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Some graph parameters and extended Fibonacci cubes

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Abstract

The fundamental component of a distributed system is the interconnection network.

The network topology is significant since the communication between processors is derived via message exchange in distributed systems. Graph can be used for modeling the interconnection network. In case, by using combinatorics and graph theory, the properties of a network can be recognized. There have been studies on various interconnection network topologies. The Extended Fibonacci Cubes are network topologies yielding useful properties for an interconnection network. In this paper, some of the vertex vulnerability parameters of the Extended Fibonacci Cubes are determined.

Keywords: Network topology, network design and communication, connectivity, vertex vulnerability parameters, Extended Fibonacci cubes.
AMS subject classification (2010): 05C40, 05C90, 68R10, 68M10, 94C15.

1. Introduction

Various interconnection topologies [6, 7, 10] have been proposed in the literature. The hypercube, or the n-cube H_n is the interconnection topology which is studied in general. Fibonacci cube deals with the

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Fibonacci numbers. Fibonacci cube has been proposed and the properties are studied by Hsu [6] as a new network topology. The Fibonacci numbers are defined as $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, for $n > 1$. The Fibonacci cube of order n has f_n nodes, $n > 1$, where f_n is the $n'th$ Fibonacci number and the nodes can be labelled with binary strings of length $n-2$ with no consecutive 1's. Two nodes are connected if their labels differ in exactly one position. Then, it can be said that the resulting graph, a Fibonacci cube is obtained from a hypercube after disruption of some nodes. The Fibonacci cube topology is generalized by Wu [10] and the series of Extended Fibonacci Cubes, (EFC_k) $k > 0$ are defined. Also by using the same recursive relation of the Fibonacci numbers, but changing the initial conditions, The Extended Fibonacci Cubes are defined. This definition increases the number of choices for the number of nodes for an interconnection network. An interconnection network consists of a set of processors. Each of the processors has a local memory and a set of bidirectional (or unidirectional) links. These links function for the exchange of data between processors. Interconnection networks are represented by undirected (or directed if the links are unidirectional) graphs $G = (V, E)$ where V is the set of nodes and E is a set of edges. Each node and each edge corresponds to a processor and a link, respectively. Various proposals have been made for measuring the reliability of a communication network. Such parameters that come to mind firstly in general for measuring the vulnerability of a graph or a network are connectivity and edge-connectivity. Connectivity gives the minimum cost to disrupt the network, but it does not take into account what remains after disruption.A number of graph parameters which attempt to cope with this weakness have been introduced [6, 10] including (edge-) toughness, (edge-) integrity, scattering number, (edge-) tenacity, rupture degree, and several variations of (edge-) connectivity [1-4, 8, 9, 11]. Computing the corresponding problems for most of these parameters are NP-hard. In this paper, some of the vertex vulnerability parameters of Extended Fibonacci Cubes such as toughness, tenacity, scattering number and rupture degree are determined.

We first recall the definitions of some graph parameters. In the following definitions we assume that $G = (V(G), E(G))$ is a noncomplete connected graph. A set $S \subseteq V$ is a cut set of G if either $G-S$ is disconnected or has only one vertex. For $S \subseteq V$, let $\omega(G-S)$ and $m(G-S)$ be denote the number of components and the order of a largest component of $G - S$, respectively.

(i) Vertex connectivity $\kappa(G)$
$\kappa(G) = \min\{\ |S| : S \subseteq V$ is a cut set of $G\ \}$;

(ii) Vertex toughness t(G) (Chvtal, 1973, [3])

t(G)= min $\{\frac{|S|}{\omega(G-S)}$:$S \subseteq V$ is a cut set of $G\ \}$;

(iii) Scattering number s(G) (Jung, 1978, [8, 11])

s(G)= max$\{\ \omega(G-S) - |S|$:$S \subseteq V$ is a cut set of $G\ \}$;

(iv) Vertex Tenacity T(G) (Cozzens, et al., 1995, [4])

T(G) = min $\{\frac{|S|+m(G-S)}{\omega(G-S)}$:$S \subseteq V$ is a cut set of $G\ \}$;

(v) Rupture degree r(G) (Li et al., [9])

r(G)= max$\{\ \omega(G-S) - |S| - m(G-S)$:$S \subseteq V, \omega(G-S) > 1\ \}$;

(vi) Vertex Neighbour-Tenacity NT(G) (Dundar et al., 1999, [5])

NT(G)= min $\{\frac{|S|+m(G-S)}{\omega(G-S)}$: $S \subseteq V$ is any vertex subversion strategy of $G\}$.

