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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF SYSTEM OF NEUTRAL FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS BY TRIPLE FIXED POINT THEOREM

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Abstract. In this paper, we establish the existence of unique solution for system of Caputo-Hadamard type fractional neutral differential equations, for $i = 1, 2, 3,\ 1 \leqslant t \leqslant \tau, \tau > 1,$

$$\mathcal{D}^{\alpha}\Big[\mathcal{D}^{\beta}\Big(\mathcal{D}^{\gamma}u_i(t) - h\big(t, u_i(t)\big)\Big) - g\big(t, u_{i+1-2\delta_{ij}}(t)\big)\Big] = f\big(t, u_{i+2-\delta_{ij}}(t)\big),$$

$$u_4 = u_1 \text{ and } j = \begin{cases} i, & \text{if } i = 2, 3, \\ 0, & \text{otherwise,} \end{cases}$$

$$u_i(1) = A,\ u_i'(1) = 0,\ u_i(\tau) = B;\ A, B \in \mathbb{R},$$

where δ_{ij} is a Kronecker delta function, $0 < \alpha,\ \beta,\ \gamma \leqslant 1$ and $\mathcal{D}^{\alpha}, \mathcal{D}^{\beta}$ and $\mathcal{D}^{\gamma}$ are Caputo-Hadamard fractional derivatives of orders α, β and γ respectively, by application of tripled fixed point theorems on cone metric spaces.

1. Introduction

The Fractional order differential equations are used to model problems in finance, fluid dynamics, and other areas of application. Recent investigations have shown that sometimes physical systems can be modeled more accurately using fractional derivative formulations [**12**]. On the other hand neutral functional differential equations appear in many mathematical models for various types of biological and physical phenomena [**9**]. The study on functional fractional neutral differential equations is very few. In [**3**], Benchohra et al. studied existence theory for the

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neutral fractional differential equation with initial condition

$$D^{\alpha}[x(t)-g(t,x_t)]=f(t,x_t),\ t\in[0,b],0<\alpha<1$$
$$x(t)=\phi(t),t\in(-\infty,0].$$

based on the Banach contraction principle and nonlinear alternative Leray-Schauder fixed point theory. Agarwal, Zhou and He [**1**] considered the initial value problem of fractional neutral functional differential equations with bounded delay of the form

$$^{c}D^{\alpha}[x(t)-g(t,x_t)]=f(t,x_t),\ t\in[0,b],0<\alpha<1$$
$$x(t)=\phi(t),\ t\in[t_0,\infty),\ t_0\geqslant 0.$$

and established existence criterion by using Krasnoselskiis fixed point theorem. Yukunthorn et al. [**16**] stadied the existence of solutions for an impulsive hybrid system of multi-orders CaputoHadamard fractional differential equations

$$\begin{aligned} &^{C}D_{t_k}^{\alpha_k}x(t)=f(t,x(t)),\ t\in[t_0,T]\backslash\{t_1,t_2,\cdots t_m\}\\ &\Delta x(t_k)=\phi_k(x(t_k)),\ k=1,2,\cdots,m,\\ &\Delta\delta x(t_k)=\phi_k^{*}(x(t_k)),\ k=1,2,\cdots,m,\\ &x(t_0)=I_{t_0}^{\mu}g(\xi,x(\xi)),\ x(T)=I_{t_m}^{\nu}h(\eta,x(\eta)),\end{aligned}$$

by using KrasnoselskiiZabreiko, Sadovski and ORegan fixed point theorems. Recently, Gambo et al. [**6**] considered generalized Caputo fractional order Cauchy problem

$$\left({}_{a}^{C}D^{\alpha,\rho}x\right)(t)=h\left[t,x(t),\left({}_{a}^{C}D^{\alpha_1,\rho}x\right)(t),\cdots,\left({}_{a}^{C}D^{\alpha_m,\rho}x\right)(t)\right]$$
$$(\gamma^k x)(a)=d_k,\ d_k\in\mathbb{R}\,(k=0,1,\cdots,n-1)\,,$$

and by using Banach fixed point theorem, the existence and uniqueness of solution established. Bashir Ahmad et al. [**2**] studied existence theory for fractional order neutral boundary value problem

$$D^{\omega}[D^{\kappa}x(t)-h(t,x(t))]=f(t,x(t)),\ t\in[0,\mathfrak{T}],\mathfrak{T}>1$$
$$x(1)=0,\ x(\mathfrak{T})=0$$

by applying various fixed point theorems.

In 2007, Huang and Zhang [**7**] generalized the concept of metric spaces, by replacing real numbers as an ordered Banach space, proved certain fixed point theorems of contractive mappings on complete cone metric space by assuming the normality of a cone. Sh. Rezapour and R. Hamlbarani [**15**] generalized the above results by omitting the assumption of normality on the cone. Subsequently many authors have generalized the results of Huang and Zhang and studied fixed point theorems for normal and non-normal cones. Recently, Kumar and Pitchiamani [**14**] defined a generalized $T-$contraction and derive some new coupled fixed point theorems in cone metric spaces with total ordering condition. Motivated by aforementioned works, in this paper we establish tripled fixed point theorems on complete cone metric spaces and then, as an application we study system of neutral fractional order boundary value problem, for $i = 1, 2, 3,\ 1 \leqslant t \leqslant \tau,\ \tau > 1$,

$$\text{(1.1)} \quad \begin{cases} \mathcal{D}^{\alpha}\Big[\mathcal{D}^{\beta}\Big(\mathcal{D}^{\gamma}u_i(t) - h\big(t, u_i(t)\big)\Big) - g\big(t, u_{i+1-2\delta_{ij}}(t)\big)\Big] = f\big(t, u_{i+2-\delta_{ij}}(t)\big), \\ \qquad u_4 = u_1 \text{ and } j = i \text{ for } i = 2, 3 \text{ and zero otherwise}, \\ \qquad\quad u_i(1) = A,\ u_i'(1) = 0,\ u_i(\tau) = B;\ A, B \in \mathbb{R}, \end{cases}$$

where δ_{ij} is a Kronecker delta function, $0 < \alpha, \beta, \gamma \leqslant 1$ and $\mathcal{D}^{\alpha}, \mathcal{D}^{\beta}$ and $\mathcal{D}^{\gamma}$ are Caputo-Hadamard fractional derivatives of orders α, β and γ respectively. We establish the existence of unique solution solution for (1.1).

