*) if and only if* $(w_1 \longrightarrow w_2) \longrightarrow w_1 = w_1$ *for all* $w_1, w_2 \in W$.

Corollary 2.2 ([**4**]). *For a bounded Hilbert algebra* W*, the following conditions are equivalent:*

(a) W *is a Boolean algebra (according to the natural ordering on* W*),*
(b) $(w_1 \longrightarrow w_2) \longrightarrow w_2 = (w_2 \longrightarrow w_1) \longrightarrow w_1,$
(c) $w_1^* \longrightarrow w_2 = w_2^* \longrightarrow w_1,$
(d) $(w_1 \longrightarrow w_2) \longrightarrow w_2 = w_1 \vee w_2,$
(e) $w_1^* \longrightarrow w_2 = w_1 \vee w_2,$

Definition 2.3. ([**18**]) A bounded Hilbert algebra W is called a weak implication algebra if it satisfies in the following condition:

(I) $\quad (w_1 \longrightarrow w_2) \longrightarrow w_1 = w_1$ for all $w_1, w_2 \in W$ such that $w_2 \neq 0$.

(If $w_2 = 0$, then $(w_1 \longrightarrow 0) \longrightarrow w_1 = w_1^{**} = w_1$, Thus, W is a Boolean algebra.)

Proposition 2.3 ([**18**]). *Every Boolean algebra is a weak implication algebra.*

Let k be a field and tensor products be defined over this field. For a *k-space* V, we denote by $\tau : V \otimes V \longrightarrow V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$ and by $I : V \longrightarrow V$ the identity map over the space V; for a *k-linear* map $R : V \otimes V \longrightarrow V \otimes V$, let $R^{12} = R \otimes I$, $R^{23} = I \otimes R$, and $R^{13} = (I \otimes \tau)(R \otimes I)(\tau \otimes I)$.

Definition 2.4. ([**12**]) A Yang-Baxter operator is *k-linear* map $R : V \otimes V \longrightarrow V \otimes V$, which is invertible, and it satisfies the braid condition called the Yang-Baxter equation:

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{1}$$

If R satisfies Equation (1), then both $R \circ \tau$ and $\tau \circ R$ satisfy the quantum Yang-Baxter equation(QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{2}$$

Lemma 2.2 ([**12**]). *Equations (1) and (2) are equivalent to each other.*

3. The Solutions to the set-theoretical Yang-Baxter Equation in a weak implication algebra

In this part, we build solutions to the set-theoretical Yang-Baxter equation in a weak implication algebra.

To do this, we give first the following definition.

Definition 3.1. ([**12**]) Let X be a set and $S : X \times X \longrightarrow X \times X$, $S(u, v) = (u', v')$ be a map. The map S is a solution of the set-theoretical Yang-Baxter equation if it satisfies the following equation:

$$S_{12} \circ S_{23} \circ S_{12} = S_{23} \circ S_{12} \circ S_{23},$$

which is also equivalent

$$S_{12} \circ S_{13} \circ S_{23} = S_{23} \circ S_{13} \circ S_{12},$$

where

$$S_{12} : X \times X \times X \longrightarrow X \times X \times X, S_{12}(u, v, w) = (u', v', w),$$

$$S_{23} : X \times X \times X \longrightarrow X \times X \times X, S_{23}(u, v, w) = (u, v', w'),$$

$$S_{13} : X \times X \times X \longrightarrow X \times X \times X, S_{13}(u, v, w) = (u', v, w').$$

Now, we find solutions of the set-theoretical Yang-Baxter equation by using the properties of a weak implication algebra.

Theorem 3.1. *Let $(W, \longrightarrow, 1)$ be a weak implication algebra. Then $S(w_1, w_2) = (w_1 \longrightarrow w_2, w_1)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.*

Proof. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = (w_1 \longrightarrow w_2, w_1, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, w_2 \longrightarrow w_3, w_2).$$

We show that the equality $(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3) = (S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3)$ holds for all $(w_1,w_2,w_3)\in W^3$:

$$\begin{aligned}
(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3) &= (S_{12}\circ S_{23})(S_{12}(w_1,w_2,w_3))\\
&= (S_{12}\circ S_{23})(w_1\longrightarrow w_2,w_1,w_3)\\
&= S_{12}(S_{23}(w_1\longrightarrow w_2,w_1,w_3))\\
&= S_{12}(w_1\longrightarrow w_2,w_1\longrightarrow w_3,w_1)\\
&= ((w_1\longrightarrow w_2)\longrightarrow (w_1\longrightarrow \\
&\quad w_3),w_1\longrightarrow w_2,w_1)\\
&= (w_1\longrightarrow (w_2\longrightarrow w_3),w_1\longrightarrow w_2,w_1) \qquad (h9)\\
&= S_{23}(w_1\longrightarrow (w_2\longrightarrow w_3),w_1,w_2)\\
&= S_{23}(S_{12}(w_1,w_2\longrightarrow w_3,w_2))\\
&= (S_{23}\circ S_{12})(w_1,w_2\longrightarrow w_3,w_2)\\
&= (S_{23}\circ S_{12})(S_{23}(w_1,w_2,w_3))\\
&= (S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3).
\end{aligned}$$

Thus, $S(w_1,w_2) = (w_1\longrightarrow w_2,w_1)$ is a solution of the set-theoretical Yang-Baxter equation in the weak implication algebra. □

THEOREM 3.2. *Let* $(W,\longrightarrow,1)$ *be a weak implication algebra. Then* $S(w_1,w_2) = (w_2^*,w_1^*)$ *is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.*