2. Extended Fibonacci Cubes

Extended Fibonacci Cubes are defined based on the same Fibonacci sequence defined as $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$ and $F_i = F_{i-1} + F_{i-2}$, for $i \geq 2$. However, their initial conditions are different. The following defines the first Extended Fibonacci Cube in the series. The symbol $||$ denotes a concatenation operation; for example, $01||\{0,1\} = \{010, 011\}$ and $01||\{\} = 01$.

Definition 2.1. ([10]) Assume $EFC_1(n) = (V_1(n), E_1(n))$, $EFC_1(n-1) = (V_1(n-1), E_1(n-1))$, and $EFC_1(n-2) = (V_1(n-2), E_1(n-2))$. Then $V_1(n) = \{0||V_1(n-1) \bigcup 10||V_1(n-2)\}$. Two nodes in $EFC_1(n)$ are connected by an edge in $E_1(n)$ if and only if their labels differ in exactly one bit position. As initial conditions for recursion, $V_1(3) = \{0, 1\}$ and $V_1(4) = \{00, 10, 11, 01\}$.

Fig. 1 shows examples of $EFC_1(n)$ of size $n, where\ n = 3, 4, 5, 6$ respectively. An $EFC_1(n)$ of size n consists of one $EFC_1(n-1)$ and one $EFC_1(n-2)$.

By changing the initial conditions for $EFC_1(n)$, another Extended Fibonacci Cube can be extracted, denoted as $EFC_2(n)$ with initial conditions $V_2(4) = \{00, 01, 10, 11\}$ and $V_2(5) = \{000, 001, 010, 011, 100, 101, 110, 111\}$. $EFC_1(n)$ is a proper subgraph of $EFC_2(n)$. Fig. 2 shows examples of $EFC_2(n)$ of size n, where $n = 4, 5, 6$ respectively.

In this paper, the number of vertices of $EFC_1(n)$ or $EFC_2(n)$ is denoted by m. For $EFC_1(n)$, $m = F_{n-1} + F_{n-2} + F_{n-3}$, where F_n denotes the $n-th$ Fibonacci number. For $EFC_2(n)$, $m = m(EFC_2(n-1)) + m(EFC_2(n-2))$, where $n \geq 6$ and $m(EFC_2(4)) = 4$, $m(EFC_2(5)) = 8$.

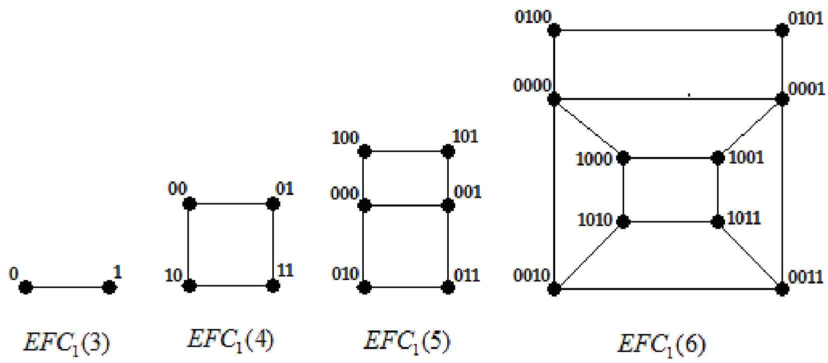

Fig. 1 $EFC_1(n)$, $n = 3,4,5,6$.