Definition 1.1. ([**7**]) Let X be a real Banach Space. A subset $\mathcal{P}$ of X is called a cone if the following conditions are satisfied

(i) $\mathcal{P}$ is closed, nonempty and $\mathcal{P} \neq \{\theta\}$;

(ii) $\alpha, \beta \in \mathbb{R}^+$ and $u, v \in \mathcal{P}$ imply that $\alpha u + \beta v \in \mathcal{P}$;

(iii) $\mathcal{P} \cup (-\mathcal{P}) = \{\theta\}$.

Given a cone $\mathcal{P}$ of X define a partial ordering $\preccurlyeq$ with respect to $\mathcal{P}$ by $u \preccurlyeq v$ if and only if $v - u \in \mathcal{P}$. We shall write $u \prec v$ to indicate that $u \preccurlyeq v$ but $u \neq v$ while $u \ll v$ will stand for $v - u \in \overset{\circ}{\mathcal{P}}$(interior of $\mathcal{P}$). A cone P is called normal if there is a number $C > 0$ such that for all $u, v \in X$,

$$\theta \preccurlyeq u \preccurlyeq v \ \text{ implies } \ \|u\| \leqslant C\|v\|,$$

or equivalently, if, for any n, $u_n \preccurlyeq v_n \preccurlyeq w_n$ and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} w_n = l$ imply

$$\lim_{n\to\infty} v_n = l.$$

The least positive number C satisfying above inequality is called the normal constant of $\mathcal{P}$.

Theorem 1.2 ([**10**]). *Let E be a vector space and $\mathcal{P}$ be a partial ordering cone with partial order $\preccurlyeq$ defined by $u \preccurlyeq v$ if and only if $v - u \in \mathcal{P}$. Then $\preccurlyeq$ is a total order on E if and only if $\mathcal{P} \cup (-\mathcal{P}) = E$.*

Definition 1.3. ([**7**]) Let X be a nonempty set and $d : X \times X \to E$ be a mapping such that the following conditions hold:

(i) $\theta \preccurlyeq d(u, v)$ for all $u, v \in X$ and $d(u, v) = \theta$ if and only if $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \preccurlyeq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.4. ([**7**]) Let (X, d) be a cone metric space. We say that $\{u_n\}$ is;

(i) a Cauchy sequence if for every $k \in X$ with $\theta \ll k$ there is N such that for all $m, n > N,\ d(u_n, u_m) \ll k$;

(ii) a convergent sequence if for every $k \in X$ with $\theta \ll k$ there is N such that for all $n > N$, $d(u_n, l) \ll c$, for some $l \in X$. We denote it by $\lim_{n\to\infty} u_n = l$ or $u_n \to l$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

THEOREM 1.5 ([**5**]). *Let (X, d) be a cone metric space. Then the following properties hold: (X need not be normal).*

(i) *If $u \preccurlyeq v$and $v \ll w$, then $u \ll w$;*

(ii) *if $\theta \preccurlyeq u \ll k$ for each $c \in \overset{\circ}{\mathcal{P}}$, then $u = \theta$;*

(iii) *if E is a real Banach space with a cone $\mathcal{P}$ and if $c \preccurlyeq \alpha c$ where $c \in \mathcal{P}$ and $0 \leqslant \alpha < 1$, then $c = \theta$.*

(iv) *if $a \in \overset{\circ}{\mathcal{P}}$, $u_n \in E$ and $u_n \to \theta$, then there exists n_0 such that for all $n > n_0$, we have $u_n \ll a$.*

DEFINITION 1.6. ([**5**]) Let (X, d) be a cone metric space, $\mathcal{P}$ be a solid cone and $f : X \to X$. Then

(i) f is said to be continuous if $\lim_{n\to\infty} u_n = u$ implies that $\lim_{n\to\infty} fu_n = fu$, for all $\{u_n\}$ in X.

(ii) f is said to be sequentially convergent if, for every sequence $\{u_n\}$, such that $\{fu_n\}$ is convergent, then $\{u_n\}$ also is convergent.

(iii) f is said to be subsequentially convergent if, for every sequence $\{u_n\}$, such that $\{fu_n\}$ is convergent, then $\{u_n\}$ has a convergent subsequence.

DEFINITION 1.7. ([**5**]) Let (X, d) be a cone metric space and $T, f : X \to X$ two mappings. A mapping f is said to be a $T-$Hardy–Rogers contraction, if there exist $a_i \geqslant 0, i = 1, 2, \cdots, 5$ with $\sum_{i=1}^{5} = a_i < 1$ such that for $u, v \in X$,

$$d(Tfu, Tfv) \preccurlyeq a_1 d(Tu, Tv) + a_2 d(Tu, Tfu) + a_3 d(Tv, Tfv) + a_4 d(Tu, Tfv) \\ + a_5 d(Tv, Tfu).$$

DEFINITION 1.8. ([**13**]) Let (X, d) be a cone metric space and $T : X \to X$ be a mapping. A mapping $S : X \times X \to X$ is called a $T-$Sabetghadam–contraction if there exist $a, b \geqslant 0$ with $a + b < 1$ such that for all $u, v \in X$

$$d(TS(u, v), TS(\tilde{u}, \tilde{v})) \preccurlyeq ad(Tu, T\tilde{u}) + bd(Tv, T\tilde{v}).$$

DEFINITION 1.9. ([**11**]) Let (X, d) be a cone metric space. An element $(u, v) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(u, v) = u$ and $F(v, u) = v$.

DEFINITION 1.10. ([**14**]) Let (X, d) be a cone metric space with $\mathcal{P} \cup (-\mathcal{P}) = E$, (i.e. $\mathcal{P}$ is a total ordering cone) and $T : X \to X$ be a mapping. A mapping $S : X \times X \to X$ is called a generalized $T-$contraction if there exists λ with $0 \leqslant \lambda < 1$ such that

$$d(TS(u, v), TS(\tilde{u}, \tilde{v})) \leqslant \lambda \max\{d(Tu, T\tilde{u}), d(Tv, T\tilde{v})\}.$$

for all $u, v, \tilde{u}, \tilde{v} \in X$,

Definition 1.11. ([**4**]) An element $(u, v, w) \in X^3$ is called a tripled fixed point of the mapping $F : X^3 \to X$ if $F(u, v, w) = u, F(v, u, w) = v$ and $F(w, v, u) = w$.

