EXISTENCE AND UNIQUENESS OF SOLUTIONS
OF SYSTEM OF NEUTRAL FRACTIONAL ORDER
BOUNDARY VALUE PROBLEMS
BY TRIPLE FIXED POINT THEOREM

Kapula Rajendra Prasad, Md. Khuddush and D. Leela

Abstract. In this paper, we establish the existence of unique solution for
system of Caputo-Hadamard type fractional neutral differential equations, for

\[ D^\alpha \left[ D^\beta \left( D^{\gamma} u_i(t) - h(t, u_i(t)) \right) \right] - g(t, u_{i+1-2\delta_{ij}}(t)) = f(t, u_{i+2-2\delta_{ij}}(t)), \]

\[ u_i(1) = A; \quad u_i'(1) = 0; \quad u_i(\tau) = B; \quad A, B \in \mathbb{R}, \]

where \( \delta_{ij} \) is a Kronecker delta function, \( 0 < \alpha, \beta, \gamma \leq 1 \) and \( D^\alpha, D^\beta \) and \( D^{\gamma} \)
are Caputo-Hadamard fractional derivatives of orders \( \alpha, \beta \) and \( \gamma \) respectively, by application of tripled fixed point theorems on cone metric spaces.

1. Introduction

The Fractional order differential equations are used to model problems in finance, fluid dynamics, and other areas of application. Recent investigations have shown that sometimes physical systems can be modeled more accurately using fractional derivative formulations [12]. On the other hand neutral functional differential equations appear in many mathematical models for various types of biological and physical phenomena [9]. The study on functional fractional neutral differential equations is very few. In [3], Benchohra et al. studied existence theory for the

2010 Mathematics Subject Classification. Primary 47H10; Secondary 54H25,34K05.
Key words and phrases. Boundary value problem, Caputo-Hadamard derivative, neutral fractional differential equation, fixed point theorem, cone metric space.

123
neutral fractional differential equation with initial condition
\[ D^\alpha[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0, b], 0 < \alpha < 1 \]
\[ x(t) = \phi(t), \quad t \in (-\infty, 0]. \]

based on the Banach contraction principle and nonlinear alternative Leray-Schauder fixed point theory. Agarwal, Zhou and He [1] considered the initial value problem of fractional neutral functional differential equations with bounded delay of the form
\[ ^cD^\alpha[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0, b], 0 < \alpha < 1 \]
\[ x(t) = \phi(t), \quad t \in [t_0, \infty), \quad t_0 \geq 0. \]

and established existence criterion by using Krasnoselski’s fixed point theorem. Yukunthorn et al. [16] studied the existence of solutions for an impulsive hybrid system of multi-orders Caputo-Hadamard fractional differential equations
\[ ^cD^\alpha x(t) = f(t, x_t), \quad t \in [t_0, T]\setminus\{t_1, t_2, \cdots, t_m\} \]
\[ \Delta x(tk) = \phi_k(x(tk)), \quad k = 1, 2, \cdots, m, \]
\[ \Delta x(tk) = \phi'_k(x(tk)), \quad k = 1, 2, \cdots, m, \]
\[ x(t_0) = I^\alpha_{t_0} g(\xi, x(\xi)), \quad x(T) = I^\alpha_{t_m} h(\eta, x(\eta)). \]

by using Krasnoselski-Zabreiko, Sadovski and ORegan fixed point theorems. Recently, Gambo et al. [6] considered generalized Caputo fractional order Cauchy problem
\[ \left( ^cD^{\alpha_k} x(t) \right)(t) = h \left[ t, x(t), \left( ^cD^{\alpha_1} x(t) \right)(t), \cdots, \left( ^cD^{\alpha_n} x(t) \right)(t) \right] \]
\[ (\gamma^k x)(a) = d_k, \quad d_k \in \mathbb{R} (k = 0, 1, \cdots, n - 1), \]

and by using Banach fixed point theorem, the existence and uniqueness of solution established. Bashir Ahmad et al. [2] studied existence theory for fractional order neutral boundary value problem
\[ D^\alpha[D^\sigma x(t) - h(t, x(t))] = f(t, x(t)), \quad t \in [0, \Xi], \Xi > 1 \]
\[ x(1) = 0, \quad x(\Xi) = 0 \]

by applying various fixed point theorems.

In 2007, Huang and Zhang [7] generalized the concept of metric spaces, by replacing real numbers as an ordered Banach space, proved certain fixed point theorems of contractive mappings on complete cone metric space by assuming the normality of a cone. Sh. Rezapour and R. Hamlbarani [15] generalized the above results by omitting the assumption of normality on the cone. Subsequently many authors have generalized the results of Huang and Zhang and studied fixed point theorems for normal and non-normal cones. Recently, Kumar and Pitchianmani [14] defined a generalized $T-$contraction and derive some new coupled fixed point theorems in cone metric spaces with total ordering condition. Motivated by aforementioned works, in this paper we establish tripled fixed point theorems on complete
EXISTENCE AND UNIQUENESS OF NEUTRAL FRACTIONAL BVP

cone metric spaces and then, as an application we study system of neutral fractional order boundary value problem, for $i = 1, 2, 3$, $1 \leq t \leq \tau$, $\tau > 1$,

$$\begin{cases}
D^{\alpha}[D^{\beta}(D^{\gamma}u_{i}(t) - h(t, u_{i}(t)))] - g(t, u_{i+1-2\delta_{ij}}(t)) = f(t, u_{i+2-\delta_{ij}}(t)), \\
u_{i}(1) = A, u'_{i}(1) = 0, u_{i}(\tau) = B; A, B \in \mathbb{R},
\end{cases}$$

where $\delta_{ij}$ is a Kronecker delta function, $0 < \alpha, \beta, \gamma \leq 1$ and $D^{\alpha}, D^{\beta}$ and $D^{\gamma}$ are Caputo-Hadamard fractional derivatives of orders $\alpha, \beta$ and $\gamma$ respectively. We establish the existence of unique solution solution for (1.1).

