Bull. Soc. Math. Banja Luka
ISSN 0354-5792 (p), ISSN 1986-521X (o)
Vol. 17 (2010), 43-53

Two-dimensional Peristaltic Motion of Blood trough a Circular Tube

Dulal Chandra Sanyal[1] and Ananda Biswas[2]

Abstract

Assuming blood to be incompressible viscous Newtonian fluid, its two-dimensional peristaltic motion through a circular tube is discussed. The problem has been solved by the separation of variable method consisting of Bessel type radial function and an unknown type axial function. The comparison of the solutions for various parameters are shown graphically. The problem and its solutions are supposed to be of practical interest in medical science.

AMS Mathematics Subject Classification (2010): 76Z05

Key words and phrases: Two-dimensional peristaltic motion, Navier-Stoke equations, Bassel function.

1 Introduction

Peristaltic motion of blood (or other fluid) in animal or human bodies has been considered by many authors. It is an important mechanism for transporting blood, where the cross-section of the artery is contracted or expanded periodically by the propagation of progressive wave. Peristaltic motion occurs widely when stenosis is formed in the functioning of ureter, chyme movement in the intestine, movement of egg in the fallopian tube, the transport of spermatozoa in the cervical canal, transport of bile in bile duct, transport of cilia etc. [1]. However, such motion in the case of blood is somewhat different from above. For blood, the motion is highly speedy, Reynolds number is large and the peristalsity is maintained automatically.

Shapiro et al. [2] introduced an idea for peristaltic pumping with long wavelength and low Reynolds number. Jaffrin and Shapiro [3] showed a comprehensive developing of a mathematical model of peristaltic flow. Takabatake et al. [4,5] applied numerical method, with upwined finite-difference technique to solve the problem of peristaltic flow in circular cylindrical tube. Burns and Parkes [6], Fung and Yih [7], Chow [8] and Misra and Pande [9] followed the perturbation technique to study the peristaltic flow in circular cylindrical tube.

[1]Department of Mathematics,University of Kalyani, Kalyani-741235, Nadia, West Bengal,India, e-mail: klyuniv@yahoo.com

[2]Department of Mathematics, University of Kalyani, Kalyani-741235, Nadia, West Bengal, India, e-mail: anand.was@gmail.com

Applying numerical technique, Brown and Hung [10] considered the curvature effect of peristaltic flow. Rao and Usha [11] studied the motion of two-layered Newtonian fluid through a circular cylindrical tube, while Utkin et al. [12] solved numerically the system of equations of motion and the wall.
Blood is a non-Newtonian fluid and is a suspension of cells in plasma. Not only blood, but many other fluids of animal or human are, in fact, non-Newtonian. However, blood can be treated as a Newtonian fluid if the radius of the artery is greater than 0.25 mm. [15]. Here we consider the blood to be an incompressible Newtonian fluid .
In this paper we study a model of peristalsis. Actually the model of peristaltic pumping of blood depends not only on sine or cosine functions or their linear combination, but may be a function like Fourier series.We consider here a very closer model of experimental results [16-17]. Noting the model of Mokhtar et al. [18], we consider here the velocity components to consist of Bessels function for radial variable and an unknown function for the axial variable. Finally, solving the unknown function we attain the results which are in close agreement with the experimental results. The results are also compared for various involved parameters.

2 Basic equations

The basic equations of motion for an incompressible viscous Newtonian fluid in cylindrical polar coordinate (r, θ, z) with axial symmetry are given by

$$\frac{\partial u}{\partial t} + u.\frac{\partial u}{\partial r} + w.\frac{\partial u}{\partial z} = -\frac{1}{\rho}.\frac{\partial p}{\partial r} + \nu[\frac{1}{r}.\frac{\partial}{\partial r}(r.\frac{\partial u}{\partial r}) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}], \tag{1}$$

$$\frac{\partial w}{\partial t} + u.\frac{\partial w}{\partial r} + w.\frac{\partial w}{\partial z} = -\frac{1}{\rho}.\frac{\partial p}{\partial z} + \nu[\frac{1}{r}.\frac{\partial}{\partial r}(r.\frac{\partial w}{\partial r}) + \frac{\partial^2 w}{\partial z^2}] \tag{2}$$

and the equation of continuity is

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{3}$$

where $(u, 0, w)$ are the velocity components along (r, θ, z) directions; ν is the kinematic coefficient of viscosity; ρ is the density of the fluid and p is the pressure.
For peristaltic motion, the boundary of the tube is taken to be