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1,w_2,w_3) = (w_2^*,w_1^*,w_3)$$

$$S_{23}(w_1,w_2,w_3) = (w_1,w_3^*,w_2^*).$$

We show that the equality $(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3) = (S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3)$ holds for all $(w_1,w_2,w_3)\in W^3$:

$$\begin{aligned}
(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3) &= (S_{12}\circ S_{23})(S_{12}(w_1,w_2,w_3))\\
&= (S_{12}\circ S_{23})(w_2^*,w_1^*,w_3)\\
&= S_{12}(S_{23}(w_2^*,w_1^*,w_3))\\
&= S_{12}(w_2^*,w_3^*,w_1^{**})\\
&= (w_3^{**},w_2^{**},w_1^{**})\\
&= S_{23}(w_3^{**},w_1^*,w_2^*)\\
&= (S_{23}(S_{12}(w_1,w_3^*,w_2^*))\\
&= (S_{23}\circ S_{12})(w_1,w_3^*,w_2^*)\\
&= (S_{23}\circ S_{12})(S_{23}(w_1,w_2,w_3))\\
&= (S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3)
\end{aligned}$$

Hence, $S(w_1,w_2) = (w_2^*,w_1^*)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra. □

THEOREM 3.3. *Let* $(W,\longrightarrow,1)$ *be a weak implication algebra. Then* $S(w_1,w_2) = ((w_1\longrightarrow w_2)\longrightarrow w_2,w_2)$ *is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.*

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = ((w_1 \longrightarrow w_2) \longrightarrow w_2, w_2, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3).$$

We show that the equality $(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) = (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3)$ holds for all $(w_1, w_2, w_3) \in W^3$. Then it is obtained from Corollary 2.2 (b), Proposition 2.3, $(h4)$ and $(h5)$ that

$$\begin{aligned}
(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) &= (S_{12} \circ S_{23})(S_{12}(w_1, w_2, w_3)) \\
&= (S_{12} \circ S_{23})((w_1 \longrightarrow w_2) \longrightarrow w_2, w_2, w_3) \\
&= S_{12}(S_{23}((w_1 \longrightarrow w_2) \longrightarrow w_2, w_2, w_3)) \\
&= S_{12}((w_1 \longrightarrow w_2) \longrightarrow w_2, \\
&\quad (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3) \\
&= ((((w_1 \longrightarrow w_2) \longrightarrow w_2) \longrightarrow ((w_2 \\
&\quad \longrightarrow w_3) \longrightarrow w_3)) \longrightarrow ((w_2 \longrightarrow w_3) \\
&\quad \longrightarrow w_3), (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3) \\
&= ((((w_1 \longrightarrow w_2) \longrightarrow w_2) \longrightarrow ((w_3 \\
&\quad \longrightarrow w_2) \longrightarrow w_2)) \longrightarrow ((w_3 \longrightarrow w_2) \\
&\quad \longrightarrow w_2), (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3) \\
&= (((w_3 \longrightarrow w_2) \longrightarrow (((w_1 \longrightarrow w_2) \\
&\quad \longrightarrow w_2) \longrightarrow w_2)) \longrightarrow ((w_3 \longrightarrow w_2) \\
&\quad \longrightarrow w_2), (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3) \\
&= (((w_3 \longrightarrow w_2) \longrightarrow (w_1 \longrightarrow \\
&\quad w_2)) \longrightarrow ((w_3 \longrightarrow w_2) \longrightarrow \\
&\quad w_2), (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3) \\
&= ((w_1 \longrightarrow ((w_2 \longrightarrow w_3) \longrightarrow \\
&\quad w_3)) \longrightarrow ((w_2 \longrightarrow w_3) \longrightarrow \\
&\quad w_3), (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3)
\end{aligned}$$

and

$$\begin{aligned}
(S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3) &= (S_{23} \circ S_{12})(S_{23}(w_1, w_2, w_3)) \\
&= (S_{23} \circ S_{12})(w_1, (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3) \\
&= S_{23}(S_{12}(w_1, (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3)) \\
&= S_{23}((w_1 \longrightarrow ((w_2 \longrightarrow w_3) \longrightarrow \\
&\quad w_3)) \longrightarrow ((w_2 \longrightarrow w_3) \longrightarrow \\
&\quad w_3), (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3) \\
&= ((w_1 \longrightarrow ((w_2 \longrightarrow w_3) \longrightarrow w_3)) \longrightarrow \\
&\quad ((w_2 \longrightarrow w_3) \longrightarrow w_3), (((w_2 \longrightarrow \\
&\quad w_3) \longrightarrow w_3) \longrightarrow w_3) \longrightarrow w_3, w_3) \\
&= ((w_1 \longrightarrow ((w_2 \longrightarrow w_3) \longrightarrow \\
&\quad w_3)) \longrightarrow ((w_2 \longrightarrow w_3) \longrightarrow w_3), \\
&\quad (w_2 \longrightarrow w_3) \longrightarrow w_3, w_3).
\end{aligned}$$

So, $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_2, w_2)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra. □

COROLLARY 3.1. *Let* $(W, \longrightarrow, 1)$ *be a weak implication algebra. Then*

$S(w_1, w_2) = ((w_2 \longrightarrow w_1) \longrightarrow w_1, w_1)$,
$S(w_1, w_2) = ((w_2 \longrightarrow w_1) \longrightarrow w_1, w_2)$, *and*
$S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_2, w_1)$

are solutions to the set-theoretical Yang-Baxter equation in the weak implication algebra.

Remark 3.1. $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_2, w_2)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra by Theorem 3.3 but it is not a solution to the set-theoretical Yang-Baxter equation in a Hilbert algebra.