The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas that provide useful information regarding the behavior of solution of the boundary value problem (1.1), then we construct the Green's function for (1.1), estimate bounds for the Green's function, and some lemmas are established which are needed in our main results. In Section 3, we establish tripled fixed point theorems on cone metric space. In Section 4, we establish existence of solution for (1.1). Finally, an example is given to demonstrate our results.

2. Green's function and bounds

In this section, we list some definitions and lemmas which are useful for our later discussions, and constructed Green's function for (1.1), and established certain lemmas for the bounds of the Green's function.

Definition 2.1. ([**8**]) The Hadamard derivative of fractional order r for a function $p : [1, \infty) \to \mathbb{R}$ is defined as

$$\mathcal{D}^r p(x) = \frac{1}{\Gamma(n-r)} \left(x \frac{d}{dx} \right)^n \int_1^x \left(\log \frac{x}{s} \right)^{n-r-1} \frac{p(s)}{s} ds,$$

$n - 1 < r < n$, $n = [r] + 1$, where $[r]$ denotes the integer part of the real number r and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. ([**8**]) The Hadamard fractional integral of order r for a function $p : [1, \infty) \to \mathbb{R}$ is defined as

$$\mathcal{I}^r p(x) = \frac{1}{\Gamma(r)} \int_1^x \left(\log \frac{x}{s} \right)^{r-1} \frac{p(s)}{s} ds, \ r > 0,$$

provided the integral exists.

Lemma 2.3. *The problem*

$$\begin{gathered} \mathcal{D}^\alpha \Big[\mathcal{D}^\beta \Big(\mathcal{D}^\gamma u(t) - h\big(t, u(t)\big) \Big) - g\big(t, v(t)\big) \Big] = f\big(t, w(t)\big), \\ u(1) = A, \ u'(1) = 0, \ u(\tau) = B; \ A, B \in \mathbb{R}, \end{gathered} \tag{2.1}$$

is equivalent to the following integral equation

$$\begin{aligned} u(t) = & \int_1^\tau G_h(t,s) h(s, u(s)) ds + \int_1^\tau G_g(t,s) g(s, v(s)) ds \\ & + \int_1^\tau G_f(t,s) f(s, w(s)) ds + \left(\frac{\log t}{\log \tau} \right)^{\beta+\gamma} (B - A) + A, \end{aligned} \tag{2.2}$$

where

$$G_h(t,s)=\begin{cases}\dfrac{\Delta_1}{s}\Bigg[\Big(\log\dfrac{t}{s}\Big)^{\gamma-1}\Big(\log\tau\Big)^{\beta+\gamma}-\Big(\log t\Big)^{\beta+\gamma}\Big(\log\dfrac{\tau}{s}\Big)^{\gamma-1}\Bigg], & s\leqslant t,\\[2ex] -\dfrac{\Delta_1}{s}\Big(\log t\Big)^{\beta+\gamma}\Big(\log\dfrac{\tau}{s}\Big)^{\gamma-1}, & t\leqslant s,\end{cases}$$

$$G_g(t,s)=\begin{cases}\dfrac{\Delta_2}{s}\Bigg[\Big(\log\dfrac{t}{s}\Big)^{\beta+\gamma-1}\Big(\log\tau\Big)^{\beta+\gamma} \\ \qquad\qquad -\Big(\log t\Big)^{\beta+\gamma}\Big(\log\frac{\tau}{s}\Big)^{\beta+\gamma-1}\Bigg], & s\leqslant t,\\[2ex] -\dfrac{\Delta_2}{s}\Big(\log t\Big)^{\beta+\gamma}\Big(\log\dfrac{\tau}{s}\Big)^{\beta+\gamma-1}, & t\leqslant s,\end{cases}$$

$$G_f(t,s)=\begin{cases}\dfrac{\Delta_3}{s}\Bigg[\Big(\log\dfrac{t}{s}\Big)^{\alpha+\beta+\gamma-1}\Big(\log\tau\Big)^{\beta+\gamma} \\ \qquad\qquad -\Big(\log t\Big)^{\beta+\gamma}\Big(\log\frac{\tau}{s}\Big)^{\alpha+\beta+\gamma-1}\Bigg], & s\leqslant t,\\[2ex] -\dfrac{\Delta_3}{s}\Big(\log t\Big)^{\beta+\gamma}\Big(\log\dfrac{\tau}{s}\Big)^{\alpha+\beta+\gamma-1}, & t\leqslant s,\end{cases}$$

$$\Delta_1=\frac{1}{\big(\log\tau\big)^{\beta+\gamma}\Gamma(\gamma)},\qquad \Delta_2=\frac{1}{\big(\log\tau\big)^{\beta+\gamma}\Gamma(\beta+\gamma)}$$

and

$$\Delta_3=\frac{1}{\big(\log\tau\big)^{\beta+\gamma}\Gamma(\alpha+\beta+\gamma)}.$$

PROOF. The solution of the Hadamard fractional differential equation in (1.1) can be written as [**8**]

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\gamma)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\gamma-1}\frac{h(s,u(s))}{s}ds+\int_1^t\Big(\log\frac{t}{s}\Big)^{\beta+\gamma-1}\frac{g(s,v(s))}{\Gamma(\beta+\gamma)s}ds\\ &+\frac{1}{\Gamma(\alpha+\beta+\gamma)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha+\beta+\gamma-1}\frac{f(s,w(s))}{s}ds+c_1\frac{(\log t)^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}\\ &+c_2\frac{(\log t)^{\gamma}}{\Gamma(\gamma+1)}+c_3,\end{aligned}\tag{2.3}$$

where c_1, c_2, $c_3 \in \mathbb{R}$ are arbitrary constants. In view of the boundary condition in (1.1), it follows from (2.3) that $c_3 = A, c_2 = 0$ and

$$\begin{aligned} c_1 = \frac{\Gamma(\beta+\gamma+1)}{(\log\tau)^{\beta+\gamma}} \bigg[&(B-A) + \frac{1}{\Gamma(\gamma)} \int_1^\tau \left(\log\frac{\tau}{s}\right)^{\gamma-1} \frac{h(s,u(s))}{s} ds \\ &+ \frac{1}{\Gamma(\beta+\gamma)} \int_1^\tau \left(\log\frac{\tau}{s}\right)^{\beta+\gamma-1} \frac{g(s,v(s))}{s} ds \\ &+ \frac{1}{\Gamma(\alpha+\beta+\gamma)} \int_1^\tau \left(\log\frac{\tau}{s}\right)^{\alpha+\beta+\gamma-1} \frac{f(s,w(s))}{s} ds \bigg]. \end{aligned}$$