$$r = a(1 + \frac{\epsilon}{k}A(\frac{Ut}{a} + \frac{\lambda z}{a})) = B(\frac{Ut}{a} + \frac{\lambda z}{a}), \texttt{(say)}, \tag{4}$$

where a is the undisturbed radius of the tube, $\epsilon (0 < \epsilon < 1)$ is a non-dimensional parameter such that $a\epsilon$ represents the amplitude of oscillation, U is the axial velocity at z=0 and t=0 and

$$A(x) = \tfrac{3}{4} sin\tfrac{\pi}{12} + sinx + \tfrac{1}{2} sin(2x + \tfrac{\pi}{4}), k = \tfrac{3}{4} sin\tfrac{\pi}{12} + \tfrac{3}{2} sin\tfrac{\pi}{4}.$$

The boundary conditions are assumed as

$$u = \frac{U}{a}.B(\frac{Ut}{a}).J_1(\frac{\alpha}{a}.B(\frac{Ut}{a}+\frac{\lambda z}{a})).e^{-\alpha\frac{z}{a}}, \tag{5}$$

$$w = \frac{U}{a}.B(\frac{Ut}{a}).J_0(\frac{\alpha}{a}.B(\frac{Ut}{a}+\frac{\lambda z}{a})).e^{-\alpha\frac{z}{a}} \tag{6}$$

at $r = B(\frac{Ut}{a}+\frac{\lambda z}{a})$ and

$$p = \rho U^2 p_0, \tag{7}$$

when $z \to \infty$ and $r = 0$ and $\alpha(> 0)$ is a non-dimensional parameter, supposed to be very small.
Now eliminating p between (1)and(2) and satisfying (3) we obtain

$$\frac{\partial\Omega}{\partial t} + r(u.\frac{\partial}{\partial r} + w.\frac{\partial}{\partial z})(\frac{\Omega}{r}) = \nu[\frac{\partial}{\partial r}(\frac{1}{r}.\frac{\partial}{\partial r}(r\Omega)) + \frac{\partial^2\Omega}{\partial z^2}] \tag{8}$$

where $\Omega = \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}$.
For convenience, we consider the following non-dimensional quantities :

$$r_1 = \frac{r}{a}, z_1 = \frac{z}{a}, t_1 = \frac{tU}{a}, u_1 = \frac{u}{U}, w_1 = \frac{w}{U}, p_1 = \frac{p}{\rho U^2}, \Omega_1 = \frac{a\Omega}{U}, B_1 = \frac{B}{a}. \tag{9}$$

Substituting (9) in (8) and (3-7) and then ommiting the suffix 1 we obtain

$$\frac{\partial\Omega}{\partial t} + r(u.\frac{\partial}{\partial r} + w.\frac{\partial}{\partial z})(\frac{\Omega}{r}) = \frac{1}{Re}[\frac{\partial}{\partial r}(\frac{1}{r}.\frac{\partial}{\partial r}(r\Omega)) + \frac{\partial^2\Omega}{\partial z^2}] \tag{10}$$

where $\Omega = \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}$ and $Re = \frac{aU}{\nu}$ = Reynolds number.
The non-dimensional equation of continuity is

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \tag{11}$$

The boundary reads as

$$r = 1 + \frac{\epsilon}{k}A(t+\lambda z) = B(t+\lambda z) \tag{12}$$

and the boundary conditions are reduced to

$$u = B(t).J_1(\alpha.B(t+\lambda z)).e^{-\alpha z}, \tag{13}$$

$$w = B(t).J_0(\alpha.B(t+\lambda z)).e^{-\alpha z} \tag{14}$$

at $r = B(t+\lambda z)$ and

$$p = p_0, \tag{15}$$

when $z \to \infty$ and $r = 0$.