Example 3.1. Consider a bounded Hilbert algebra $(W, \longrightarrow, 1)$ with following Hasse diagram where $W = \{0, x, y, z, 1\}$:

Figure 1

The binary operation $\longrightarrow$ on W has the Cayley table as below:

$\longrightarrow$	0	x	y	z	1
0	1	1	1	1	1
x	0	1	1	1	1
y	0	x	1	1	1
z	0	x	y	1	1
1	0	x	y	z	1

But this algebra is not a weak implication algebra since $(y \longrightarrow x) \longrightarrow y = 1 \neq y$. Because

$$\begin{aligned}(S_{12} \circ S_{23} \circ S_{12})(x, 0, z) &= ((((x \longrightarrow 0) \longrightarrow 0) \longrightarrow ((0 \longrightarrow z) \longrightarrow z)) \\ &\quad \longrightarrow ((0 \longrightarrow z) \longrightarrow z), (0 \longrightarrow z) \longrightarrow z, z) \\ &= ((1 \longrightarrow z) \longrightarrow z, z, z) \\ &= (1, z, z)\end{aligned}$$

and

$$\begin{aligned}(S_{23} \circ S_{12} \circ S_{23})(x, 0, z) &= ((x \longrightarrow ((0 \longrightarrow z) \longrightarrow z)) \longrightarrow ((0 \\ &\quad \longrightarrow z) \longrightarrow z), (0 \longrightarrow z) \longrightarrow z, z) \\ &= ((x \longrightarrow z) \longrightarrow z, z, z) \\ &= (z, z, z)\end{aligned}$$

that is, $(S_{12} \circ S_{23} \circ S_{12})(x, 0, z) \neq (S_{23} \circ S_{12} \circ S_{23})(x, 0, z)$, $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_2, w_2)$ is not a solution to the set-theoretical Yang-Baxter equation in this bounded Hilbert algebra.

Lemma 3.1. *Let $(W, \longrightarrow, 1)$ be a weak implication algebra. Then*

(a) $S(w_1, w_2) = (w_1, w_1)$,

(b) $S(w_1, w_2) = (w_1, w_2)$

are solutions to the set-theoretical Yang-Baxter equation in the weak implication algebra.

Proof. (a) S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = (w_1, w_1, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, w_2, w_2).$$

We show that the equality $(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) = (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3)$ holds for all $(w_1, w_2, w_3) \in W^3$:

$$\begin{aligned}(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) &= (S_{12} \circ S_{23})(S_{12}(w_1, w_2, w_3)) \\ &= (S_{12} \circ S_{23})(w_1, w_1, w_3) \\ &= S_{12}(S_{23}(w_1, w_1, w_3)) \\ &= S_{12}(w_1, w_1, w_1) \\ &= (w_1, w_1, w_1) \\ &= S_{23}(w_1, w_1, w_2) \\ &= S_{23}(S_{12}(w_1, w_2, w_2)) \\ &= (S_{23} \circ S_{12})(w_1, w_2, w_2) \\ &= (S_{23} \circ S_{12})(S_{23}(w_1, w_2, w_3)) \\ &= (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3).\end{aligned}$$

Therefore, $S(w_1, w_2) = (w_1, w_1)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.

(b) S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = (w_1, w_2, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, w_2, w_3).$$

Since

$$(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) = (w_1, w_2, w_3) = (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3)$$

for all $(w_1, w_2, w_3) \in W^3$, then $S(w_1, w_2) = (w_1, w_2)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra. □

Theorem 3.4. *Let $(W, \longrightarrow, 1)$ be a weak implication algebra. Then*

(a) $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_1, w_1)$,

(b) $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_1, w_2)$

are solutions of the set-theoretical Yang-Baxter equation in the weak implication algebra.

PROOF. Since $(W, \longrightarrow, 1)$ is a weak implication algebra, $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_1, w_1) = (w_1, w_1)$ and $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_1, w_2) = (w_1, w_2)$. Therefore, they are solutions of the set-theoretical Yang-Baxter equation in the weak implication algebra by Lemma 3.1. □

REMARK 3.2. $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_1, w_1)$ and $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_1, w_2)$ are solutions to the set-theoretical Yang-Baxter equation in the weak implication algebra by Theorem 3.4 while they are not solutions to the set-theoretical Yang-Baxter equation in a Hilbert algebra.

EXAMPLE 3.2. Let $W = [0, 1]$ with $0 < w_1 < w_2 < w_3 < 1$, and we define $w_1 \longrightarrow w_2 = min(1, 1 - w_1 + w_2)$ for all $w_1, w_2 \in W$. Then $(W, \longrightarrow, 1)$ is a bounded Hilbert algebra. For some $(w_1, w_3, w_2) \in W^3$,

$$(S_{12} \circ S_{23} \circ S_{12})(w_1, w_3, w_2) = (w_1, w_1, w_1)$$

and

$$\begin{aligned}(S_{23} \circ S_{12} \circ S_{23})(w_1, w_3, w_2) &= ((w_1 \longrightarrow ((w_3 \longrightarrow w_2) \\ &\quad \longrightarrow w_3)) \longrightarrow w_1, w_1, w_1) \\ &= ((w_1 \longrightarrow min(1, 2w_3 \\ &\quad -w_2)) \longrightarrow w_1, w_1, w_1) \\ &= (min(1, 2w_1 - min(1, \\ &\quad 2w_3 - w_2)), w_1, w_1)\end{aligned}$$

To examine that $(S_{12} \circ S_{23} \circ S_{12})(w_1, w_3, w_2) = (S_{23} \circ S_{12} \circ S_{23})(w_1, w_3, w_2)$, it is sufficient to show that

$$min(1, 2w_1 - min(1, 2w_3 - w_2)) = w_1.$$

Then there exist two cases.

Case I $w_1 = 1$ which is a contradiction by the hypothesis.

Case II Let $2w_1 - min(1, 2w_3 - w_2) = w_1$, i.e., $min(1, 2w_3 - w_2) = w_1$. So, there exist two subcases.

(a) $w_1 = 1$ which is a contradiction by the hypothesis.

(b) $2w_3 - w_2 = w_1$, i.e., $w_3 = \frac{w_1+w_2}{2}$ which is a contradiction by the hypothesis.

Therefore, $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_1, w_1)$ is not a solution to the set-theoretical Yang-Baxter equation in this Hilbert algebra. Similarly, it can be seen that $S(w_1, w_2) = ((w_1 \longrightarrow w_2) \longrightarrow w_1, w_2)$ is also not a solution to the set-theoretical Yang-Baxter equation in this Hilbert algebra.