**Definition 1.1.** ([7]) Let $X$ be a real Banach Space. A subset $P$ of $X$ is called a cone if the following conditions are satisfied

(i) $P$ is closed, nonempty and $P \neq \{\theta\}$;
(ii) $\alpha, \beta \in \mathbb{R}^+$ and $u, v \in P$ imply that $\alpha u + \beta v \in P$;
(iii) $P \cup (-P) = \{\theta\}$.

Given a cone $P$ of $X$ define a partial ordering $\preceq$ with respect to $P$ by $u \preceq v$ if and only if $v \in P$ and $u \in P$. We shall write $u \prec v$ to indicate that $u \preceq v$ but $u \neq v$ while $u \ll v$ will stand for $v - u \in P(\text{interior of } P)$. A cone $P$ is called normal if there is a number $C > 0$ such that for all $u, v \in X$,

$$\theta \ll u \ll v \implies \|u\| \leq C\|v\|,$$

or equivalently, if, for any $n$, $u_{n} \ll v_{n} \ll w_{n}$ and $\lim_{n \to \infty} u_{n} = \lim_{n \to \infty} w_{n} = l$ imply

$$\lim_{n \to \infty} v_{n} = l.$$

The least positive number $C$ satisfying above inequality is called the normal constant of $P$.

**Theorem 1.2** ([10]). Let $E$ be a vector space and $P$ be a partial ordering cone with partial order $\preceq$ defined by $u \preceq v$ if and only if $v - u \in P$. Then $\preceq$ is a total order on $E$ if and only if $P \cup (-P) = E$.

**Definition 1.3.** ([7]) Let $X$ be a nonempty set and $d : X \times X \to E$ be a mapping such that the following conditions hold:

(i) $\theta \ll d(u, v)$ for all $u, v \in X$ and $d(u, v) = \theta$ if and only if $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \ll d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

**Definition 1.4.** ([7]) Let $(X, d)$ be a cone metric space. We say that $\{u_{n}\}$ is;

(i) a Cauchy sequence if for every $k \in X$ with $\theta \ll k$ there is $N$ such that for all $m, n > N$, $d(u_{m}, u_{n}) \ll k$;
(ii) A convergent sequence if for every $k \in X$ with $\theta \ll k$ there is $N$ such that for all $n > N$, $d(u_n, l) \ll c$, for some $l \in X$. We denote it by $\lim_{n \to \infty} u_n = l$ or $u_n \to l$.

A cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

**Theorem 1.5** ([5]). Let $(X, d)$ be a cone metric space. Then the following properties hold: (X need not be normal).

(i) If $u \ll v \ll w$, then $u \ll w$;

(ii) if $\theta \ll u \ll k$ for each $c \in \mathcal{P}$, then $u = \theta$;

(iii) if $E$ is a real Banach space with a cone $\mathcal{P}$ and if $c \ll \alpha c$ where $c \in \mathcal{P}$ and $0 \leq \alpha < 1$, then $c = \theta$.

(iv) if $a \in \mathcal{P}$, $u_n \in E$ and $u_n \to \theta$, then there exists $n_0$ such that for all $n > n_0$, we have $u_n \ll a$.

**Definition 1.6.** ([5]) Let $(X, d)$ be a cone metric space, $\mathcal{P}$ be a solid cone and $f : X \to X$. Then

(i) $f$ is said to be continuous if $\lim_{n \to \infty} u_n = u$ implies that $\lim_{n \to \infty} f u_n = f u$, for all $\{u_n\}$ in $X$.

(ii) $f$ is said to be sequentially convergent if, for every sequence $\{u_n\}$, such that $\{f u_n\}$ is convergent, then $\{u_n\}$ also is convergent.

(iii) $f$ is said to be subsequentially convergent if, for every sequence $\{u_n\}$, such that $\{f u_n\}$ is convergent, then $\{u_n\}$ has a convergent subsequence.

**Definition 1.7.** ([5]) Let $(X, d)$ be a cone metric space and $T, f : X \to X$ two mappings. A mapping $f$ is said to be a $T$–Hardy–Rogers contraction, if there exist $a_i \geq 0$, $i = 1, 2, \cdots, 5$ with $\sum_{i=1}^{5} a_i < 1$ such that for $u, v \in X$,

$$d(T f u, T f v) \leq a_1 d(T u, T v) + a_2 d(T u, T f u) + a_3 d(T v, T f v) + a_4 d(T u, T f v) + a_5 d(T v, T f v).$$

**Definition 1.8.** ([13]) Let $(X, d)$ be a cone metric space and $T : X \to X$ be a mapping. A mapping $S : X \times X \to X$ is called a $T$–Sabetzghadam–contraction if there exist $a, b \geq 0$ with $a + b < 1$ such that for all $u, v \in X$

$$d(T S(u, v), T S(u, v)) \leq a d(T u, T u) + b d(T v, T v).$$

**Definition 1.9.** ([11]) Let $(X, d)$ be a cone metric space. An element $(u, v) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(u, v) = u$ and $F(v, u) = v$.

**Definition 1.10.** ([14]) Let $(X, d)$ be a cone metric space with $\mathcal{P} \cup (-\mathcal{P}) = E$, (i.e. $\mathcal{P}$ is a total ordering cone) and $T : X \to X$ be a mapping. A mapping $S : X \times X \to X$ is called a generalized $T$–contraction if there exists $\lambda$ with $0 \leq \lambda < 1$ such that

$$d(T S(u, v), T S(u, v)) \leq \lambda \max\{d(T u, T u), d(T v, T v)\}.$$
for all \( u, v, \bar{u}, \bar{v} \in X \),

**Definition 1.11.** ([4]) An element \( (u, v, w) \in X^3 \) is called a tripled fixed point of the mapping \( F : X^3 \to X \) if \( F(u, v, w) = u, F(v, u, w) = v \) and \( F(w, v, u) = w \).

