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A CONNECTION BETWEEN QUASI-ANTIORDERS AND PAIRS OF COEQUALITIES[1]

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Abstract

A connection of lattice of quasi-antiorders of set with apartness with the direct square of lattice of coequalities of set with apartness is obtained.

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Dedicated to professor Siniša Crvenkovć on his 60th birthday

1 Introduction

This investigation is in Bishop's constructive mathematics. Let us consider a set with apartness $(S, =, \neq)$. A binary relation $\sigma \subseteq S \times S$ is called a quasi-antiorder of S if σ is consistent and cotransitive. If we set $\sigma' = \sigma^{-1} = \{(x, y) : (y, x) \in \sigma\}$, then $\Im(S)$, the family of all quasi-antiorders of S, becomes an involution lattice $\Im(S) = (\Im(S), \wedge, \vee, ')$ where $\wedge$ is the cotransitive fulfillment of the intersection, and $\vee$ is the union. The sublattice $\{\sigma \in \Im(S) : \sigma' = \sigma\} = \mathbf{Q}(S)$ is just the coequality lattice of S. Let $\mathbf{Q}(S) \times \mathbf{Q}(S)$ denote the direct square of the lattice $\mathbf{Q}(S)$ equipped with the involution defined by $(\alpha, \beta)' = (\beta, \alpha)$. Then $\mathbf{Q}(S) \times \mathbf{Q}(S)$ is an involution lattice.

For anti-ordered sets $(S, =, \neq, \alpha)$ and $(T, =, \neq, \beta)$, and a mapping $\varphi : S \longrightarrow T$, let

$$\Phi : \Im(T) \ni \sigma \longmapsto \{(x, x') \in S \times S : (\varphi(x), \varphi(x')) \in \sigma\} = \varphi^{-1}(\sigma) \in \Im(S)$$

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and

$$F : \mathbf{Q}(T) \times \mathbf{Q}(T) \ni (q, q') \longmapsto (\varphi^{-1}(q), \varphi^{-1}(q')) \in \mathbf{Q}(S) \times \mathbf{Q}(S).$$

For lattices $\Im(S)$ and $\mathbf{Q}(S)$ consider the following mappings ϑ, $\mu : \Im(S) \longrightarrow \mathbf{Q}(S)$, defined by

$$\vartheta(\sigma) = c(\sigma \cap \alpha) \cup c(\sigma^{-1}\alpha^{-1}), \quad \mu(\sigma) = c(\sigma \cap \alpha^{-1}) \cup c(\sigma^{-1} \cap \alpha),$$

and $\Psi : \mathbf{Q}(S) \times \mathbf{Q}(S) \longrightarrow \Im(S)$ defined by

$$(q_1, q_2) = c(q_1 \cap \alpha) \cup c(q_2 \cap \alpha^{-1}).$$

To prove that $\Theta : \Im(S) \longrightarrow \mathbf{Q}(S) \times \mathbf{Q}(S)$, defined by

$$\Theta(\sigma) = (\vartheta(\sigma), \mu(\sigma))$$

for any $\sigma \in \Im(S)$, and Ψ are natural transformations, let $\varphi : S \longrightarrow T$ be a surjective strongly extensional isotone and reverse isotone mapping between anti-order relational systems. We have to show that the following equality holds

$$\Theta_S \circ \Phi = F \circ \Theta_T \ ,$$

i.e. that the following diagram

$$\begin{array}{ccc} \Im(T) & \longrightarrow & \Im(S) \\ \downarrow & & \downarrow \\ \mathbf{Q}(T) \times \mathbf{Q}(T) & \longrightarrow & \mathbf{Q}(S) \times \mathbf{Q}(S) \end{array}$$

commutes. Following ideas presented in article [4] we analyze similar situation on connection between quasi-antiorders and coequality relations on anti-ordered sets with apartness. This article is a continuation of the second author's forthcoming article [8].

2 Preliminaries

Let $(A, =, \neq)$ be a set in the sense of [1], [2], [3] and [9], where '$\neq$' is a binary relation on A which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ \ x \neq z \Longrightarrow x \neq y \ \vee\ y \neq z,$$
$$x \neq y \ \wedge\ y = z \Longrightarrow x \neq z,$$

called *apartness* (A. Heyting). Let Y be a subset of A and $x \in A$. The subset Y of A is *strongly extensional* in A if and only if $y \in Y \Longrightarrow y \neq x \vee x \in Y$ ([1], [9]).