THEOREM 3.5. *Let $(W, \longrightarrow, 1)$ be a weak implication algebra. Then $S(w_1, w_2) = ((w_1 \longrightarrow w_2)^{**}, w_1)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.*

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = ((w_1 \longrightarrow w_2)^{**}, w_1, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, (w_2 \longrightarrow w_3)^{**}, w_2).$$

We show that the equality $(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) = (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3)$ holds for all $(w_1, w_2, w_3) \in W_3$:

$$\begin{aligned}
(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) &= (S_{12} \circ S_{23})(S_{12}(w_1, w_2, w_3) \\
&= (S_{12} \circ S_{23})((w_1 \longrightarrow w_2)^{**}, w_1, w_3) \\
&= S_{12}(S_{23}((w_1 \longrightarrow w_2)^{**}, w_1, w_3)) \\
&= S_{12}((w_1 \longrightarrow w_2)^{**}, (w_1 \longrightarrow w_3)^{**}, w_1) \\
&= (((w_1 \longrightarrow w_2)^{**} \longrightarrow (w_1 \longrightarrow \\
&\quad w_3)^{**})^{**}, (w_1 \longrightarrow w_2)^{**}, w_1) \\
&= (((w_1 \longrightarrow w_2) \longrightarrow (w_1 \longrightarrow \\
&\quad w_3))^{****}, (w_1 \longrightarrow w_2)^{**}, w_1) && (bh7) \\
&= (((w_1 \longrightarrow w_2) \longrightarrow (w_1 \longrightarrow \\
&\quad w_3))^{**}, (w_1 \longrightarrow w_2)^{**}, w_1) && (bh6) \\
&= ((w_1 \longrightarrow (w_2 \longrightarrow w_3))^{**}, \\
&\quad (w_1 \longrightarrow w_2)^{**}, w_1) && (h9)
\end{aligned}$$

and

$$\begin{aligned}
(S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3) &= (S_{23} \circ S_{12})(S_{23}(w_1, w_2, w_3)) \\
&= (S_{23} \circ S_{12})(w_1, (w_2 \longrightarrow w_3)^{**}, w_2) \\
&= S_{23}(S_{12}(w_1, (w_2 \longrightarrow w_3)^{**}, w_2)) \\
&= S_{23}((w_1 \longrightarrow (w_2 \longrightarrow w_3)^{**})^{**}, w_1, w_2) \\
&= ((w_1 \longrightarrow (w_2 \longrightarrow w_3)^{**})^{**}, \\
&\quad (w_1 \longrightarrow w_2)^{**}, w_1) \\
&= ((w_1 \longrightarrow (w_2 \longrightarrow w_3))^{****}, \\
&\quad (w_1 \longrightarrow w_2)^{**}, w_1) && (bh7) \\
&= ((w_1 \longrightarrow (w_2 \longrightarrow w_3))^{**}, \\
&\quad (w_1 \longrightarrow w_2)^{**}, w_1). && (bh6)
\end{aligned}$$

Thus, $S(w_1, w_2) = ((w_1 \longrightarrow w_2)^{**}, w_1)$ is a solution of the set-theoretical Yang-Baxter equation in the weak implication algebra. □

THEOREM 3.6. *Let $(W, \longrightarrow, 1)$ be a weak implication algebra. Then $S(w_1, w_2) = (w_2, w_1^*)$ is a solution of the set-theoretical Yang-Baxter equation in the weak implication algebra.*

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = (w_2, w_1^*, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, w_3, w_2^*)$$

We show that the equality $(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) = (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3)$ holds for all $(w_1, w_2, w_3) \in W^3$:

$$
\begin{aligned}
(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3) &= (S_{12}\circ S_{23})(S_{12}(w_1,w_2,w_3))\\
&= (S_{12}\circ S_{23})(w_2,w_1^*,w_3)\\
&= S_{12}(S_{23}(w_2,w_1^*,w_3))\\
&= S_{12}(w_2,w_3,w_1^{**})\\
&= (w_3,w_2^*,w_1^{**})\\
&= S_{23}(w_3,w_1^*,w_2^*)\\
&= S_{23}(S_{12}(w_1,w_3,w_2^*))\\
&= (S_{23}\circ S_{12})(w_1,w_3,w_2^*)\\
&= (S_{23}\circ S_{12})(S_{23}(w_1,w_2,w_3))\\
&= (S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3).
\end{aligned}
$$

Hence, $S(w_1,w_2)=(w_2,w_1^*)$ is a solution of the set-theoretical Yang-Baxter equation in the weak implication algebra. □

THEOREM 3.7. *Let* $(W,\longrightarrow,1)$ *be a weak implication algebra. Then* $S(w_1,w_2)=((w_1\longrightarrow w_2)^{**},w_1^{**})$ *is a solution of the set-theoretical Yang-Baxter equation in the weak implication algebra.*

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1,w_2,w_3)=((w_1\longrightarrow w_2)^{**},w_1^{**},w_3)$$

$$S_{23}(w_1,w_2,w_3)=(w_1,(w_2\longrightarrow w_3)^{**},w_2^{**}).$$

We show that the equality $(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3)=S({}_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3)$ holds for all $(w_1,w_2,w_3)\in W^3$. Then it follows from $(bh6)$, $(bh7)$ and $(h9)$ that

$$
\begin{aligned}
(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3) &= (S_{12}\circ S_{23})(S_{12}(w_1,w_2,w_3))\\
&= (S_{12}\circ S_{23})((w_1\longrightarrow w_2)^{**},w_1^{**},w_3)\\
&= S_{12}(S_{23}((w_1\longrightarrow w_2)^{**},w_1^{**},w_3))\\
&= S_{12}((w_1\longrightarrow w_2)^{**},(w_1^{**}\longrightarrow w_3)^{**},w_1^{****})\\
&= S_{12}((w_1\longrightarrow w_2)^{**},w_1^{****}\longrightarrow w_3^{**},w_1^{****})\\
&= S_{12}((w_1\longrightarrow w_2)^{**},(w_1\longrightarrow w_3)^{**},w_1^{**})\\
&= (((w_1\longrightarrow w_2)^{**}\longrightarrow (w_1\longrightarrow\\
&\quad w_3)^{**})^{**},(w_1\longrightarrow w_2)^{****},w_1^{**})\\
&= (((w_1\longrightarrow w_2)\longrightarrow (w_1\longrightarrow\\
&\quad w_3))^{****},(w_1\longrightarrow w_2)^{****},w_1^{**})\\
&= ((w_1\longrightarrow (w_2\longrightarrow w_3))^{**},\\
&\quad (w_1\longrightarrow w_2)^{**},w_1^{**})
\end{aligned}
$$