3 The solutions of the problem

Let us consider

$$u = -\sum_n C_n t^n f_n'(z).J_1(\alpha_n r) \tag{16}$$

where C_n, α_n are constants, f_n is an unknown function of z with $f_n'(z) = \frac{\partial f_n}{\partial z}$. Since w is finite when $r = 0$, so by (11) we obtain

$$w = \sum_n C_n \alpha_n t^n f_n(z).J_0(\alpha_n r). \tag{17}$$

Now by (16) and (17) we have

$$\Omega = -\sum_n C_n t^n J_1(\alpha_n r)[\alpha_n^2 f_n(z) - f_n''(z)]. \tag{18}$$

Putting (18) in (10) we get

$$\begin{aligned} &- \sum_n (n+1) C_{n+1} t^n J_1(\alpha_{n+1} r)[\alpha_{n+1}^2 f_{n+1}(z) - f_{n+1}''(z)] \\ &+ \sum_n C_n t^n f_n'(z) J_1(\alpha_n r). \sum_n C_n t^n J_2(\alpha_n r)[\alpha_n^2 f_n(z) - f_n''(z)] \\ &- \sum_n C_n \alpha_n t^n f_n(z) J_0(\alpha_n r). \sum_n C_n t^n J_1(\alpha_n r)[\alpha_n^2 f_n'(z) - f_n'''(z)] \\ &= \frac{1}{Re} \sum_n C_n t^n J_1(\alpha_n r)[\alpha_n^4 f_n(z) - 2\alpha_n^2 f_n''(z) + f_n^{1v}(z)] \end{aligned} \tag{19}$$

It is clear that separation of the functions containing the variables r and z is very much difficult. However, it is easy to see that the relation

$$\alpha_n^2 f_n(z) - f_n''(z) = 0, \forall n \tag{20}$$

is satisfied identically. So from (20), we may take the solution as

$$f_n(z) = A_1 e^{-\alpha_n z} + A_2 e^{\alpha_n z}.$$

Since $f_n(z)$ must be finite as $z \to \infty$ we get $A_2 = 0$ and so

$$f_n(z) = A_1 e^{-\alpha_n z}.$$

Assuming $\alpha_n = \alpha$, $\forall\ n$, we have

$$f_n(z) = A_1 e^{-\alpha z}, \forall n \tag{21}$$

and thus from (16) and (17) we have

$$u = \alpha J_1(\alpha r) e^{-\alpha z} \sum_n \alpha A_1 C_n t^n, \tag{22}$$

$$w = \alpha J_0(\alpha r) e^{-\alpha z} \sum_n \alpha A_1 C_n t^n. \tag{23}$$

After reducing the equations (1) and (2) in non-dimensional form and putting the values of u and w from (22,23) we obtain

$$\frac{\partial p}{\partial r} = -\phi'(t)e^{-\alpha z}J_1(\alpha r) - \phi^2(t)e^{-2\alpha z}[J_1(\alpha r)J_1'(\alpha r) - \alpha J_0(\alpha r)J_1(\alpha r)] \tag{24}$$

$$\frac{\partial p}{\partial z} = -\phi'(t)e^{-\alpha z}J_0(\alpha r) + \alpha\phi^2(t)e^{-2\alpha z}[J_1^2(\alpha r) + J_0^2(\alpha r)]. \tag{25}$$

Solving these equations by satisfying (15) we obtain

$$p = p_0 + \frac{1}{\alpha}\phi'(t)e^{-\alpha z}J_0(\alpha r) - \frac{1}{2}.\phi^2(t)e^{-2\alpha z}[J_1^2(\alpha r) + J_0^2(\alpha r)] \tag{26}$$

where $\phi(t) = \sum_n \alpha A_1 C_n t^n$.
The solutions (22,23) and (26) give an exact solutions of **Navier-Stokes** equation.
Now satisfying the boundary conditions (13,14), the solutions (22,23) and (26) are given by

$$u = B(t)J_1(\alpha r)e^{-\alpha z} \tag{27}$$

$$w = B(t)J_0(\alpha r)e^{-\alpha z} \tag{28}$$

and

$$p = p_0 + \frac{\epsilon}{\alpha k}A'(t)J_0(\alpha r)e^{-\alpha z} - \frac{1}{2}.B^2(t)[J_1^2(\alpha r) + J_0^2(\alpha r)]e^{-2\alpha z}. \tag{29}$$