The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas that provide useful information regarding the behavior of solution of the boundary value problem (1.1), then we construct the Green’s function for (1.1), estimate bounds for the Green’s function, and some lemmas are established which are needed in our main results. In Section 3, we establish tripled fixed point theorems on cone metric space. In Section 4, we establish existence of solution for (1.1). Finally, an example is given to demonstrate our results.

### 2. Green’s function and bounds

In this section, we list some definitions and lemmas which are useful for our later discussions, and constructed Green’s function for (1.1), and established certain lemmas for the bounds of the Green’s function.

**Definition 2.1.** ([8]) The Hadamard derivative of fractional order \( r \) for a function \( p : [1, \infty) \to \mathbb{R} \) is defined as

\[
D^r p(x) = \frac{1}{\Gamma(n-r)} \left( x \frac{d}{dx} \right)^n \int_1^x \left( \log \frac{x}{s} \right)^{n-r-1} \frac{p(s)}{s} ds,
\]

\( n - 1 < r < n, n = [r] + 1 \), where \([r]\) denotes the integer part of the real number \( r \) and \( \log(\cdot) = \log_e(\cdot) \).

**Definition 2.2.** ([8]) The Hadamard fractional integral of order \( r \) for a function \( p : [1, \infty) \to \mathbb{R} \) is defined as

\[
I^r p(x) = \frac{1}{\Gamma(r)} \int_1^x \left( \log \frac{x}{s} \right)^{r-1} \frac{p(s)}{s} ds, \quad r > 0,
\]

provided the integral exists.

**Lemma 2.3.** The problem

\[
D^\alpha \left[ D^\beta \left( D^\gamma u(t) - h(t, u(t)) \right) - g(t, v(t)) \right] = f(t, w(t)),
\]

\( u(1) = A, u'(1) = 0, u(\tau) = B; A, B \in \mathbb{R} \),

is equivalent to the following integral equation

\[
u(t) = \int_1^t G_h(t, s)h(s, u(s))ds + \int_1^t G_g(t, s)g(s, v(s))ds
\]

\[
+ \int_1^t G_f(t, s)f(s, w(s))ds + \left( \frac{\log t}{\log \tau} \right)^{\beta+\gamma} (B - A) + A,
\]
where

\[
G_h(t, s) = \begin{cases}
\frac{\Delta_1}{s} \left[ \left( \log \frac{t}{s} \right)^{\gamma^{-1}} \left( \log \tau \right)^{\beta+\gamma} - \left( \log t \right)^{\beta+\gamma} \left( \log \frac{\tau}{s} \right)^{\gamma^{-1}} \right], & s \leq t, \\
-\frac{\Delta_1}{s} \left( \log t \right)^{\beta+\gamma} \left( \log \frac{\tau}{s} \right)^{\gamma^{-1}}, & t < s,
\end{cases}
\]

\[
G_g(t, s) = \begin{cases}
\frac{\Delta_2}{s} \left[ \left( \log \frac{t}{s} \right)^{\beta+\gamma-1} \left( \log \tau \right)^{\beta+\gamma} - \left( \log t \right)^{\beta+\gamma} \left( \log \frac{\tau}{s} \right)^{\beta+\gamma-1} \right], & s \leq t, \\
-\frac{\Delta_2}{s} \left( \log t \right)^{\beta+\gamma} \left( \log \frac{\tau}{s} \right)^{\beta+\gamma-1}, & t < s,
\end{cases}
\]

\[
G_f(t, s) = \begin{cases}
\frac{\Delta_3}{s} \left[ \left( \log \frac{t}{s} \right)^{\alpha+\beta+\gamma-1} \left( \log \tau \right)^{\beta+\gamma} - \left( \log t \right)^{\beta+\gamma} \left( \log \frac{\tau}{s} \right)^{\alpha+\beta+\gamma-1} \right], & s \leq t, \\
-\frac{\Delta_3}{s} \left( \log t \right)^{\beta+\gamma} \left( \log \frac{\tau}{s} \right)^{\alpha+\beta+\gamma-1}, & t < s,
\end{cases}
\]

\[
\Delta_1 = \frac{1}{(\log \tau)^{\beta+\gamma} \Gamma(\gamma)}, \quad \Delta_2 = \frac{1}{(\log \tau)^{\beta+\gamma} \Gamma(\beta + \gamma)}
\]

and

\[
\Delta_3 = \frac{1}{(\log \tau)^{\beta+\gamma} \Gamma(\alpha + \beta + \gamma)}
\]

**Proof.** The solution of the Hadamard fractional differential equation in (1.1) can be written as [8]

\[
u(t) = \frac{1}{\Gamma(\gamma)} \int_1^t \left( \log \frac{t}{s} \right)^{\gamma^{-1}} h(s, u(s)) ds + \int_1^t \left( \log \frac{t}{s} \right)^{\beta+\gamma-1} \frac{g(s, v(s))}{\Gamma(\beta + \gamma)s} ds
\]