and

$$\begin{aligned}(S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3) &= (S_{23} \circ S_{12})(S_{23}(w_1, w_2, w_3)) \\ &= (S_{23} \circ S_{12})(w_1, (w_2 \longrightarrow w_3)^{**}, w_2^{**}) \\ &= S_{23}(S_{12}(w_1, (w_2 \longrightarrow w_3)^{**}, w_2^{**})) \\ &= S_{23}((w_1 \longrightarrow (w_2 \longrightarrow w_3)^{**})^{**}, w_1^{**}, w_2^{**}) \\ &= S_{23}((w_1 \longrightarrow (w_2 \longrightarrow w_3))^{**}, w_1^{**}, w_2^{**}) \\ &= ((w_1 \longrightarrow (w_2 \longrightarrow w_3))^{**}, (w_1^{**} \longrightarrow w_2^{**})^{**}, w_1^{****}) \\ &= ((w_1 \longrightarrow (w_2 \longrightarrow w_3))^{**}, (w_1 \longrightarrow w_2)^{**}, w_1^{**}).\end{aligned}$$

So, $S(w_1, w_2) = ((w_1 \longrightarrow w_2)^{**}, w_1^{**})$ is a solution of the set-theoretical Yang-Baxter equation in the weak implication algebra. ☐

THEOREM 3.8. *Let $(W, \longrightarrow, 1)$ be a weak implication algebra. Then $S(w_1, w_2) = ((w_1 \longrightarrow w_2^*)^*, w_1)$ is a solution of the set-theoretical Yang-Baxter equation in the weak implication algebra.*

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = ((w_1 \longrightarrow w_2^*)^*, w_1, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, (w_2 \longrightarrow w_3^*)^*, w_2).$$

We show that the equality $(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) = S(_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3)$ holds for all $(w_1, w_2, w_3) \in W^3$. Then it is obtained from $(bh2)$, $(bh6)$, $(bh7)$, $(h3)$, $(h5)$, Theorem 2.1(a) and Proposition 2.3 that

$$\begin{aligned}(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) &= (S_{12} \circ S_{23})(S_{12}(w_1, w_2, w_3)) \\ &= (S_{12} \circ S_{23})((w_1 \longrightarrow w_2^*)^*, w_1, w_3) \\ &= S_{12}(S_{23}((w_1 \longrightarrow w_2^*)^*, w_1, w_3)) \\ &= S_{12}((w_1 \longrightarrow w_2^*)^*, (w_1 \longrightarrow w_3^*)^*, w_1) \\ &= (((w_1 \longrightarrow w_2^*)^* \longrightarrow (w_1 \longrightarrow w_3^*)^{**})^*, (w_1 \longrightarrow w_2^*)^*, w_1) \\ &= (((w_1 \longrightarrow w_2^*)^* \longrightarrow (w_1 \longrightarrow w_3^*))^*, (w_1 \longrightarrow w_2^*)^*, w_1) \\ &= ((w_1 \longrightarrow ((w_1 \longrightarrow w_2^*)^* \longrightarrow w_3^*))^*, (w_1 \longrightarrow w_2^*)^*, w_1) \\ &= ((w_1 \longrightarrow (w_3 \longrightarrow (w_1 \longrightarrow w_2^*)))^*, (w_1 \longrightarrow w_2^*)^*, w_1) \\ &= ((w_1 \longrightarrow (w_1 \longrightarrow (w_3 \longrightarrow w_2^*)))^*, (w_1 \longrightarrow w_2^*)^*, w_1) \\ &= ((w_1 \longrightarrow (w_3 \longrightarrow w_2^*))^*, (w_1 \longrightarrow w_2^*)^*, w_1) \\ &= ((w_1 \longrightarrow (w_2 \longrightarrow w_3^*))^*, (w_1 \longrightarrow w_2^*)^*, w_1))\end{aligned}$$

and

$$\begin{aligned}(S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3) &= (S_{23}\circ S_{12})(S_{23}(w_1,w_2,w_3))\\ &= (S_{23}\circ S_{12})(w_1,(w_2\longrightarrow w_3^*)^*,w_2)\\ &= S_{23}(S_{12}(w_1,(w_2\longrightarrow w_3^*)^*,w_2))\\ &= S_{23}((w_1\longrightarrow (w_2\longrightarrow w_3^*)^{**})^*,w_1,w_2)\\ &= ((w_1\longrightarrow (w_2\longrightarrow w_3^*)^{**})^*,(w_1\longrightarrow w_2^*)^*,w_1)\\ &= ((w_1\longrightarrow (w_2\longrightarrow w_3^*))^*,(w_1\longrightarrow w_2^*)^*,w_1)\end{aligned}$$

Hence, $S(w_1,w_2)=((w_1\longrightarrow w_2^*)^*,w_1)$ is a solution of the set-theoretical Yang-Baxter equation in the weak implication algebra. □

COROLLARY 3.2. *Let $(W,\longrightarrow,1)$ be a weak implication algebra. Then*
$S(w_1,w_2)=((w_2\longrightarrow w_1^*)^*,w_1)$,
$S(w_1,w_2)=((w_2\longrightarrow w_1^*)^*,w_2)$, *and*
$S(w_1,w_2)=((w_1\longrightarrow w_2^*)^*,w_2)$.
are solutions to the set-theoretical Yang-Baxter equation in the weak implication algebra.