Substituting the value of c_1, c_2 and c_3 in (2.3), we get the solution (2.2). By direct computation, we can show that $u(t)$ given by (2.2) satisfies the problem (2.1). This completes the proof. □

LEMMA 2.4. *Let $G_h(t,s)$, $G_g(t,s)$ and $G_f(t,s)$ be given in the Lemma 2.3. Then*

(i) $\displaystyle\max_{t\in[1,\tau]} \int_1^\tau |G_h(t,s)|ds = \frac{2\big(\log\tau\big)^\gamma}{\Gamma(\gamma+1)}$,

(ii) $\displaystyle\max_{t\in[1,\tau]} \int_1^\tau |G_g(t,s)|ds = \frac{2\big(\log\tau\big)^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}$,

(iii) $\displaystyle\max_{t\in[1,\tau]} \int_1^\tau |G_f(t,s)|ds = \frac{2\big(\log\tau\big)^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}$.

PROOF. By direct integration, we can get the identities. So, we omit the details here. □

3. Tripled fixed point theorems

In this Section we establish tripled fixed point theorems which will be useful in main results. For this, we define the following contraction condition.

Let (X,d) be a cone metric space with $\mathcal{P}\cup(-\mathcal{P}) = E, (\mathcal{P}$ is a total ordering cone) and $\mathcal{S} : X \to X$ be a mapping. A mapping $F : X^3 \to X$ is called a generalized $\mathcal{S}-$contraction if there exists α with $0 \leqslant \alpha < 1$ such that

$$d\big(\mathcal{S}F(u,v,w), \mathcal{S}F(\tilde u,\tilde v,\tilde w)\big) \preccurlyeq \alpha \max\Big\{d(\mathcal{S}u,\mathcal{S}\tilde u), d(\mathcal{S}v,\mathcal{S}\tilde v), d(\mathcal{S}w,\mathcal{S}\tilde w)\Big\}, \tag{3.1}$$

for all $u, v, w, \tilde u, \tilde v, \tilde w \in X$.

THEOREM 3.1. *Let (X,d) be a complete cone metric space, $\mathcal{P}$ be a solid cone with $\mathcal{P}\cup(\mathcal{P}) = E$ and $\mathcal{S} : X \to X$ be a continuous, one-to-one mapping and $F : X^3 \to X$ be a mapping such that (3.1) holds for all $u, v, w, \tilde u, \tilde v, \tilde w \in X$. Then*

(i) *there exist $\tilde u_0, \tilde v_0, \tilde w_0 \in X$ such that*

$$\lim_{n\to\infty} \mathcal{S}F^n(u_0,v_0,w_0) = \tilde u_0,\ \lim_{n\to\infty} \mathcal{S}F^n(v_0,u_0,w_0) = \tilde v_0$$

and

$$\lim_{n\to\infty} \mathcal{S}F^n(w_0,v_0,u_0) = \tilde w_0,$$

where $F^n(u_0, v_0, w_0) = u_n, F^n(v_0, u_0, w_0) = v_n$ *and* $F^n(w_0, v_0, u_0) = w_n$ *are the iterative sequences;*

(ii) *if* $\mathcal{S}$ *is subsequentially convergent, then* $\{F^n(u_0, v_0, w_0)\}$, $\{F^n(v_0, u_0, w_0)\}$ *and* $\{F^n(w_0, v_0, u_0)\}$ *have a convergent subsequence;*

(iii) *there exist unique* $\hat{u}_0, \hat{v}_0, \hat{w}_0 \in X$ *such that*

$$F(\hat{u}_0, \hat{v}_0, \hat{w}_0) = \hat{u}_0,\ F(\hat{v}_0, \hat{u}_0, \hat{w}_0) = \hat{v}_0$$

and

$$F(\hat{w}_0, \hat{v}_0, \hat{u}_0) = \hat{w}_0;$$

(iv) *if* $\mathcal{S}$ *is a sequentially convergent, then for every* $u_0, v_0, w_0 \in X$, *the sequence* $\{F^n(u_0, v_0, w_0)\}$ *converges to* $\hat{u}_0 \in X$, *the sequence* $\{F^n(v_0, u_0, w_0)\}$ *converges to* $\hat{v}_0 \in X$ *and the sequence* $\{F^n(w_0, v_0, u_0)\}$ *converges to* $\hat{w}_0 \in X$.

Proof. For $u_0, v_0 \in X$, define

$$\begin{aligned} u_{n+1} &= F(u_n, v_n, w_n) = F^{n+1}(u_0, v_0, w_0), \forall n = 0, 1, 2, \cdots \\ v_{n+1} &= F(v_n, u_n, w_n) = F^{n+1}(v_0, u_0, w_0), \forall n = 0, 1, 2, \cdots \\ w_{n+1} &= F(w_n, v_n, u_n) = F^{n+1}(w_0, v_0, u_0), \forall n = 0, 1, 2, \cdots \end{aligned}$$

By (3.1) , we have

$$\begin{aligned} d(\mathcal{S}u_n, \mathcal{S}u_{n+1}) &= d\big(\mathcal{S}F(u_{n-1}, v_{n-1}, w_{n-1}), \mathcal{S}F(u_n, v_n, w_n)\big) \\ &\preccurlyeq \alpha \max\Big\{d(\mathcal{S}u_{n-1}, \mathcal{S}u_n), d(\mathcal{S}v_{n-1}, \mathcal{S}v_n), d(\mathcal{S}w_{n-1}, \mathcal{S}w_n)\Big\}, \\ d(\mathcal{S}v_n, \mathcal{S}v_{n+1}) &= d\big(\mathcal{S}F(v_{n-1}, u_{n-1}, w_{n-1}), \mathcal{S}F(v_n, u_n, w_n)\big) \\ &\preccurlyeq \alpha \max\Big\{d(\mathcal{S}v_{n-1}, \mathcal{S}v_n), d(\mathcal{S}u_{n-1}, \mathcal{S}u_n), d(\mathcal{S}w_{n-1}, \mathcal{S}w_n)\Big\} \end{aligned}$$

and

$$\begin{aligned} d(\mathcal{S}w_n, \mathcal{S}w_{n+1}) &= d\big(\mathcal{S}F(w_{n-1}, v_{n-1}, u_{n-1}), \mathcal{S}F(w_n, v_n, u_n)\big) \\ &\preccurlyeq \alpha \max\Big\{d(\mathcal{S}w_{n-1}, \mathcal{S}w_n), d(\mathcal{S}v_{n-1}, \mathcal{S}v_n), d(\mathcal{S}u_{n-1}, \mathcal{S}u_n)\Big\} \end{aligned}$$