\[
+ \frac{1}{\Gamma(\alpha + \beta + \gamma)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha+\beta+\gamma-1} \frac{f(s, w(s))}{s} ds + c_1 \left( \log t \right)^{\beta+\gamma} + c_2 \left( \log t \right)^{\beta+\gamma} + c_3,
\]
where $c_1, c_2, c_3 \in \mathbb{R}$ are arbitrary constants. In view of the boundary condition in (1.1), it follows from (2.3) that $c_3 = A, c_2 = 0$ and
\[
c_1 = \frac{\Gamma(\beta + \gamma + 1)}{(\log \tau)^{\beta + \gamma}} \left[ (B - A) + \frac{1}{\Gamma(\gamma)} \int_1^\tau \left( \log \frac{s}{\tau} \right)^{\gamma-1} \frac{h(s, u(s))}{s} ds \right]
+ \frac{1}{\Gamma(\beta + \gamma)} \int_1^\tau \left( \log \frac{s}{\tau} \right)^{\beta + \gamma - 1} \frac{g(s, v(s))}{s} ds
+ \frac{1}{\Gamma(\alpha + \beta + \gamma)} \int_1^\tau \left( \log \frac{s}{\tau} \right)^{\alpha + \beta + \gamma - 1} \frac{f(s, w(s))}{s} ds.
\]
Substituting the value of $c_1, c_2$ and $c_3$ in (2.3), we get the solution (2.2). By direct computation, we can show that $u(t)$ given by (2.2) satisfies the problem (2.1). This completes the proof. □

**Lemma 2.4.** Let $G_h(t, s), G_g(t, s)$ and $G_f(t, s)$ be given in the Lemma 2.3. Then
\begin{itemize}
  \item[(i)] $\max_{t \in [1, \tau]} \int_1^\tau |G_h(t, s)| ds = 2 \left( \frac{\log \tau}{\Gamma(\gamma + 1)} \right)^\gamma,$
  \item[(ii)] $\max_{t \in [1, \tau]} \int_1^\tau |G_g(t, s)| ds = 2 \left( \frac{\log \tau}{\Gamma(\beta + \gamma + 1)} \right)^{\beta + \gamma},$
  \item[(iii)] $\max_{t \in [1, \tau]} \int_1^\tau |G_f(t, s)| ds = 2 \left( \frac{\log \tau}{\Gamma(\alpha + \beta + \gamma + 1)} \right)^{\alpha + \beta + \gamma}.$
\end{itemize}

**Proof.** By direct integration, we can get the identities. So, we omit the details here. □

3. Tripled fixed point theorems

In this Section we establish tripled fixed point theorems which will be useful in main results. For this, we define the following contraction condition.

Let $(X, d)$ be a cone metric space with $\mathcal{P} \cup (-\mathcal{P}) = E, (\mathcal{P}$ is a total ordering cone) and $S : X \to X$ be a mapping. A mapping $F : X^3 \to X$ is called a generalized $S-$contraction if there exists $\alpha$ with $0 \leq \alpha < 1$ such that
\[
(3.1) \quad d(SF(u, v, w), SF(\bar{u}, \bar{v}, \bar{w})) \leq \alpha \max \left\{ d(Su, S\bar{u}), d(Sv, S\bar{v}), d(Sw, S\bar{w}) \right\},
\]
for all $u, v, w, \bar{u}, \bar{v}, \bar{w} \in X.$

**Theorem 3.1.** Let $(X, d)$ be a complete cone metric space, $\mathcal{P}$ be a solid cone with $\mathcal{P} \cup (-\mathcal{P}) = E$ and $S : X \to X$ be a continuous, one-to-one mapping and $F : X^3 \to X$ be a mapping such that (3.1) holds for all $u, v, w, \bar{u}, \bar{v}, \bar{w} \in X.$ Then
\begin{itemize}
  \item[(i)] there exist $\bar{u}_0, \bar{v}_0, \bar{w}_0 \in X$ such that
    \[
    \lim_{n \to \infty} SF^n(u_0, v_0, w_0) = \bar{u}_0, \quad \lim_{n \to \infty} SF^n(v_0, u_0, w_0) = \bar{v}_0
    \]
    and
    \[
    \lim_{n \to \infty} SF^n(w_0, v_0, u_0) = \bar{w}_0,
    \]
\end{itemize}
where $F^n(u_0, v_0, w_0) = u_n, F^n(v_0, u_0, w_0) = v_n$ and $F^n(w_0, v_0, u_0) = w_n$ are the iterative sequences;

(ii) if $S$ is subsequentially convergent, then $\{F^n(u_0, v_0, w_0)\}, \{F^n(v_0, u_0, w_0)\}$ and $\{F^n(w_0, v_0, u_0)\}$ have a convergent subsequence;

(iii) there exist unique $\hat{u}_0, \hat{v}_0, \hat{w}_0 \in X$ such that

$$F(\hat{u}_0, \hat{v}_0, \hat{w}_0) = \hat{u}_0, F(\hat{v}_0, \hat{u}_0, \hat{w}_0) = \hat{v}_0$$
and

$$F(\hat{w}_0, \hat{v}_0, \hat{u}_0) = \hat{w}_0;$$

(iv) if $S$ is a sequentially convergent, then for every $u_0, v_0, w_0 \in X$, the sequence $\{F^n(u_0, v_0, w_0)\}$ converges to $\hat{u}_0 \in X$, the sequence $\{F^n(v_0, u_0, w_0)\}$ converges to $\hat{v}_0 \in X$ and the sequence $\{F^n(w_0, v_0, u_0)\}$ converges to $\hat{w}_0 \in X$.