Let $\varphi : (A, =, \neq) \longrightarrow (B, =, \neq)$ be a mapping. We say that :

(a) φ is *strongly extensional* if and only if

$$(\forall a, b \in A)(\varphi(a) \neq \varphi(b) \Longrightarrow a \neq b);$$

(b) φ is an *embedding* if and only if

$$(\forall a, b \in A)(a \neq b \Longrightarrow \varphi(a) \neq \varphi(b)).$$

Let $\alpha \subseteq A \times B$ and $\beta \subseteq B \times C$ be relations. The *filled product* ([5]) of relations α and β is the relation

$$\beta * \alpha = \{(a, c) \in A \times C : (\forall b \in B)((a, b) \in \alpha \vee (b, c) \in \beta\}.$$

It is easy to check that the filled product is associative. (See, for example, [6]) For $\beta = \alpha$ we put ${}^2\alpha = \alpha * \alpha$, and for given natural n, by induction, we define

$${}^{n+1}\alpha = {}^n\alpha * \alpha \, (= \alpha * {}^n\alpha), \ \ {}^1\alpha = \alpha.$$

Besides, for any relation $\alpha \subseteq X \times X$, we can construct the relation

$$c(\alpha) = \bigcap_{n \in N} {}^n\alpha.$$

It is clear that $c(\alpha) \subseteq \alpha$ and the following $c(\alpha) \subseteq c(\alpha) * c(\alpha)$ is valid. It is called *cotransitive internal fulfilment* of α. This notion was studied by the second author in his articles [5] and [6]. If α is a consistent relation on set A, then $c(\alpha)$ is the maximal quasi-antiorder on A under α (see, for example, article [5] or in [6])

A relation α on A is *anti-order* on A ([7], [8]) if and only if

$$\alpha \subseteq \neq, \ \alpha \subseteq \alpha * \alpha, \ \neq \subseteq \alpha \cup \alpha^{-1}.$$

For anti-order on set A we say that it is *complete* anti-order if $\alpha \cap \alpha^{-1} = \emptyset$ holds. As in [7], a relation $\tau \subseteq A \times A$ is a *quasi-antiorder* on A if and only if

$$\tau \subseteq \alpha \, (\subseteq \neq), \ \ \tau \subseteq \tau * \tau.$$

Quasi-antiorder τ is *complete* if $\tau \cap \tau^{-1} = \emptyset$ holds. Let us note that if α and β are (quasi-)anti-orders on a set, then the union $\alpha \cup \beta$ is also a (quasi-)anti-order. Finally, relation $q \subseteq A \times A$ is a *coequality relation* on A if

$$q \subseteq \neq, \ q^{-1} = q, \ \ q \subseteq q * q.$$

Let $\varphi : ((A, =, \neq), \alpha) \longrightarrow ((B, =, \neq), \beta)$ be a strongly extensional mapping of relational systems. φ is called *isotone* if

$$(\forall x, y \in A)((x, y) \in \alpha \Longrightarrow (\varphi(x), \varphi(y)) \in \beta);$$

φ is called *reverse isotone* if and only if

$$(\forall x, y \in A)((\varphi(x), \varphi(y)) \in \beta \Longrightarrow (x, y) \in \alpha).$$

Let us note that if $\varphi : ((A, =, \neq), \alpha) \longrightarrow ((B, =, \neq), \beta)$ is a strongly extensional mapping of quasi-antiorder systems then the relation $\varphi^{-1}(\beta) = \{(x, y) \in A \times A : (\varphi(x), \varphi(y)) \in \beta\}$ is a quasi-antiorder too. It is easy to verify the following:

(1) φ is isotone if $\alpha \subseteq \varphi^{-1}(\beta)$ and

(2) φ is reverse isotone if $\varphi^{-1}(\beta) \subseteq \alpha$.

3 Proofs

1. Φ is an isotone mapping between lattices $\Im(T)$ and $\Im(S)$. Indeed, if σ and τ are quasi-antiorders on semigroup T, then $\varphi^{-1}(\sigma)$ and $\varphi^{-1}(\tau)$ are quasi-antiorders on S. Suppose that $\sigma \subseteq \tau$. If $(a, a') \in \varphi^{-1}(\sigma)$, i.e. if $(\varphi(a), \varphi(b)) \in \sigma \subseteq \tau$, then $(a, b) \in \varphi^{-1}(\tau)$ and, hence, Φ is an isotone mapping.

2. F is an isotone mapping from lattice $\mathbf{Q}(T) \times \mathbf{Q}(T)$ to lattice $\mathbf{Q}(S) \times \mathbf{Q}(S)$. In fact, if q and q' are a pair of coequalities on semigroup T, then $\varphi^{-1}(q)$ and $\varphi^{-1}(q')$ are coequalities on S.