Let $d_n = \max\{d(\mathcal{S}u_n, \mathcal{S}u_{n+1}), d(\mathcal{S}v_n, \mathcal{S}v_{n+1}), d(\mathcal{S}w_n, \mathcal{S}w_{n+1})\}$. Then

$$\begin{aligned} d_n &\preccurlyeq \alpha \max\{d(\mathcal{S}u_{n-1}, \mathcal{S}u_n), d(\mathcal{S}v_{n-1}, \mathcal{S}v_n), d(\mathcal{S}w_{n-1}, \mathcal{S}w_n)\} \\ &= \alpha d_{n-1}. \end{aligned}$$

Continuing in this way, we get

$$\theta \preccurlyeq d_n \preccurlyeq \alpha d_{n-1} \preccurlyeq \cdots \preccurlyeq \alpha^n d_0.$$

If $d_0 = \theta$, then (u_0, v_0, w_0) is a tripled fixed point of F. Assume that $d_0 > \theta$ and for $n > l$, we have

$$d(\mathcal{S}u_l, \mathcal{S}u_n) \preccurlyeq d(\mathcal{S}u_l, \mathcal{S}u_{l+1}) + d(\mathcal{S}u_{l+1}, \mathcal{S}u_{l+2}) + \cdots + d(\mathcal{S}u_{n-1}, \mathcal{S}u_n), \tag{3.2}$$

$$d(\mathcal{S}v_l, \mathcal{S}v_n) \preccurlyeq d(\mathcal{S}v_l, \mathcal{S}v_{l+1}) + d(\mathcal{S}v_{l+1}, \mathcal{S}v_{l+2}) + \cdots + d(\mathcal{S}v_{n-1}, \mathcal{S}v_n), \tag{3.3}$$

and

$$d(\mathcal{S}w_l, \mathcal{S}w_n) \preccurlyeq d(\mathcal{S}w_l, \mathcal{S}w_{l+1}) + d(\mathcal{S}w_{l+1}, \mathcal{S}w_{l+2}) + \cdots + d(\mathcal{S}w_{n-1}, \mathcal{S}w_n). \tag{3.4}$$

From (3.2)-(3.4), we obtain

$$\begin{aligned}
\max\{d(\mathcal{S}u_l, \mathcal{S}u_n), d(\mathcal{S}v_l, \mathcal{S}v_n), d(\mathcal{S}w_l, \mathcal{S}w_n)\}& \\
\preccurlyeq \max\{d(\mathcal{S}u_l, \mathcal{S}u_{l+1}), d(\mathcal{S}v_l, \mathcal{S}v_{l+1}), d(\mathcal{S}w_l, \mathcal{S}w_{l+1})\} + \cdots& \\
+ \max\{d(\mathcal{S}u_{n-1}, \mathcal{S}u_n), d(\mathcal{S}v_{n-1}, \mathcal{S}v_n), d(\mathcal{S}w_{n-1}, \mathcal{S}w_n)\}& \\
= d_l + d_{l+1} + \cdots + d_{n-1}& \\
\preccurlyeq (\alpha^l + \alpha^{l+1} + \cdots + \alpha^{n-1}) d_0& \\
\preccurlyeq \frac{\alpha^l}{1-\alpha} d_0.&
\end{aligned}$$

Now from Theorem 1.5, we have for every $a \in \overset{\circ}{\mathcal{P}}$, there exists a positive integer N such that

$$\max\{d(\mathcal{S}u_l, \mathcal{S}u_n), d(\mathcal{S}v_l, \mathcal{S}v_n), \mathcal{S}w_l, \mathcal{S}w_n\} \ll a \ \forall\ n > l > N$$

which implies that $\{\mathcal{S}u_n\}, \{\mathcal{S}v_n\}$ and $\{\mathcal{S}w_n\}$ are Cauchy sequences in X. By the completeness of X, we can find $\tilde{u}_0, \tilde{v}_0,\ \tilde{w}_0 \in X$ such that

$$\lim_{n\to\infty} \mathcal{S}F^n(u_0, v_0, w_0) = \tilde{u}_0,\ \lim_{n\to\infty} \mathcal{S}F^n(v_0, u_0, w_0) = \tilde{v}_0$$

and

$$\lim_{n\to\infty} \mathcal{S}F^n(w_0, v_0, u_0) = \tilde{w}_0.$$

If $\mathcal{S}$ is subsequentially convergent, then $F^n(u_0, v_0, w_0), F^n(v_0, u_0, w_0)$ and $F^n(w_0, v_0, u_0)$ have convergent subsequences. Thus, there exist $\hat{u}_0, \hat{v}_0, \hat{w}_0$ in X and sequences $\{u_{n_k}\}, \{v_{n_k}\}$ and $\{w_{n_k}\}$ such that

$$\begin{cases} \lim\limits_{k\to\infty} F^{n_k}(u_0,v_0, w_0) = \hat{u}_0,\ \lim\limits_{k\to\infty} F^{n_k}(v_0, u_0, w_0) = \hat{v}_0, \\ \qquad \lim\limits_{k\to\infty} F^{n_k}(w_0, v_0, u_0) = \hat{w}_0. \end{cases} \tag{3.5}$$

Since $\mathcal{S}$ is continuous, we have

$$\begin{cases} \lim\limits_{k\to\infty} \mathcal{S}F^{n_k}(u_0,v_0, w_0) = \mathcal{S}\hat{u}_0,\ \lim\limits_{k\to\infty} \mathcal{S}F^{n_k}(v_0, u_0, w_0) = \mathcal{S}\hat{v}_0 \\ \qquad \lim\limits_{k\to\infty} \mathcal{S}F^{n_k}(w_0, v_0, u_0) = \mathcal{S}\hat{w}_0. \end{cases} \tag{3.6}$$