**Proof.** For $u_0, v_0 \in X$, define

$$u_{n+1} = F(u_n, v_n, w_n) = F^{n+1}(u_0, v_0, w_0), \forall n = 0, 1, 2, \ldots$$

$$v_{n+1} = F(v_n, u_n, w_n) = F^{n+1}(v_0, u_0, w_0), \forall n = 0, 1, 2, \ldots$$

$$w_{n+1} = F(w_n, v_n, u_n) = F^{n+1}(w_0, v_0, u_0), \forall n = 0, 1, 2, \ldots$$

By (3.1), we have

$$d(Su_n, Su_{n+1}) = d(SF(u_{n-1}, v_{n-1}, w_{n-1}), SF(u_n, v_n, w_n))$$

$$\leq \alpha \max \left\{d(Su_{n-1}, Su_n), d(Sv_{n-1}, Sv_n), d(sw_{n-1}, Sw_n)\right\},$$

$$d(Sv_n, Sv_{n+1}) = d(SF(v_{n-1}, u_{n-1}, w_{n-1}), SF(v_n, u_n, w_n))$$

$$\leq \alpha \max \left\{d(Sv_{n-1}, Sv_n), d(Su_{n-1}, Su_n), d(sw_{n-1}, Sw_n)\right\}$$

and

$$d(sw_n, sw_{n+1}) = d(SF(w_{n-1}, v_{n-1}, u_{n-1}), SF(w_n, v_n, u_n))$$

$$\leq \alpha \max \left\{d(sw_{n-1}, sw_n), d(sv_{n-1}, sv_n), d(sw_{n-1}, sw_n)\right\}$$

Let $d_n = \max\{d(Su_n, Su_{n+1}), d(Sv_n, Sv_{n+1}), d(sw_n, sw_{n+1})\}$. Then

$$d_n \leq \alpha \max\{d(Su_{n-1}, Su_n), d(Sv_{n-1}, Sv_n), d(sw_{n-1}, sw_n)\}$$

$$= \alpha d_{n-1}.$$ 

Continuing in this way, we get

$$\theta \leq d_n \leq \alpha d_{n-1} \leq \cdots \leq \alpha^n d_0.$$ 

If $d_0 = \theta$, then $(u_0, v_0, w_0)$ is a tripled fixed point of $F$. Assume that $d_0 > \theta$ and for $n > l$, we have

$$d(Su_l, Su_{l+1}) \leq d(Su_l, Su_{l+1}) + d(Su_{l+1}, Su_{l+2}) + \cdots + d(Su_{n-1}, Su_n),$$

(3.2) $$d(Sv_l, Sv_{l+1}) \leq d(Sv_l, Sv_{l+1}) + d(Sv_{l+1}, Sv_{l+2}) + \cdots + d(Sv_{n-1}, Sv_n),$$

(3.3) $$d(sw_l, sw_{l+1}) \leq d(sw_l, sw_{l+1}) + d(sw_{l+1}, sw_{l+2}) + \cdots + d(sw_{n-1}, sw_n),$$

(3.4) $$d(sw_l, sw_{l+1}) \leq d(sw_l, sw_{l+1}) + d(sw_{l+1}, sw_{l+2}) + \cdots + d(sw_{n-1}, sw_n).$$
From (3.2)-(3.4), we obtain

\[ X; \text{completeness of } fS \text{ which implies that } \]

\[ a \text{ such that } \]

\[ d \]

and

\[ (3.4) \quad d(Sw_t, Sw_u) \leq d(Sw_t, Sw_{t+1}) + d(Sw_{t+1}, Sw_{t+2}) + \cdots + d(Sw_{n-1}, Sw_n). \]

From (3.2)-(3.4), we obtain

\[
\begin{align*}
\max\{d(Su_t, Su_n), d(Sv_t, Sv_n), d(Sw_t, Sw_n)\} \\
\leq \max\{d(Su_t, Su_{t+1}), d(Sv_t, Sv_{t+1}), d(Sw_t, Sw_{t+1})\} + \cdots \\
+ \max\{d(Su_{n-1}, Su_n), d(Sv_{n-1}, Sv_n), d(Sw_{n-1}, Sw_n)\} \\
= d_t + d_{t+1} + \cdots + d_{n-1} \\
\leq (\alpha^t + \alpha^{t+1} + \cdots + \alpha^{n-1})d_0 \\
\leq \frac{\alpha^t}{1 - \alpha}d_0.
\end{align*}
\]

Now from Theorem 1.5, we have for every \( a \in \mathfrak{P} \), there exists a positive integer \( N \) such that

\[
\max\{d(Su_t, Su_n), d(Sv_t, Sv_n), d(Sw_t, Sw_n)\} \leq a \quad \forall \, n > t > N
\]

which implies that \( \{Su_n\}, \{Sv_n\} \) and \( \{Sw_n\} \) are Cauchy sequences in \( X \). By the completeness of \( X \), we can find \( \tilde{u}_0, \tilde{v}_0, \tilde{w}_0 \in X \) such that

\[
\lim_{n \to \infty} SF^n(u_0, v_0, w_0) = \tilde{u}_0, \quad \lim_{n \to \infty} SF^n(v_0, u_0, w_0) = \tilde{v}_0
\]

and

\[
\lim_{n \to \infty} SF^n(w_0, v_0, u_0) = \tilde{w}_0.
\]

If \( S \) is subsequentially convergent, then \( F^n(u_0, v_0, w_0), F^n(v_0, u_0, w_0) \) and \( F^n(w_0, v_0, u_0) \) have convergent subsequences. Thus, there exist \( \hat{u}_0, \hat{v}_0, \hat{w}_0 \) in \( X \) and sequences \( \{u_{n_k}\}, \{v_{n_k}\} \) and \( \{w_{n_k}\} \) such that

\[
(3.5) \quad \begin{cases} 
\lim_{k \to \infty} F^{n_k}(u_0, v_0, w_0) = \hat{u}_0, \quad \lim_{k \to \infty} F^{n_k}(v_0, u_0, w_0) = \hat{v}_0, \\
\lim_{k \to \infty} F^{n_k}(w_0, v_0, u_0) = \hat{w}_0.
\end{cases}
\]