THEOREM 3.9. *Let $(W,\longrightarrow,1)$ be a weak implication algebra. If $(w_1\longrightarrow w_2)^*=w_2\longrightarrow w_1$ for all $w_1,w_2\in W$, then $S(w_1,w_2)=((w_1\longrightarrow w_2)^*,w_1)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.*

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1,w_2,w_3)=((w_1\longrightarrow w_2)^*,w_1,w_3)$$

$$S_{23}(w_1,w_2,w_3)=(w_1,(w_2\longrightarrow w_3)^*,w_2).$$

We show that the equality $(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3)=(S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3)$ holds for all $(w_1,w_2,w_3)\in W_3$:

$$\begin{aligned}(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3) &= (S_{12}\circ S_{23})(S_{12}(w_1,w_2,w_3))\\ &= (S_{12}\circ S_{23})((w_1\longrightarrow w_2)^*,w_1,w_3)\\ &= S_{12}(S_{23}((w_1\longrightarrow w_2)^*,w_1,w_3))\\ &= S_{12}((w_1\longrightarrow w_2)^*,(w_1\longrightarrow w_3)^*,w_1)\\ &= (((w_1\longrightarrow w_2)^*\longrightarrow (w_1\longrightarrow\\ &\quad w_3)^*)^*,(w_1\longrightarrow w_2)^*,w_1)\\ &= (((w_1\longrightarrow w_3)\longrightarrow (w_1\longrightarrow\\ &\quad w_2)^{**})^*,(w_1\longrightarrow w_2)^*,w_1) \quad (bh2)\\ &= (((w_1\longrightarrow w_3)\longrightarrow (w_1\longrightarrow\\ &\quad w_2))^{***},(w_1\longrightarrow w_2)^*,w_1) \quad (bh7)\\ &= ((w_1\longrightarrow (w_3\longrightarrow w_2))^*,\\ &\quad (w_1\longrightarrow w_2)^*,w_1) \quad ((h9)\ and\ (bh6))\\ &= ((w_1\longrightarrow (w_2\longrightarrow w_3)^*)^*,\\ &\quad (w_1\longrightarrow w_2)^*,w_1) \quad (hyp.)\end{aligned}$$

and

$$\begin{aligned}(S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3) &= (S_{23}\circ S_{12})(S_{23}(w_1,w_2,w_3))\\ &= (S_{23}\circ S_{12})(w_1,(w_2\longrightarrow w_3)^*,w_2)\\ &= S_{23}(S_{12}(w_1,(w_2\longrightarrow w_3)^*,w_2))\\ &= S_{23}((w_1\longrightarrow (w_2\longrightarrow w_3)^*)^*,w_1,w_2)\\ &= ((w_1\longrightarrow (w_2\longrightarrow w_3)^*)^*,(w_1\longrightarrow w_2)^*,w_1).\end{aligned}$$

Thus, $S(w_1,w_2)=((w_1\longrightarrow w_2)^*,w_1)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra. □

THEOREM 3.10. *Let* $(W,\longrightarrow,1)$ *be a weak implication algebra. Then*

$$S(w_1,w_2)=(w_1^*\longrightarrow w_2,0)$$

is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1,w_2,w_3)=(w_1^*\longrightarrow w_2,0,w_3)$$

$$S_{23}(w_1,w_2,w_3)=(w_1,w_2^*\longrightarrow w_3,0).$$

We show that the equality $(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3)=(S_{23}\circ S_{12}\circ S_{23})(w_1,w_2,w_3)$ holds for all $(w_1,w_2,w_3)\in W_3$. Then it follows from $(bh1)$, $(h1)$, $(h5)$, Corollary 2.2 (c) and Proposition 2.3 that

$$\begin{aligned}(S_{12}\circ S_{23}\circ S_{12})(w_1,w_2,w_3) &= (S_{12}\circ S_{23})(S_{12}(w_1,w_2,w_3))\\ &= (S_{12}\circ S_{23})(w_1^*\longrightarrow w_2,0,w_3)\\ &= S_{12}(S_{23}(w_1^*\longrightarrow w_2,0,w_3))\\ &= S_{12}(w_1^*\longrightarrow w_2,w_3,0)\\ &= ((w_1^*\longrightarrow w_2)^*\longrightarrow w_3,0,0)\\ &= (w_3^*\longrightarrow (w_1^*\longrightarrow w_2),0,0)\\ &= (w_1^*\longrightarrow (w_3^*\longrightarrow w_2),0,0)\\ &= (w_1^*\longrightarrow (w_2^*\longrightarrow w_3),0,0)\\ &= (w_1^*\longrightarrow (w_2^*\longrightarrow w_3),0^*\longrightarrow 0,0)\\ &= S_{23}(w_1^*\longrightarrow (w_2^*\longrightarrow w_3),0,0)\\ &= S_{23}(S_{12}(w_1,w_2^*\longrightarrow w_3,0))\\ &= S_{23}\circ S_{12}(w_1,w_2^*\longrightarrow w_3,0)\\ &= S_{23}\circ S_{12}(S_{23}(w_1,w_2,w_3))\\ &= S_{23}\circ S_{12}\circ S_{23}(w_1,w_2,w_3)\end{aligned}$$

So, $S(w_1,w_2)=(w_1^*\longrightarrow w_2,0)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra. □

COROLLARY 3.3. *Let* $(W,\longrightarrow,1)$ *be a weak implication algebra. Then*

$$S(w_1,w_2)=(w_2^*\longrightarrow w_1,0)$$

is a solution to the set-theoretical Yang-Baxter equation in a weak implication algebra.

REMARK 3.3. $S(w_1, w_2) = (w_1^* \longrightarrow w_2, 0)$ is a solution to the set-theoretical Yang-Baxter equation in a weak implication algebra by Theorem 3.10 while it is not a solution to the set-theoretical Yang-Baxter equation in a bounded Hilbert algebra.