3. The mappings ϑ, $\mu : \Im(S) \longrightarrow \mathbf{Q}(S)$ are correctly defined isotone mappings. Let σ be a quasi-antiorder on semigroup S. It is clear that $\vartheta(\sigma) = c(\sigma \cap \alpha) \cup c(\sigma^{-1} \cap \alpha^{-1})$ and $\mu(\sigma) = c(\sigma \cap \alpha^{-1}) \cup c(\sigma^{-1} \cap \alpha)$ are consistent and cotransitive relations on S.

3.1. Let (a, b) be an arbitrary element of $\vartheta(\sigma)$. Then $(a, b) \in c(\sigma \cap \alpha)$ or $(a, b) \in c(\sigma^{-1} \cap \alpha^{-1})$. Assume that $(a, b) \in c(\sigma \cap \alpha)$. Then $(b, a) \in \sigma^{-1} \cap \alpha^{-1}$. Further on, out of $(a, b) \in {}^2(\sigma \cap \alpha)$, i.e. out of $(\forall t)((a, t) \in \sigma \cap \alpha \vee (t, b) \in \sigma \cap \alpha)$ we have $(\forall t)((b, t) \in \sigma^{-1} \cap \alpha^{-1} \vee (t, a) \in \sigma^{-1} \cap \alpha^{-1})$. Hence, we have $(b, a) \in {}^2(\sigma^{-1} \cap \alpha^{-1})$. Suppose that the implication $(a, b) \in {}^n(\sigma \cap \alpha) \Longrightarrow (b, a) \in {}^n(\sigma^{-1} \cap \alpha^{-1})$ holds for natural number n. Now, out of $(a, b) \in {}^{n+1}(\sigma \cap \alpha) = (\sigma \cap \alpha) * {}^n(\sigma \cap \alpha)$, i.e. from $(\forall t)((a, t) \in (\sigma \cap \alpha) \vee (t, b) \in {}^n(\sigma \cap \alpha))$ we have $(\forall t)((b, t) \in \sigma^{-1} \cap \alpha^{-1} \vee (t, a) \in {}^n(\sigma^{-1} \cap \alpha^{-1}))$. This means that $(b, a) \in {}^{n+1}(\sigma^{-1} \cap \alpha^{-1})$. So, by induction, we have the implication $(a, b) \in c(\sigma \cap \alpha) \Longrightarrow (b, a) \in c(\sigma^{-1} \cap \alpha^{-1})$. For the implication $(a, b) \in c(\sigma^{-1} \cap \alpha^{-1}) \Longrightarrow (b, a) \in (\sigma \cap \alpha)$ the proof is analogous to the previous proof. Therefore, finally, we have that $(a, b) \in \vartheta(\sigma) \Longrightarrow (b, a) \in \vartheta(\sigma)$.

3.2. The proof that relation $\mu(\sigma) = c(\sigma \cap \alpha^{-1}) \cup c(\sigma^{-1} \cap \alpha)$ is a coequality relation on the set S is analogous.

4. Let q_1 and q_2 be coequalities on set S. It is easy to check that the relation $\Psi(q_1, q_2) = c(q_1 \cap \alpha) \cup c(q_2 \cap \alpha^{-1})$ is a quasi-antiorder relation on S.

5. Let φ be an isotone and reverse isotone mapping from anti-ordered set $(S, =, \neq, \alpha)$ onto anti-ordered set $(T, =, \neq, \beta)$, and let σ be an arbitrary quasi-antiorder relation on set T. Then, we have

$$(\Theta_S \circ \Phi)(\sigma) = (F \circ \Theta_T)(\sigma).$$

This means

$$(\vartheta(\varphi^{-1}(\sigma)), \mu(\varphi^{-1}(\sigma))) = (\varphi^{-1}(\vartheta(\sigma)), \varphi^{-1}(\mu(\sigma)))$$

i.e. we need to prove the following $\vartheta(\varphi^{-1}(\sigma)) = \varphi^{-1}(\vartheta(\sigma))$ and $\mu(\varphi^{-1}(\sigma)) = \varphi^{-1}(\mu(\sigma))$.