Since \( S \) is continuous, we have

\[
(3.6) \quad \begin{cases} 
\lim_{k \to \infty} SF^{n_k}(u_0, v_0, w_0) = S\hat{u}_0, \quad \lim_{k \to \infty} SF^{n_k}(v_0, u_0, w_0) = S\hat{v}_0 \\
\lim_{k \to \infty} SF^{n_k}(w_0, v_0, u_0) = S\hat{w}_0.
\end{cases}
\]

Hence, from (3.5) and (3.6), we have

\[
S\hat{u}_0 = \tilde{u}_0, S\hat{v}_0 = \tilde{v}_0 \text{ and } S\hat{w}_0 = \tilde{w}_0.
\]
Next, by triangle inequality and (3.1), we have
\[ d(SF(\bar{u}_0, \bar{v}_0, \bar{w}_0), \mathcal{S}\bar{u}_0) \leq d(SF(\bar{u}_0, \bar{v}_0, \bar{w}_0), SF(u_{n_k}, v_{n_k}, w_{n_k})) \]
\[ + d(SF(u_{n_k}, v_{n_k}, w_{n_k}), \bar{u}_0) \]
\[ \leq \alpha \max \left\{ d(S\bar{u}_0, S\bar{u}_{n_k}), d(S\bar{v}_0, S\bar{v}_{n_k}), d(S\bar{w}_0, S\bar{w}_{n_k}) \right\} \]
\[ + d(SF(u_{n_k}, v_{n_k}, w_{n_k}), \mathcal{S}\bar{u}_0) \]
\[ \leq \alpha \max \left\{ d(S\bar{u}_0, S\bar{u}_{n_k}), d(S\bar{v}_0, S\bar{v}_{n_k}), d(S\bar{w}_0, S\bar{w}_{n_k}) \right\} \]
\[ + d(S\bar{u}_{n_k+1}, \mathcal{S}\bar{u}_0) \]

Applying Theorem 1.5, we have \( d(SF(\bar{u}_0, \bar{v}_0, \bar{w}_0), \mathcal{S}\bar{u}_0) = \theta \), i.e., \( SF(\bar{u}_0, \bar{v}_0, \bar{w}_0) = \mathcal{S}\bar{u}_0 \). As \( \mathcal{S} \) is one-to-one, we have \( F(\bar{u}_0, \bar{v}_0, \bar{w}_0) = \bar{u}_0 \). Similarly, we can prove \( F(\bar{v}_0, \bar{u}_0, \bar{w}_0) = \bar{v}_0 \) and \( F(\bar{w}_0, \bar{v}_0, \bar{u}_0) = \bar{w}_0 \). Therefore, \( (\bar{u}_0, \bar{v}_0, \bar{w}_0) \) is a tripled fixed point of \( F \).

Now, suppose \( (\bar{u}_0, \bar{v}_0, \bar{w}_0) \) is another tripled fixed point of \( F \), then
\[ d(S\bar{u}_0, \mathcal{S}\bar{u}_0) = d(SF(\bar{u}_0, \bar{v}_0, \bar{w}_0), SF(\bar{u}_0, \bar{v}_0, \bar{w}_0)) \]
\[ \leq \max \{ d(S\bar{u}_0, S\bar{u}_0), d(S\bar{v}_0, S\bar{v}_0), d(S\bar{w}_0, S\bar{w}_0) \} \]

Similarly,
\[ d(S\bar{v}_0, S\bar{v}_0) \leq \max \{ d(S\bar{v}_0, S\bar{v}_0), d(S\bar{u}_0, S\bar{u}_0), d(S\bar{w}_0, S\bar{w}_0) \} \]
and
\[ d(S\bar{w}_0, S\bar{w}_0) \leq \max \{ d(S\bar{w}_0, S\bar{w}_0), d(S\bar{v}_0, S\bar{v}_0), d(S\bar{u}_0, S\bar{u}_0) \} \]
which implies that
\[ \max \{ d(S\bar{u}_0, S\bar{u}_0), d(S\bar{v}_0, S\bar{v}_0), d(S\bar{w}_0, S\bar{w}_0) \} \]
\[ \leq \alpha \max \left\{ d(S\bar{u}_0, S\bar{u}_0), d(S\bar{v}_0, S\bar{v}_0), d(S\bar{w}_0, S\bar{w}_0) \right\} \]

Since \( \alpha < 1 \), \( d(S\bar{u}_0, S\bar{u}_0) = d(S\bar{v}_0, S\bar{v}_0) = d(S\bar{w}_0, S\bar{w}_0) \), i.e., \( S\bar{u}_0 = S\bar{u}_0 \), \( S\bar{v}_0 = S\bar{v}_0 \) and \( S\bar{w}_0 = S\bar{w}_0 \). Since \( \mathcal{S} \) is one-to-one, we have \( (\bar{u}_0, \bar{v}_0, \bar{w}_0) = (\bar{u}_0, \bar{v}_0, \bar{w}_0) \). Further, if \( \mathcal{S} \) is sequentially convergent, by replacing \( n \) by \( n_k \), we get
\[ \lim_{n \to \infty} F^n(t_0, v_0, w_0) = \bar{u}_0, \quad \lim_{n \to \infty} F^n(t_0, u_0, w_0) = \bar{v}_0 \]
and
\[ \lim_{n \to \infty} F^n(w_0, v_0, u_0) = \bar{w}_0. \]
This completes the proof. \( \square \)

**Corollary 3.2.** Let \( (X, d) \) be a complete cone metric space, \( \mathcal{P} \) be a solid cone and \( \mathcal{S} : X \to X \) be continuous, one-to-one mapping and \( F : X^3 \to X \) be a mapping such that
\[ d(SF(u, v, w), SF(\bar{u}, \bar{v}, \bar{w})) \leq \alpha \max \left\{ d(Su, S\bar{u}), d(Sv, S\bar{v}), d(Sw, S\bar{w}) \right\} \]
\[ + \frac{d(Su, S\bar{u}) + d(Sv, S\bar{v}) + d(Sw, S\bar{w})}{3} \]
for all \( u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X \) where \( 0 \leq \alpha < 1 \). Then the conclusions (i) – (iv) of Theorem 3.1 hold.