EXAMPLE 3.3. Consider a bounded Hilbert algebra $(W, \longrightarrow, 1)$ with the following Hasse diagram where $W = \{0, x, y, z, t, 1\}$:

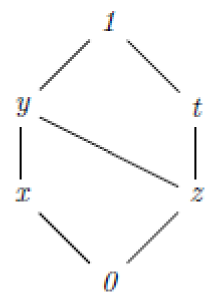

FIGURE 2

The binary operation $\longrightarrow$ on W has the Cayley table as follow:

$\longrightarrow$	0	x	y	z	t	1
0	1	1	1	1	1	1
x	0	1	1	z	t	1
y	0	x	1	z	t	1
z	0	x	1	1	1	1
t	0	x	y	z	1	1
1	0	x	y	z	t	1

However this algebra is not a weak implication algebra because $(y \longrightarrow z) \longrightarrow y = 1 \neq y$. Since

$$\begin{aligned}(S_{12} \circ S_{23} \circ S_{12})(0, y, 0) &= (0^* \longrightarrow (0^* \longrightarrow y), 0, 0)\\ &= (y, 0, 0) \qquad ((bh1) \textit{ and } (h1))\end{aligned}$$

and

$$\begin{aligned}(S_{23} \circ S_{12} \circ S_{23})(0, y, 0) &= (0^* \longrightarrow (y^* \longrightarrow 0), 0, 0)\\ &= (1 \longrightarrow (0 \longrightarrow 0), 0, 0) \qquad (bh1)\\ &= (1, 0, 0), \qquad (h1)\end{aligned}$$

i. e., $S_{12} \circ S_{23} \circ S_{12})(0, y, 0) \neq (S_{23} \circ S_{12} \circ S_{23})(0, y, 0)$, $S(w_1, w_2) = (w_1^* \longrightarrow w_2, 0)$ is not a solution to the set-theoretical Yang-Baxter equation in this algebra.

THEOREM 3.11. *Let* $(W, \longrightarrow, 1)$ *be a weak implication algebra. Then*

$$S(w_1, w_2) = (w_1^* \longrightarrow w_2, w_2)$$

is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = (w_1^* \longrightarrow w_2, w_2, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, w_2^* \longrightarrow w_3, w_3).$$

We show that the equality $(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) = (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3)$ holds for all $((w_1, w_2, w_3) \in W^3$. Then it is ontained from $(h3)$, $(h5)$, Corollary 2.2 (c) and Proposition 2.3 that

$$\begin{aligned}
(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) &= (S_{12} \circ S_{23})(S_{12}(w_1, w_2, w_3)) \\
&= (S_{12} \circ S_{23})(w_1^* \longrightarrow w_2, w_2, w_3) \\
&= S_{12}(S_{23}(w_1^* \longrightarrow w_2, w_2, w_3)) \\
&= ((w_1^* \longrightarrow w_2)^* \longrightarrow (w_2^* \\
&\quad \longrightarrow w_3), w_2^* \longrightarrow w_3, w_3) \\
&= ((w_2^* \longrightarrow w_3)^* \longrightarrow (w_1^* \\
&\quad \longrightarrow w_2), w_2^* \longrightarrow w_3, w_3) \\
&= (w_1^* \longrightarrow ((w_2^* \longrightarrow w_3)^* \\
&\quad \longrightarrow w_2), w_2^* \longrightarrow w_3, w_3) \\
&= (w_1^* \longrightarrow (w_2^* \longrightarrow (w_2^* \longrightarrow \\
&\quad w_3)), w_2^* \longrightarrow w_3, w_3) \\
&= (w_1^* \longrightarrow (w_2^* \longrightarrow w_3), \\
&\quad w_2^* \longrightarrow w_3, w_3)
\end{aligned}$$

and

$$\begin{aligned}
(S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3) &= (S_{23} \circ S_{12})(S_{23}(w_1, w_2, w_3)) \\
&= (S_{23} \circ S_{12})(w_1, w_2^* \longrightarrow w_3, w_3) \\
&= S_{23}(S_{12}(w_1, w_2^* \longrightarrow w_3, w_3)) \\
&= S_{23}(w_1^* \longrightarrow (w_2^* \longrightarrow w_3), w_2^* \longrightarrow w_3, w_3) \\
&= (w_1^* \longrightarrow (w_2^* \longrightarrow w_3), \\
&\quad (w_2^* \longrightarrow w_3)^* \longrightarrow w_3, w_3) \\
&= (w_1^* \longrightarrow (w_2^* \longrightarrow w_3), w_3^* \\
&\quad \longrightarrow (w_3^* \longrightarrow w_2), w_3) \\
&= (w_1^* \longrightarrow (w_2^* \longrightarrow w_3), \\
&\quad w_3^* \longrightarrow w_2, w_3) \\
&= (w_1^* \longrightarrow (w_2^* \longrightarrow \\
&\quad w_3), w_2^* \longrightarrow w_3, w_3)
\end{aligned}$$

Therefore, $S(w_1, w_2) = (w_1^* \longrightarrow w_2, w_2)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra. □

THEOREM 3.12. *Let* $(W, \longrightarrow, 1)$ *be a weak implication algebra. Then*

$$S(w_1, w_2) = (w_2^{**}, w_1^*)$$

is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra.