3. Vertex vulnerability parameters of Extended Fibonacci Cubes

In this section, we will first review some of the known results about vertex toughness t(G) and scattering number s(G). Then, we will give general results some vertex vulnerability parameters, namely toughness, tenacity, scattering number, rupture degree and neighbour tenacity of Extended Fibonacci Cubes. For a graph G, let $\alpha(G)$ and $\beta(G)$ denote the independence number and covering number of G respectively. The following proposition is obvious.

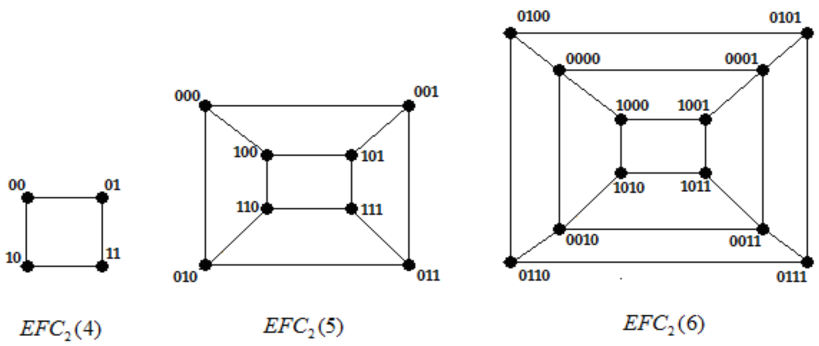

Fig. 2 $EFC_2(n)$, $n = 4,5,6$.

Proposition 3.1. *Let $EFC_1(n)$ be Extended Fibonacci Cube of order m. Then $\alpha(EFC_1(n)) = \frac{m}{2}$ and $\beta(EFC_1(n)) = \frac{m}{2}$.*

Theorem 3.1. ([3]) *Let G be a graph of order n. Then $t(G) \leq \frac{n-\alpha(G)}{\alpha(G)}$*

Theorem 3.2. ([11]) *Let G be a graph of order n. Then*

$$2\alpha(G) - n \leq s(G) \leq \alpha(G) - \kappa(G)$$

Theorem 3.3. *The scattering number of $EFC_1(n)$ is $s(EFC_1(n)) = 0$.*

P r o o f. By Proposition 3.1 and the left inequality of the Theorem 3.2 , we have

$$2.\frac{m}{2} - m \leq s(EFC_1(n))$$
$$0 \leq s(EFC_1(n)).....(3.1)$$

On the other hand, let S be a vertex cut of $EFC_1(n)$ and $|S| = r$. If we remove r vertices from $EFC_1(n)$, then $\omega(EFC_1(n) - S) \leq |S|$. Since $\omega(EFC_1(n) - S) - |S| \leq |S| - |S|$, we have

$$s(EFC_1(n)) \leq 0.....(3.2)$$

By (3.1) and (3.2), we have $s(EFC_1(n)) = 0$.
The proof is completed. ■

Theorem 3.4. *The toughness of $EFC_1(n)$ is $t(EFC_1(n)) = 1$.*

P r o o f. By Proposition 3.1 and Theorem 3.1 , we have

$$t(EFC_1(n)) \leq \frac{m - \frac{m}{2}}{\frac{m}{2}}$$

$$t(EFC_1(n)) \leq 1.....(3.3)$$

On the other hand, let S be a vertex cut of $EFC_1(n)$ and $|S| = r$. If we remove r vertices from $EFC_1(n)$, we have at most r components. From the definition of toughness

$$t(G) = min\{\frac{|S|}{\omega(G-S)} \geq min\{\frac{r}{r}\}$$

$$t(EFC_1(n)) \geq 1.....(3.4)$$

By (3.3) and (3.4), we have $t(EFC_1(n)) = 0$.
The proof is completed. ∎

Theorem 3.5. *The tenacity of $EFC_1(n)$ is $T(EFC_1(n)) \geq 1 - \frac{1}{4m}$.*

P r o o f. Let S be a vertex cut of $EFC_1(n)$ and $|S| = r$.If we remove r vertices from $EFC_1(n)$, we have at most r components. On the other hand, one of the remaining connected components has at least $\frac{m-r}{r}$. Then we have, function $f(r) = \frac{r^2-r+m}{r^2}$. It takes its maximum value at $r = 2m$. Thus, the minimal value of $f(r) = 1 - \frac{1}{4m}$.
Then, we have $T(EFC_1(n)) \geq 1 - \frac{1}{4m}$.
The proof is completed. ∎