**Theorem 3.3.** Let \((X, d)\) be a complete cone metric space, \(\mathcal{P}\) be a solid cone with \(\mathcal{P} \cup (\mathcal{P}) = E\) and \(\mathcal{S} : X \to X\) be a continuous, one-to-one mapping and \(F : X^3 \to X\) be a mapping such that

\[
d(SF(u, v, w), SF(\tilde{u}, \tilde{v}, \tilde{w})) \leq \alpha \max \left\{d(SF(u, v, w), Su), d(SF(\tilde{u}, \tilde{v}, \tilde{w}), S\tilde{u})\right\},
\]

for all \( u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X \) where \( 0 \leq \alpha < 1 \). Then conclusions (i) – (iv) of Theorem 3.1 hold.

**Proof.** The proof is similar to that of Theorem 3.1.

**Corollary 3.4.** Let \((X, d)\) be a complete cone metric space, \(\mathcal{P}\) be a solid cone and \(\mathcal{S} : X \to X\) be continuous, one-to-one mapping and \(F : X^3 \to X\) be a mapping such that

\[
d(SF(u, v, w), SF(\tilde{u}, \tilde{v}, \tilde{w})) \leq \alpha \max \left\{d(SF(u, v, w), Su), d(SF(\tilde{u}, \tilde{v}, \tilde{w}), S\tilde{u}) + \frac{d(SF(u, v, w), Su), d(SF(\tilde{u}, \tilde{v}, \tilde{w}), S\tilde{u})}{2}\right\},
\]

for all \( u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X \) where \( 0 \leq \alpha < 1 \). Then conclusions (i) – (iv) of Theorem 3.1 hold.

**Theorem 3.5.** Let \((X, d)\) be a complete cone metric space, \(\mathcal{P}\) be a solid cone with \(\mathcal{P} \cup (\mathcal{P}) = E\) and \(\mathcal{S} : X \to X\) be a continuous, one-to-one mapping and \(F : X^3 \to X\) be a mapping such that

\[
d(SF(u, v, w), SF(\tilde{u}, \tilde{v}, \tilde{w})) \leq \alpha \max \{d(SF(u, v, w), Su), d(SF(\tilde{u}, \tilde{v}, \tilde{w}), S\tilde{u})\},
\]

for all \( u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X \) where \( 0 \leq \alpha < 1 \). Then conclusions (i) – (iv) of Theorem 3.1 hold.

**Corollary 3.6.** Let \((X, d)\) be a complete cone metric space, \(\mathcal{P}\) be a solid cone and \(\mathcal{S} : X \to X\) be continuous, one-to-one mapping and \(F : X^3 \to X\) be a mapping such that

\[
d(SF(u, v, w), SF(\tilde{u}, \tilde{v}, \tilde{w})) \leq \alpha \max \left\{d(SF(u, v, w), Su), d(SF(\tilde{u}, \tilde{v}, \tilde{w}), S\tilde{u}) + \frac{d(SF(u, v, w), Su), d(SF(\tilde{u}, \tilde{v}, \tilde{w}), S\tilde{u})}{2}\right\},
\]

for all \( u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X \) where \( 0 \leq \alpha < 1 \). Then conclusions (i) – (iv) of Theorem 3.1 hold.
4. An application to system of neutral fractional order boundary value problem

In this section, we study the existence of solution for system of neutral boundary value problem (1.1) using the results we obtained.

Let \( X = C([1, \tau], \mathbb{R}), \tau > 1 \) be together with the metric given by

\[
d(u, v) = \sup_{t \in [1, \tau]} |u(t) - v(t)|, \forall u, v \in X.
\]

**Theorem 4.1.** Suppose that the following hold:

(\(H_1\)) For every \( u, v, w, x, y, z \in X \) and \( t \in [1, \tau] \), we have

\[
|h(t, u(t)) - h(t, x(t))| \leq \alpha_1|u(t) - x(t)|,
\]

\[
|g(t, v(t)) - g(t, y(t))| \leq \alpha_2|v(t) - y(t)|,
\]

\[
|f(t, w(t)) - f(t, z(t))| \leq \alpha_3|w(t) - z(t)|,
\]

where \( \alpha_1, \alpha_2, \alpha_3 \) are nonnegative constants with \( \alpha = \max\{\alpha_1, \alpha_2, \alpha_3\} < 1 \).

(\(H_2\)) \( \frac{\log \tau^\gamma}{\Gamma(\gamma + 1)} + \frac{\log \tau^{\beta + \gamma}}{\Gamma(\beta + \gamma + 1)} + \frac{\log \tau^{\alpha + 2\gamma}}{\Gamma(\alpha + \beta + \gamma + 1)} \leq \frac{1}{2} \).

Then the system (1.1) has unique solution in \( C^2([1, \tau], \mathbb{R}) \).