Hence, from (3.5) and (3.6), we have

$$\mathcal{S}\hat{u}_0 = \tilde{u}_0, \mathcal{S}\hat{v}_0 = \tilde{v}_0 \text{ and } \mathcal{S}\hat{w}_0 = \tilde{w}_0.$$

Next, by triangle inequality and (3.1), we have

$$\begin{aligned}
d(\mathcal{S}F(\hat{u}_0,\hat{v}_0,\hat{w}_0),\mathcal{S}\hat{u}_0) &\preccurlyeq d\big(\mathcal{S}F(\hat{u}_0,\hat{v}_0,\hat{w}_0),\mathcal{S}F(u_{n_k},v_{n_k},w_{n_k})\big)\\
&\quad + d\big(\mathcal{S}F(u_{n_k},v_{n_k},w_{n_k}),\hat{u}_0\big)\\
&\preccurlyeq \alpha\max\Big\{d(\mathcal{S}\hat{u}_0,\mathcal{S}u_{n_k}),d(\mathcal{S}\hat{v}_0,\mathcal{S}v_{n_k}),d(\mathcal{S}\hat{w}_0,\mathcal{S}w_{n_k})\Big\}\\
&\quad + d\big(\mathcal{S}F(u_{n_k},v_{n_k},w_{n_k}),\mathcal{S}\hat{u}_0\big)\\
&\preccurlyeq \alpha\max\Big\{d(\mathcal{S}\hat{u}_0,\mathcal{S}u_{n_k}),d(\mathcal{S}\hat{v}_0,\mathcal{S}v_{n_k}),d(\mathcal{S}\hat{w}_0,\mathcal{S}w_{n_k})\Big\}\\
&\quad + d\big(\mathcal{S}u_{n_k+1},\mathcal{S}\hat{u}_0\big)
\end{aligned}$$

Applying Theorem1.5 , we have $d(\mathcal{S}F(\hat{u}_0,\hat{v}_0,\hat{w}_0),\mathcal{S}\hat{u}_0)=\theta$, i.e., $\mathcal{S}F(\hat{u}_0,\hat{v}_0,\hat{w}_0)=\mathcal{S}\hat{u}_0$. As $\mathcal{S}$ is one-to-one, we have $F(\hat{u}_0,\hat{v}_0,\hat{w}_0)=\hat{u}_0$. Similarly, we can prove $F(\hat{v}_0,\hat{u}_0,\hat{w}_0)=\hat{v}_0$ and $F(\hat{w}_0,\hat{v}_0,\hat{u}_0)=\hat{w}_0$. Therefore, $(\hat{u}_0,\hat{v}_0,\hat{w}_0)$ is a tripled fixed point of F.

Now, suppose $(\breve{u}_0,\breve{v}_0,\breve{w}_0)$ is another tripled fixed point of F, then

$$\begin{aligned}
d(\mathcal{S}\hat{u}_0,\mathcal{S}\breve{u}_0) &= d\big(\mathcal{S}F(\hat{u}_0,\hat{v}_0,\hat{w}_0),\mathcal{S}F(\breve{u}_0,\breve{v}_0,\breve{w}_0)\big)\\
&\preccurlyeq \max\{d(\mathcal{S}\hat{u}_0,\mathcal{S}\breve{u}_0),d(\mathcal{S}\hat{v}_0,\mathcal{S}\breve{v}_0),d(\mathcal{S}\hat{w}_0,\mathcal{S}\breve{w}_0)\}.
\end{aligned}$$

Similarly,

$$d(\mathcal{S}\hat{v}_0,\mathcal{S}\breve{v}_0)\preccurlyeq \max\{d(\mathcal{S}\hat{v}_0,\mathcal{S}\breve{v}_0),d(\mathcal{S}\hat{u}_0,\mathcal{S}\breve{u}_0),d(\mathcal{S}\hat{w}_0,\mathcal{S}\breve{w}_0)\}$$

and

$$d(\mathcal{S}\hat{w}_0,\mathcal{S}\breve{w}_0)\preccurlyeq \max\{d(\mathcal{S}\hat{w}_0,\mathcal{S}\breve{w}_0),d(\mathcal{S}\hat{v}_0,\mathcal{S}\breve{v}_0),d(\mathcal{S}\hat{u}_0,\mathcal{S}\breve{u}_0)\},$$

which implies that

$$\begin{aligned}
&\max\{d(\mathcal{S}\hat{u}_0,\mathcal{S}\breve{u}_0),d(\mathcal{S}\hat{v}_0,\mathcal{S}\breve{v}_0),d(\mathcal{S}\hat{w}_0,\mathcal{S}\breve{w}_0)\}\\
&\qquad\preccurlyeq \alpha\max\Big\{d(\mathcal{S}\hat{u}_0,\mathcal{S}\breve{u}_0),d(\mathcal{S}\hat{v}_0,\mathcal{S}\breve{v}_0),d(\mathcal{S}\hat{w}_0,\mathcal{S}\breve{w}_0)\Big\}.
\end{aligned}$$

Since $\alpha<1$, $d(\mathcal{S}\hat{u}_0,\mathcal{S}\breve{u}_0)=d(\mathcal{S}\hat{v}_0,\mathcal{S}\breve{v}_0)=d(\mathcal{S}\hat{w}_0,\mathcal{S}\breve{w}_0)$. i.e., $\mathcal{S}\hat{u}_0=\mathcal{S}\breve{u}_0,\mathcal{S}\hat{v}_0=\mathcal{S}\breve{v}_0$ and $\mathcal{S}\hat{w}_0=\mathcal{S}\breve{w}_0$. Since $\mathcal{S}$ is one-to-one, we have $(\hat{u}_0,\hat{v}_0,\hat{w}_0)=(\breve{u}_0,\breve{v}_0,\breve{w}_0)$. Further, if $\mathcal{S}$ is sequentially convergent, by replacing n by n_k, we get

$$\lim_{n\to\infty}F^n(u_0,v_0,w_0)=\hat{u}_0,\ \lim_{n\to\infty}F^n(v_0,u_0,w_0)=\hat{v}_0$$

and

$$\lim_{n\to\infty}F^n(w_0,v_0,u_0)=\hat{w}_0.$$

This completes the proof. □

Corollary 3.2. *Let (X,d) be a complete cone metric space, $\mathcal{P}$ be a solid cone and $\mathcal{S}:X\to X$ be continuous, one-to-one mapping and $F:X^3\to X$ be a mapping such that*

$$\begin{aligned}
d\big(\mathcal{S}F(u,v,w),\mathcal{S}F(\tilde{u},\tilde{v},\tilde{w})\big)\preccurlyeq \alpha\max\Bigg\{&d(\mathcal{S}u,\mathcal{S}\tilde{u}),d(\mathcal{S}v,\mathcal{S}\tilde{v}),d(\mathcal{S}w,\mathcal{S}\tilde{w}),\\
&\frac{d(\mathcal{S}u,\mathcal{S}\tilde{u})+d(\mathcal{S}v,\mathcal{S}\tilde{v})+d(\mathcal{S}w,\mathcal{S}\tilde{w})}{3}\Bigg\},
\end{aligned}$$

for all $u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X$ *where* $0 \leqslant \alpha < 1$. *Then the conclusions* $(i) - (iv)$ *of Theorem 3.1 hold.*