5.1. Let (x, z) be an arbitrary element of $\varphi^{-1}(\vartheta(\sigma))$. Then,

$$(\varphi(x), \varphi(z)) \in \vartheta(\sigma) = c(\sigma \cap \beta) \cup c(\sigma^{-1} \cap \beta^{-1})$$

and hence

$$(\varphi(x), \varphi(z)) \in c(\sigma \cap \beta) \ \vee \ (\varphi(x), \varphi(z)) \in c(\sigma^{-1} \cap \beta^{-1}).$$

Assume $(\varphi(x), \varphi(z)) \in c(\sigma\cap\beta) = \bigcap {}^n(\sigma\cap\beta)$. Out of $(\varphi(x), \varphi(z)) \in \sigma\cap\beta$ we have $(x, z) \in \varphi^{-1}(\sigma) \cap \alpha$, because φ is a reverse isotone mapping. If $(\varphi(x), \varphi(z)) \in {}^2(\sigma \cap \beta)$, i.e. if $(\varphi(x), t) \in \sigma \cap \beta$ holds or $(t, \varphi(z)) \in \sigma \cap \beta$ holds for any t of T, then, since φ is surjective holds $(\varphi(x), \varphi(s)) \in \sigma \cap \beta$ or $(\varphi(s), \varphi(z)) \in \sigma \cap \beta$ for any s of S. Thus, $(x, s) \in \varphi^{-1}(\sigma) \cap \alpha$ or $(s, z) \in \varphi^{-1}(\sigma) \cap \alpha$ and, by definition of filed product, $(x, z) \in {}^2(\varphi^{-1}(\sigma) \cap \alpha)$. Suppose that the following implication is valid:

$$(\varphi(x), \varphi(z)) \in c(\sigma \cap \beta) \Longrightarrow (x, z) \in {}^n(\varphi^{-1}(\sigma) \cap \alpha)$$

for any x, z of S. Now, as in the case of $(\varphi(x), \varphi(z)) \in {}^2(\sigma \cap \beta)$ out of $(\varphi(x), \varphi(z)) \in c(\sigma \cap \beta)$, we prove that $(\varphi(x), \varphi(z)) \in {}^{n+1}(\varphi^{-1}(\sigma) \cap \alpha)$. So, by induction, we have that the implication

$$(\varphi(x), \varphi(z)) \in c(\sigma \cap \beta) \Longrightarrow (x, z) \in c(\varphi^{-1}(\sigma) \cap \alpha)$$

is valid. The implication

$$(\varphi(x), \varphi(z)) \in c(\sigma^{-1} \cap \beta^{-1}) \Longrightarrow (x, z) \in c((\varphi^{-1}(\sigma))^{-1} \cap \alpha^{-1})$$

has analogous proof. Therefore, we have $\vartheta(\varphi^{-1}(\sigma)) = \varphi^{-1}(\vartheta(\sigma))$.

5.2. The equality $\mu(\varphi^{-1}(\sigma)) = \varphi^{-1}(\mu(\sigma))$ we proved similarly.

6. For the mapping $\Theta_T : \Im(T) \longrightarrow \mathbf{Q}(T)\times\mathbf{Q}(T)$, defined by $\Theta(\sigma) = (\vartheta(\sigma), \mu(\sigma))$ for any $\sigma \in \Im(T)$, and the mapping $\Psi : \mathbf{Q}(T) \times \mathbf{Q}(T) \longrightarrow \Im(T)$, defined by $\Psi(q_1, q_2) = c(q_1 \cap \alpha) \cup c(q_2 \cap \alpha^{-1})$ for any pair of coequalities $q_1, q_2 \in \mathbf{Q}(T)$, has the following properties: If relation $\sigma \in \Im(T)$ is compatible, i.e. if $\sigma \cap \sigma^{-1} = \emptyset$, we have:

$(\Psi \circ \Theta)(\sigma) = \Psi(\Theta(\sigma)) = \Psi((\vartheta(\sigma), \mu(\sigma)))$
$= c(\vartheta(\sigma) \cap \beta) \cup c(\mu(\sigma) \cap \beta^{-1})$
$= c((c(\sigma \cap \beta) \cup c(\sigma^{-1} \cap \beta^{-1})) \cap \beta) \cup c((c(\sigma \cap \beta^{-1}) \cup c(\sigma^{-1} \cap \beta)) \cap \beta^{-1})$
$= c(c(\sigma \cap \beta)) \cup c(c(\sigma \cap \beta^{-1}))$
$= c(\sigma \cap \beta) \cup c(\sigma \cap \beta^{-1})$
$\subseteq (\sigma \cap \beta) \cup (\sigma \cap \beta^{-1})$
$= \sigma \cap (\beta \cup \beta^{-1})$
$= \sigma\cap \neq = \sigma.$

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