**Proof.** From Lemma 2.3, the fractional order boundary value problem (1.1) has an integral formulation given by

\[
u_1(t) = \int_1^\tau G_h(t, s)h(s, u_1(s))ds + \int_1^\tau G_g(t, s)g(s, u_2(s))ds + \int_1^\tau G_f(t, s)f(s, u_3(s))ds + \left( \frac{\log t}{\log \tau} \right)^{\beta + \gamma} (B - A) + A,
\]

\[
u_2(t) = \int_1^\tau G_h(t, s)h(s, u_2(s))ds + \int_1^\tau G_g(t, s)g(s, u_2(s))ds + \int_1^\tau G_f(t, s)f(s, u_3(s))ds + \left( \frac{\log t}{\log \tau} \right)^{\beta + \gamma} (B - A) + A,
\]

\[
u_3(t) = \int_1^\tau G_h(t, s)h(s, u_3(s))ds + \int_1^\tau G_g(t, s)g(s, u_2(s))ds + \int_1^\tau G_f(t, s)f(s, u_1(s))ds + \left( \frac{\log t}{\log \tau} \right)^{\beta + \gamma} (B - A) + A.
\]

Define an operators \( F, F : X^3 \to X \) by

\[
F(u_1, u_2, u_3) = (F(u_1, u_2, u_3), F(u_2, u_1, u_3), F(u_3, u_2, u_1)),
\]

where

\[
F(x, y, z) = \int_1^\tau G_h(t, s)h(s, x(s))ds + \int_1^\tau G_g(t, s)g(s, y(s))ds + \left( \frac{\log t}{\log \tau} \right)^{\beta + \gamma} (B - A) + A.
\]
Then \((u, v, w)\) is the solution of (1.1) if and only if \(F(u, v, w) = (u, v, w)\), i.e., \(F(u, v, w) = u, F(v, u, w) = v\) and \(F(w, v, u) = w\). In other words \((u, v, w)\) is the solution of (1.1) if and only if \((u, v, w)\) is a tripled fixed point of \(F\). Existence of such a point follows from Theorem 3.1, by taking \(S\) as identity mapping. For \(u, v, w, x, y, z \in X\) and \(t \in [1, \tau]\), we have

\[
|F(u, v, w)(t) - F(x, y, z)(t)| \leq \int_1^\tau |G_h(t, s)||h(s, u(s)) - h(s, x(s))|ds \\
+ \int_1^\tau |G_g(t, s)||g(s, v(s)) - g(s, y(s))|ds \\
+ \int_1^\tau |G_f(t, s)||f(s, w(s)) - f(s, z(s))|ds.
\]

Now using \((H_1)\), we get

\[
|F(u, v, w)(t) - F(x, y, z)(t)| \\
\leq \alpha \left[ \int_1^\tau |G_h(t, s) + G_g(t, s) + G_f(t, s)| \max \left\{|u(s) - x(s)|, |v(s) - y(s)|, |w(s) - z(s)|\right\} ds \right] \\
\leq \alpha \alpha \left[ \int_1^\tau |G_h(t, s) + G_g(t, s) + G_f(t, s)| ds \right] \max \left\{d(u, x), d(v, y), d(w, z)\right\} \\
\leq \alpha \left[ \frac{2(\log \tau)^\gamma}{\Gamma(\gamma + 1)} + \frac{2(\log \tau)^{\beta + \gamma}}{\Gamma(\beta + \gamma + 1)} + \frac{2(\log \tau)^{\alpha + \beta + \gamma}}{\Gamma(\alpha + \beta + \gamma + 1)} \right] \max \left\{d(u, x), d(v, y), d(w, z)\right\}.
\]

Thus,

\[
d(F(u, v, w), F(x, y, z)) \leq \alpha \max \left\{d(u, x), d(v, y), d(w, z)\right\}
\]

for all \(u, v, w, x, y, z \in X\). Which shows that the contraction condition of Theorem 3.1 holds. Therefore, \(F\) has a unique tripled fixed point \((\bar{u}, \bar{v}, \bar{w}) \in C^3([1, \tau], \mathbb{R})\) which is the unique solution of (1.1).

**Example 4.2.** Consider the problem (1.1) with

\[
\alpha = \frac{1}{5}, \beta = \frac{1}{2}, \gamma = \frac{7}{10}, \tau = 1.15,
\]

\[
f(t, x) = f_1(t) + \frac{1}{7} \sin x, \ g(t, x) = g_1(t) + \frac{3}{7} \cos x
\]
and
\[ h(t,x) = h_1(t) + \frac{2}{7} e^{-x} \]
where \( f_1, g_1, h_1 \) are any real value continuous functions on \([1, \tau]\). Then it can be seen by direct calculations that
\[
|h(t, u(t)) - h(t, x(t))| \leq \frac{2}{7} |u(t) - x(t)|,
\]
\[
|g(t, v(t)) - g(t, y(t))| \leq \frac{3}{7} |v(t) - y(t)|,
\]
\[
|f(t, w(t)) - f(t, z(t))| \leq \frac{1}{7} |w(t) - z(t)|,
\]
and
\[
\left( \log \tau \right)^{\gamma} \left( \log \tau \right)^{\beta+\gamma} \left( \log \tau \right)^{\alpha+\beta+\gamma} \frac{1}{\Gamma(\gamma+1)} \frac{1}{\Gamma(\beta+\gamma+1)} \frac{1}{\Gamma(\alpha+\beta+\gamma+1)} \approx 0.4143646338 < \frac{1}{2}.
\]
Thus, all the conditions of Theorem 4.1 are satisfied. Therefore, the problem (1.1) with above choices has a unique solution on \([1, 1.15]\).

\begin{acknowledgement}
The authors thank the referee for his valuable suggestions. The author Khudush Mahammad is thankful to UGC, Government of India, New Delhi for awarding SRF under MANF No. F1-17.1/2016-17/MANF-2015-17-AND-54483 and Leela is thankful to DST-INSPIRE, Government of India, New Delhi for awarding JRF No. 2017/IF170318.
\end{acknowledgement}

References


K. Rajendra Prasad: Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, India - 530003

E-mail address: rajendra92@rediffmail.com

Md. Khuddush: Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, India - 530003

E-mail address: khuddush89@gmail.com

D. Leela: Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, India - 530003

E-mail address: leelaravidadi@gmail.com