THEOREM 3.3. *Let* (X, d) *be a complete cone metric space,* $\mathcal{P}$ *be a solid cone with* $\mathcal{P} \cup (\mathcal{P}) = E$ *and* $\mathcal{S} : X \to X$ *be a continuous, one-to-one mapping and* $F : X^3 \to X$ *be a mapping such that*

$$d\big(\mathcal{S}F(u,v,w), \mathcal{S}F(\tilde{u},\tilde{v},\tilde{w})\big) \preccurlyeq \alpha \max\Big\{d\big(\mathcal{S}F(u,v,w), \mathcal{S}u\big), d\big(\mathcal{S}F(\tilde{u},\tilde{v},\tilde{w}), \mathcal{S}\tilde{u}\big)\Big\},$$

for all $u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X$ *where* $0 \leqslant \alpha < 1$. *Then conclusions* $(i) - (iv)$ *of Theorem 3.1 hold.*

PROOF. The proof is similar to that of Theorem 3.1. □

COROLLARY 3.4. *Let* (X, d) *be a complete cone metric space,* $\mathcal{P}$ *be a solid cone and* $\mathcal{S} : X \to X$ *be continuous, one-to-one mapping and* $F : X^3 \to X$ *be a mapping such that*

$$\begin{aligned} d\big(\mathcal{S}F(u,v,w), \mathcal{S}F(\tilde{u},\tilde{v},\tilde{w})\big) \preccurlyeq \alpha \max\Big\{&d\big(\mathcal{S}F(u,v,w), \mathcal{S}u\big), d\big(\mathcal{S}F(\tilde{u},\tilde{v},\tilde{w}), \mathcal{S}\tilde{u}\big) \\ &+ \frac{d\big(\mathcal{S}F(u,v,w), \mathcal{S}u\big), d\big(\mathcal{S}F(\tilde{u},\tilde{v},\tilde{w}), \mathcal{S}\tilde{u}\big)}{2}\Big\}, \end{aligned}$$

for all $u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X$ *where* $0 \leqslant \alpha < 1$. *Then conclusions* $(i) - (iv)$ *of Theorem 3.1 hold.*

THEOREM 3.5. *Let* (X, d) *be a complete cone metric space,* $\mathcal{P}$ *be a solid cone with* $\mathcal{P} \cup (\mathcal{P}) = E$ *and* $\mathcal{S} : X \to X$ *be a continuous, one-to-one mapping and* $F : X^3 \to X$ *be a mapping such that*

$$d\big(\mathcal{S}F(u,v,w), \mathcal{S}F(\tilde{u},\tilde{v},\tilde{w})\big) \preccurlyeq \alpha \max\{d\big(\mathcal{S}F(u,v,w), \mathcal{S}\tilde{u}\big), d\big(\mathcal{S}F(\tilde{u},\tilde{v},\tilde{w}), \mathcal{S}u\big)\},$$

for all $u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X$ *where* $0 \leqslant \alpha < 1$. *Then conclusions* $(i) - (iv)$ *of Theorem 3.1 hold.*

COROLLARY 3.6. *Let* (X, d) *be a complete cone metric space,* $\mathcal{P}$ *be a solid cone and* $\mathcal{S} : X \to X$ *be continuous, one-to-one mapping and* $F : X^3 \to X$ *be a mapping such that*

$$\begin{aligned} d\big(\mathcal{S}F(u,v,w), \mathcal{S}F(\tilde{u},\tilde{v},\tilde{w})\big) \preccurlyeq \alpha \max\Big\{&d\big(\mathcal{S}F(u,v,w), \mathcal{S}\tilde{u}\big), d\big(\mathcal{S}F(\tilde{u},\tilde{v},\tilde{w}), \mathcal{S}u\big) \\ &+ \frac{d\big(\mathcal{S}F(u,v,w), \mathcal{S}\tilde{u}\big), d\big(\mathcal{S}F(\tilde{u},\tilde{v},\tilde{w}), \mathcal{S}u\big)}{2}\Big\}, \end{aligned}$$

for all $u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in X$ *where* $0 \leqslant \alpha < 1$. *Then conclusions* $(i) - (iv)$ *of Theorem 3.1 hold.*

4. An application to system of nuetral fractional order boundary value problem

In this section, we study the existence of solution for system of nuetral boundary value problem (1.1) using the results we obtained.

Let $X = C([1,\tau],\mathbb{R}), \tau > 1$ be together with the metric given by

$$d(u,v) = \sup_{t\in[1,\tau]} |u(t) - v(t)|, \forall u, v \in X.$$

THEOREM 4.1. *Suppose that the following hold:*

(H_1) *For every* $u, v, w, x, y, z \in X$ *and* $t \in [1,\tau]$, *we have*

$$|h(t,u(t)) - h(t,x(t))| \leqslant \alpha_1 |u(t) - x(t)|,$$
$$|g(t,v(t)) - g(t,y(t))| \leqslant \alpha_2 |v(t) - y(t)|,$$
$$|f(t,w(t)) - f(t,z(t))| \leqslant \alpha_3 |w(t) - z(t)|,$$

where $\alpha_1, \alpha_2, \alpha_3$ *are nonnegative constants with* $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\} < 1$.

(H_2) $\dfrac{\big(\log\tau\big)^{\gamma}}{\Gamma(\gamma+1)} + \dfrac{\big(\log\tau\big)^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + \dfrac{\big(\log\tau\big)^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} \leqslant \dfrac{1}{2}.$

Then the system (1.1) *has unique solution in* $C^3([1,\tau],\mathbb{R})$.

