Bull. Soc. Math. Banja Luka
ISSN 0354-5792
Vol. 15 (2008), 5-7

A PROOF OF GENERALIZED LAPLACE'S EXPANSION THEOREM

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Abstract

Expansion theorems for determinants are usually proved by using other fundamental properties of determinants. When determinants are defined by expanding across the first row, then the standard Laplace's rule may be obtained directly from the definition. In this note we prove the generalized Laplace's expansion theorem from the standard Laplace's rule, and no other properties of determinants are used.

1 Introduction

The following notation will be used. If D is a determinant of the order n and $1 \leq r \leq n$, by $D\begin{bmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{bmatrix}$ will be denoted the minor of the order r lying in the intersection of i_1-th, i_2-th,...,i_r-th rows and j_1-th, j_2-th,...,j_r-th columns of D and by $\overline{D}\begin{bmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{bmatrix}$ will be denoted its complement minor, that is, the minor of the order $n-r$ lying in the intersection of remaining $n-r$ rows and columns of D. If $n = r$ we then put $\overline{D} = 1$. For the set $\{i_1, \ldots, i_s\}$ of natural numbers we use abbreviation $I_s = i_1 + \cdots + i_s$. For two natural numbers j and k we define

(1) $$j(k) = \begin{cases} j & j \leq k, \\ j-1 & j > k \end{cases}.$$

The Laplace's rule says that D may be expanded along arbitrary row (or column), that is, the following formula holds

(2) $$D = \sum_{1 \leq k \leq n} (-1)^{i+k} a_{ik} \overline{D}\begin{bmatrix} i \\ k \end{bmatrix}, \ (i = 1, 2, \ldots, n).$$

A simple proof is given in [1].

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2 Main Result

In this note we shall prove so called generalized Laplace's expansion theorem, that is, that the determinant D of the order n may be expanded along i_1-th, ..., i_r-th rows, where $1 \le i_1 < \cdots < i_r \le n$ are arbitrary.

The proof will run as follows. The determinant D of the order n will be expanded along the i_r-th row. Thus obtained determinants of the order $n-1$ will be expanded along i_1-th, ..., i_{r-1}-th rows, by the induction hypothesis. In the last step of the proof the terms in thus obtained sum will be rearranged to obtain the desired result.

Theorem 2.1 *Let D be a determinant of the order n, then for every $1 \le i_1 < i_2 < \cdots < i_r \le n$ holds*

$$D = \sum_{1 \le j_1 < \cdots < j_r \le n} (-1)^{I_r + J_r} D \begin{bmatrix} i_1 & i_2 & \dots & i_r \\ j_1 & j_2 & \dots & j_r \end{bmatrix} \cdot \overline{D} \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_r \end{bmatrix},$$

where $j_1, \dots, j_r$ run trough all subsets of $\{1, 2, \dots, n\}$ with exactly r elements.

Proof. We use induction on the order of determinant. For determinants of the order 1 and 2 there is nothing to prove. Suppose that our claim holds for determinants of the order $n-1$ and let D be a determinant of the order n. Let $1 \le i_1 < i_2 < \cdots < i_r \le n$ be arbitrary. Using (2) we may expand D along i_r-th row to obtain

$$D = \sum_{1 \le k \le n} (-1)^{i_r + k} a_k^{i_r} \cdot \overline{D} \begin{bmatrix} i_r \\ k \end{bmatrix}.$$

By (1) the number $j(k)$ is the index of the column of $\overline{D} \begin{bmatrix} i_r \\ k \end{bmatrix}$ in which lie elements of j-th column of D. By the induction hypothesis we may expand each determinant $\overline{D} \begin{bmatrix} i_r \\ k \end{bmatrix}$ along i_1-th, ..., i_{r-1}-th rows to obtain

$$D = \sum_{1 \le k \le n} \sum_{\substack{1 \le l_1 < \cdots < l_{r-1} \le n \\ l_1, \dots, l_{r-1} \ne k}} (-1)^{I_r + k + l_1(k) + \cdots + l_{r-1}(k)} \cdot a_k^{i_r} \cdot$$

$$\cdot D \begin{bmatrix} i_1 & i_2 & \dots & i_{r-1} \\ l_1 & l_2 & \dots & l_{r-1} \end{bmatrix} \cdot \overline{D} \begin{bmatrix} i_1 & i_2 & \dots & i_r \\ k & l_1 & \dots & l_{r-1} \end{bmatrix}.$$

We will rearrange terms on the right side of the preceding equation in the following way. Let $1 \le j_1 < \cdots < j_r \le n$ be arbitrary. Collect summands which contain the factor $\overline{D} \begin{bmatrix} i_1 & i_2 & \dots & i_r \\ j_1 & j_2 & \dots & j_r \end{bmatrix}$. This factor appears in exactly r summands, that are obtained when k takes one of the values $j_1, \dots, j_r$ and $l_1, \dots, l_{r-1}$ take the rest of these values. We conclude that

$$D = \sum_{1 \le j_1 < \cdots < j_r \le n} \left\{ \sum_{1 \le t \le r} (-1)^{j_1(j_t) + \cdots + j_r(j_t)} a_{j_t}^{i_r} D \begin{bmatrix} i_1 & \dots & \underline{i_t} & \dots & i_{r-1} \\ j_1 & \dots & \overline{j_t} & \dots & j_r \end{bmatrix} \right\}.$$

$$\cdot(-1)^{I_r}\cdot\overline{D}\left[\begin{array}{cccc} i_1 & i_2 & \dots & i_r \\ j_1 & j_2 & \dots & j_r \end{array}\right]$$

holds. Here $\overline{j_t}$ means that columns of $D\left[\begin{array}{ccccc} i_1 & \cdots & i_t & \dots & i_{r-1} \\ j_1 & \dots & \overline{j_t} & \dots & j_r \end{array}\right]$ belong to the columns of D which indices are in $\{j_1, ..., j_r\} \setminus \{j_t\}$ $(t = 1, \dots, r)$.
Since

$$j_1(j_t) + \cdots + j_r(j_t) = j_1 + \cdots + j_t + (j_{t+1} - 1) + \cdots + (j_r - 1) =$$

$$= j_1 + \cdots + j_r + r - t = J_r + r - t,$$

and

$$(-1)^{r-t} = (-1)^{r+t},$$

we have

$$D = \sum_{1\le j_1<\cdots<j_r\le n} (-1)^{I_r+J_r} \left\{ \sum_{1\le t\le r} (-1)^{r+t} a^{i_r}_{j_t} D \left[\begin{array}{ccccc} i_1 & \cdots & i_t & \dots & i_{r-1} \\ j_1 & \dots & \overline{j_t} & \dots & j_r \end{array}\right] \right\}\cdot$$

$$\cdot\overline{D}\left[\begin{array}{cccc} i_1 & i_2 & \dots & i_r \\ j_1 & j_2 & \dots & j_r \end{array}\right].$$

The sum $\sum_{1\le t\le r}(-1)^{r+t} a^{i_r}_{j_t} D\left[\begin{array}{ccccc} i_1 & \dots & i_t & \dots & i_{r-1} \\ j_1 & \dots & \overline{j_t} & \dots & j_{r-1} \end{array}\right]$ is in fact the determinant $D\left[\begin{array}{ccccc} i_1 & \dots & i_t & \dots & i_r \\ j_1 & \dots & j_t & \dots & j_r \end{array}\right]$ expanded along the r-th row and the proof is complete.

References

[1] Janjić: *A note on Laplace's expansion theorem*, International Journal of Mathematical Education in Science and Technology, 36 (2005), 696-697.

Received by the editors July 31, 2008