Theorem 3.6. *The rupture degree of $EFC_1(n)$ is $r(EFC_1(n)) = -1$.*

P r o o f. Let S be a vertex cut of $EFC_1(n)$ and $|S| = r$.If we remove r vertices from $EFC_1(n)$, we have at most r components. If $r \leq \frac{m}{2}$ (also called independent number), then $\omega(EFC_1(n) - S) \leq r$. Furthermore we have, $m(EFC_1(n) - S) \geq \lceil \frac{m-r}{r} \rceil$ From the definition of rupture degree we have,
r(G)= max$\{$ $\omega(G-S) - |S| - m(G-S)$:$S \subseteq V, \omega(G-S) > 1$ $\}$

$$\leq r - r - \lceil \frac{m-r}{r} \rceil$$

$\leq -\lceil \frac{m-r}{r} \rceil$

$\leq -1.....(3.5)$

If $r \geq \frac{m}{2}$, then $\omega(EFC_1(n) - S) \leq m - r$ and $m(EFC_1(n) - S) = 1$. From the definition of rupture degree we have,
r(G)= max{ $\omega(G - S) - |S| - m(G - S)$:$S \subseteq V, \omega(G - S) > 1$ }

$\leq m - r - r - 1$

$\leq m - 2r - 1$

$\leq -1.....(3.6)$.

By (3.5) and (3.6), we obtain

$$r(EFC_1(n)) \leq -1.....(3.7)$$

It's clearly that, there is a vertex cut S (also called independent set) of $EFC_1(n)$ such that $|S| = \frac{m}{2}$, $\omega(EFC_1(n) - S) = \frac{m}{2}$ and $m(EFC_1(n) - S) = 1$.From the definition of rupture degree we have,
r(G)= max{ $\omega(G - S) - |S| - m(G - S)$:$S \subseteq V, \omega(G - S) > 1$ }

$= \frac{m}{2} - \frac{m}{2} - 1$

$= -1...(3.8)$
By (3.7) and (3.8), we have $r(EFC_1(n)) = -1$.
The proof is completed. ∎

Theorem 3.7. *The neigbour-tenacity of* $EFC_1(n)$ *is* $NT(EFC_1(n)) \geq \frac{1}{2} - \frac{\Delta^2}{4m}$.
Proof. Let S be a vertex cut of $EFC_1(n)$ and $|S| = r$.If we remove r vertices from $EFC_1(n)$, we have at most $2r$ components. On the other hand, one of the remaining connected components has at least $\frac{m-\Delta r}{r}$, where Δ denotes the maximal number of its adjacent vertices for any vertex v of $r(EFC_1(n))$.The function $f(r) = \frac{2r^2+m-\Delta r}{4r^2}$. It takes its maximum value at $r = r_1 = \frac{2m}{\Delta}$. Thus, the minimal value of $f(r) = \frac{1}{2} - \frac{\Delta^2}{4m}$. Then, we have $NT(EFC_1(n)) \geq \frac{1}{2} - \frac{\Delta^2}{4m}$.
The proof is completed. ∎

Proposition 3.2. *The results of the vertex vulnerability parameters for the $EFC_2(n)$ are the same with the results of the same vertex vulnerability parameters for the Extended Fibonacci Cubes $EFC_1(n)$ given above.*

4. Conclusion

The capacity of an interconnection network to pretend a basis topology as hamiltonian path or cycle and 2D mesh, hypercubes and trees, is crucial when the nodes or links of the network are faulty. Then the network can keep functioning with a relatively small number of nodes or links. The Extended Fibonacci Cubes can actively pretend as these basic topologies. The Extended Fibonacci Cubes are network topologies yielding efficient properties for an interconnection network relative to diameter, node degree, recursive decomposition, embeddability and communication algorithms.All of these properties motivated us to investigate some of the vertex vulnerability parameters of the Extended Fibonacci Cubes.

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