PROOF. From Lemma 2.3, the fractional order boundary value problem (1.1) has an integral formulation given by

$$\begin{aligned}
u_1(t) &= \int_1^\tau G_h(t,s)h(s,u_1(s))ds + \int_1^\tau G_g(t,s)g(s,u_2(s))ds \\
&\quad + \int_1^\tau G_f(t,s)f(s,u_3(s))ds + \left(\frac{\log t}{\log\tau}\right)^{\beta+\gamma}(B-A) + A, \\
u_2(t) &= \int_1^\tau G_h(t,s)h(s,u_2(s))ds + \int_1^\tau G_g(t,s)g(s,u_1(s))ds \\
&\quad + \int_1^\tau G_f(t,s)f(s,u_3(s))ds + \left(\frac{\log t}{\log\tau}\right)^{\beta+\gamma}(B-A) + A \\
u_3(t) &= \int_1^\tau G_h(t,s)h(s,u_3(s))ds + \int_1^\tau G_g(t,s)g(s,u_2(s))ds \\
&\quad + \int_1^\tau G_f(t,s)f(s,u_1(s))ds + \left(\frac{\log t}{\log\tau}\right)^{\beta+\gamma}(B-A) + A.
\end{aligned}$$

Define an operators $\mathcal{F}, F : X^3 \to X$ by

$$\mathcal{F}(u_1,u_2,u_3) = (F(u_1,u_2,u_3), F(u_2,u_1,u_3), F(u_3,u_2,u_1),$$

where

$$\begin{aligned}
F(x,y,z) &= \int_1^\tau G_h(t,s)h(s,x(s))ds + \int_1^\tau G_g(t,s)g(s,y(s))ds \\
&\quad + \int_1^\tau G_f(t,s)f(s,z(s))ds + \left(\frac{\log t}{\log\tau}\right)^{\beta+\gamma}(B-A) + A.
\end{aligned}$$

Then (u, v, w) is the solution of (1.1) if and only if $\mathcal{F}(u, v, w) = (u, v, w)$, i.e., $F(u, v, w) = u, F(v, u, w) = v$ and $F(w, v, u) = w$. In other words (u, v, w) is the solution of (1.1) if and only if (u, v, w) is a tripled fixed point of F. Existence of such a point follows from Theorem 3.1, by taking $\mathcal{S}$ as identity mapping. For $u, v, w, x, y, z \in X$ and $t \in [1, \tau]$, we have

$$\begin{aligned}|F(u,v,w)(t) - F(x,y,z)(t)| \leqslant &\int_1^\tau |G_h(t,s)||h(s,u(s)) - h(s,x(s))|ds \\ &+ \int_1^\tau |G_g(t,s)||g(s,v(s)) - g(s,y(s))|ds \\ &+ \int_1^\tau |G_f(t,s)||f(s,w(s)) - f(s,z(s))|ds.\end{aligned}$$

Now using (H_1), we get

$$\begin{aligned}&|F(u,v,w)(t) - F(x,y,z)(t)| \\ &\quad\leqslant \alpha_1 \int_1^\tau |G_h(t,s)||u(s) - x(s)|ds + \alpha_2 \int_1^\tau |G_g(t,s)||v(s) - y(s)|ds \\ &\qquad\qquad + \alpha_3 \int_1^\tau |G_f(t,s)||w(s)) - z(s)|ds \\ &\quad\leqslant \alpha \Big[\int_1^\tau |G_h(t,s) + G_g(t,s) + G_f(t,s)\Big] \\ &\qquad\qquad \times \max \Big\{ ||u(s) - x(s)|, ||v(s) - y(s)|, ||w(s) - z(s)| \Big\} ds \\ &\quad\leqslant \alpha \Big[\int_1^\tau |G_h(t,s) + G_g(t,s) + G_f(t,s)\Big] ds \times \max \Big\{ d(u,x), d(v,y), d(w,z) \Big\} \\ &\quad\leqslant \alpha \Big[\frac{2\big(\log\tau\big)^\gamma}{\Gamma(\gamma+1)} + \frac{2\big(\log\tau\big)^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + \frac{2\big(\log\tau\big)^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} \Big] \\ &\qquad\qquad \times \max \Big\{ d(u,x), d(v,y), d(w,z) \Big\} \\ &\quad\leqslant \alpha \max \Big\{ d(u,x), d(v,y), d(w,z) \Big\}.\end{aligned}$$

Thus,

$$d(F(u,v,w), F(x,y,z)) \leqslant \alpha \max \Big\{ d(u,x), d(v,y), d(w,z) \Big\}$$

for all $u, v, w, x, y, z \in X$. Which shows that the contraction condition of Theorem 3.1 holds. Therefore, F has a unique tripled fixed point $(\tilde{u}, \tilde{v}, \tilde{w}) \in C^3([1, \tau], \mathbb{R})$ which is the unique solution of (1.1).

Example 4.2. Consider the problem (1.1) with

$$\alpha = \frac{1}{5},\ \beta = \frac{1}{2},\ \gamma = \frac{7}{10},\ \tau = 1.15,$$

$$f(t,x) = f_1(t) + \frac{1}{7}\sin x,\ g(t,x) = g_1(t) + \frac{3}{7}\cos x$$

and

$$h(t,x) = h_1(t) + \frac{2}{7}e^{-x}$$

where f_1, g_1, h_1 are any real value continuous functions on $[1,\tau]$. Then it can be seen by direct calculations that

$$|h(t,u(t)) - h(t,x(t))| \leqslant \frac{2}{7}|u(t) - x(t)|,$$
$$|g(t,v(t)) - g(t,y(t))| \leqslant \frac{3}{7}|v(t) - y(t)|,$$
$$|f(t,w(t)) - f(t,z(t))| \leqslant \frac{1}{7}|w(t) - z(t)|,$$

and

$$\frac{\big(\log\tau\big)^{\gamma}}{\Gamma(\gamma+1)} + \frac{\big(\log\tau\big)^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + \frac{\big(\log\tau\big)^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)} \approx 0.4143646338 < \frac{1}{2}.$$

Thus, all the conditions of Theorem 4.1 are satisfied. Therefore, the problem (1.1) with above choices has a unique solution on $[1, 1.15]$.

□

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