PROOF. S_{12} and S_{23} are defined in the following forms:

$$S_{12}(w_1, w_2, w_3) = (w_2^{**}, w_1^*, w_3)$$

$$S_{23}(w_1, w_2, w_3) = (w_1, w_3^{**}, w_2^*)$$

We show that the equality $(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) = (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3)$ holds for all $(w_1, w_2, w_3) \in W^3$:

$$\begin{aligned}
(S_{12} \circ S_{23} \circ S_{12})(w_1, w_2, w_3) &= (S_{12} \circ S_{23})(S_{12}(w_1, w_2, w_3)) \\
&= (S_{12} \circ S_{23})(w_2^{**}, w_1^*, w_3) \\
&= S_{12}(S_{23}(w_2^{**}, w_1^*, w_3)) \\
&= S_{12}(w_2^{**}, w_3^{**}, w_1^{**}) \\
&= (w_3^{****}, w_2^{***}, w_1^{**}) \\
&= S_{23}(w_3^{****}, w_1^*, w_2^*) \\
&= S_{23}(S_{12}(w_1, w_3^{**}, w_2^*)) \\
&= (S_{23} \circ S_{12})(w_1, w_3^{**}, w_2^*) \\
&= (S_{23} \circ S_{12})(S_{23}(w_1, w_2, w_3)) \\
&= (S_{23} \circ S_{12} \circ S_{23})(w_1, w_2, w_3).
\end{aligned}$$

Therefore, $S(w_1, w_2) = (w_2^{**}, w_1^*)$ is a solution to the set-theoretical Yang-Baxter equation in the weak implication algebra. □

EXAMPLE 3.4. Consider a weak implication algebra $(W, \longrightarrow, 0, 1)$ where $W = \{0, x, y, z, t, 1\}$ is a set and the binary operation $\longrightarrow$ on W has the Cayley table as follow:

$\longrightarrow$	0	w_1	w_2	1
0	1	1	1	1
w_1	w_2	1	w_2	1
w_2	w_1	w_1	1	1
1	0	w_1	w_2	1

Then all found solutions are provided in this algebra.

REMARK 3.4. Since every Boolean algebra is a weak implication algebra, all of solutions of the set-theoretical Yang-Baxter equation in a Boolean algebra [**11**] are also solutions in a weak implication algebra.

4. Conclusion

In the study, we present a weak implication algebra which is a bounded Hilbert algebra $(H, \longrightarrow, 1)$ with the condition $(x \longrightarrow y) \longrightarrow x = x$ for all $x, y \in H$ such that $y \neq 0$ (because H is a Boolean algebra when $y = 0$), and the Yang-Baxter equation which is commonly used in various scientific, technological and industrial areas. After giving definitions and notions related to this algebraic structure and the equation, it is searched solutions to the set-theoretical Yang-Baxter equation in this algebraic structure. In fact, we find some solutions that are not mostly solutions to the set-theoretical Yang-Baxter equation in a Hilbert or a bounded Hilbert algebra but are solutions in a weak implication algebra, and exemplified that they are not solutions to the set-theoretical Yang-Baxter equation in a Hilbert or a bounded Hilbert algebra.

References

[1] R. J. Baxter. *Exactly Solved Models in Statical Mechanics*. Academy Press, London, UK, 1982.

[2] R. J. Baxter. Partition function of the eight-vertex lattice model. *Ann. Phys.* **70**(1)(1972), 193–228.

[3] D. Busneag. *Categories of algebraic logic*. Academiei Române, Bucharest, 2006.

[4] D. Busneag and M. Ghiţă. Some lattices properties of Hilbert algebras. *Bull. Math. Soc. Sci. Math. Roumanie*, **53(101)**(2)(2010), 87–107.

[5] A. Diego. Sur les algebras de Hilbert. *Collction de Logique Math., Serie A*, No. 21, Gauthiers-Villars, Paris, 1966.

[6] L. Henkin. An algebraic characterization of quantifiers. *Fund. Math.*, **37** (1950), 63–74.

[7] M. Jimbio. *Yang-Baxter Equation in Integrable Systems*, Volume 10, Advanced Series in Mathematical Physics, World Scientific Publishing Co. Inc., Singapore, 1990.

[8] M. Jimbio. Introduction to the Yang-Baxter Equation. *Int. J. Mod. Phys.*, **4**(15)(1989): 3759–3777.

[9] J. -H. Lu, M. Yan and Y. C. Zhu. On the set-theoretical Yang-Baxter equation. *Duke Math. J.*, **104**(1)(2000), 1–18.

[10] F. F. Nichita. *Hopf algebras, Quantum Groups and Yang-Baxter Equations*, (2014) (Special Issue), available at http://www.mdpi.com/journal/axioms/special_issue/hopf_algebras_2014 (accessed on 22 June 2017).

[11] F. F. Nichita. On the set-theoretical Yang-Baxter equation. *Acta Univ. Apulensis Math. Inf.*, **5**(2003), 97–100.

[12] F. F. Nichita. Yang-Baxter equations, computational methods and applications. *Axioms*, **4**(4)(2015), 423–435.

[13] T. Oner and T. Katican. On solutions to the On solutions to the set-theoretical Yang-Baxter equation in Wajsberg-algebras. *Axioms*, **7**(1)(2018): 6.

[14] T. Oner and T. Katican. On solution to the set-theoretical Yang-Baxter equation via BL-algebras. *Bull. Int. Math. Virtual Inst.*, **9**(2)(2019), 207–217.

[15] T. Oner and T. Kalkan. Yang-Baxter equations in MTL-algebras. *Bull. Int. Math. Virtual Inst.*, **9**(3)(2019), 599–607.

[16] J. H. H. Perk and Y. H. Au. *Yang-Baxter Equations*, in: J. -P. Françoise, G. L. Naber, S. T. Tsou (eds.), *Encyclopedia of Mathematical Physics*, Elseiver, Oxford, UK, 2006, 465–473.

[17] H. Rasiowa. *An algebraic approach to non-classical logics*, Studies in Logic and the Foundations of Mathematics 78, North-Holland and PWN, 1974.

[18] A. S. Nasab and A. B. Saeid. On weak implication algebra. *Soft Computing*, **23**(14)(2019), 5393–5400.

[19] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.*, **19**(23)(1967), 1312–1315.

Received by editors 25.10.2019; Revised version 10.12.2019; Available online 16.12.2019.

Department of Mathematics, Ege University, Izmir, Turkey
E-mail address: tahsin.oner@ege.edu.tr

Department of Mathematics, Ege University, Izmir, Turkey
E-mail address: tugcektcn@gmail.com

Department of Accounting and Tax Practices, Ege University, Izmir, Turkey
E-mail address: necla.kircali.gursoy@ege.edu.tr