4 Discussions

From our solutions we see that the results depend mainly on the parameters $\lambda, \alpha, \epsilon$. It is also clear that the Reynolds number (Re) does not effect the solutions. Due to these restrictions, the Reynolds number (Re) of blood is very high, but the blood flow is not turbulent as can be seen from experimental point of view. Again we conclude from our solutions that if α is very small, then $e^{-\alpha z} \approx 1$ and $J_0(\alpha r) \approx 1$ and so the blood velocity, blood pressure remain nearly unchanged. For this reason, the blood pressure approximately remains the same in all blood circulation.
Again from equations (22), (23) and (26) we see that the motion is satisfied for arbitrary time function. For the case of peristaltic motion the boundary must be a periodic function of time, which is considered as a very closer of natural periodicity of blood artery. When blood passes through the artery, it obviously satisfy the continuity condition with artery, so we chose the boundary condition as (5), (6). For this reason, if the peristalsis is changed, the blood flow remain continued.
We show in the fig. 1 the curve of peristalsis for various time i.e., the structure of the boundary of the artery for various time. In all other figures, the nature

of velocity and pressure are shown for various parameters. In a cross-sectional area of an artery the velocity components u and w are respectively shown w.r.t. it's radial distance(r) in fig. 2 and fig. 3 for different values of α. It is clear from these figures that if α is very small then u is negligible and w changes very slightly for varying r. From fig. 4 we can see the pulsation of w and the fig.5 shows that the amplitude of the periodicity of w and it's value decreases when z increases. It is seen from fig. 6 that if ϵ increases i.e., the amplitude of the peristalsis increases then pressure increases. Again, fig. 7 gives that the variation of pressure on the artery for peristaltic pumping, where the disturbance of the pressure decreases when z increases.

References

[1] Abd El Hakeem, Abd El Naby and I.I.E.El Shamy, *Slip effects on peristaltic transport of power-law fluid through an inclined tube*, Applied Mathematical Science, 1(2007), 2967-2980.

[2] A.H.Shapiro, M.Y.Jaffrin and S.L.Weinberg, *Peristaltic pumping with long wavelengths at low Reynolds number*, J.Fluid Mech., 37(1969), 799-825.

[3] M.Y.Jaffrin and A.H.Shapiro, *Peristaltic pumping*, Ann.Rev.Fluid Mech., 3(1971), 13-36.

[4] S.Takabatake, K.Ayukawa and A.Mori, *Peristaltic pumping in circular cylindrical tube: a numerical study of fluid transport and its efficiency*, J.Fluid Mech., 193(1988), 269-283.

[5] S.Takabatake and K.Ayukawa, *Numerical study of two-dimensional peristaltic flows*, J.Fluid Mech. 122(1982), 439-465.

[6] J.C.Burns and T.Parkes, *Peristaltic motion*, J.Fluid Mech., 29(1967), 731-743.

[7] Y.C.Fung and C.S.Yih, *Peristaltic transport*, Trans ASME J.Appl.Mech., 35(1968), 669-675.

[8] T.S.Chow, *Peristaltic transport in a circular cylindrical pipe*, Trans ASME J.Appl.Mech., 37(1970), 901-905.

[9] J.C.Misra and S.K.Pandey, *Peristaltic transport in a tapered tube*, Math.Comput. Modelling, 22(1995), 137-151.

[10] T.D.Brown and T.K.Hung, *Computational and experimental investigations of two-dimensional non-linear peristaltic flows*, J.Fluid Mech., 83(1974), 249-272.

[11] S.Usha and A.Ramchandra Rao, *Peristaltic transport of two-layered power-law fluids*, Trans ASME J.Biomech.Eng., 199(1997), 483-488.

[12] A.V.Utkin, I.M.Starobin and V.M.Zaiko, *Numerical modeling of the peristaltic motion of a fluid in a tube with a wall that derorms according to an harmonic law*, Mechanics of Composite Materials, 15(1980), 597-600.

[13] L.P.Srivastava and V.P.Srivastava, *Peristaltic transport of blood: Casson model II*, J.Biomech., 17(1984), 821-830.

[14] A.Ramchandra Rao and S.Usha, *Peristaltic transport of two immiscible viscous fluid in a circular tube*, J.Fluid Mech., 298(1995), 271-285.

[15] D.A.Mc.Donald, **Blood flow in arteries**,*Edward Arnold Ltd.*, 133.

[16] S.K.Chaudhuri, **Concise medical physiology**, *New Central Book Agency*, 221.

[17] Michael Swash, **Hutchison Clinical Methods**, *Bailliere Tindall*, 227.

[18] Mokhtar A.Abd. Elnaby, Nabil T.M.Eldabe and Mohammed Y.Abou Zeid, *Mathematical Analysis on MHD pulsatile flow of a non-Newtonian fluid through a tube with varying cross-section*, J.Inst.Math.Comp.Sci., 20(2007), 29-42.

Received by the editors on June 23, 2010.

Figure 1
The boundary of the peristaltic flow on vaious time

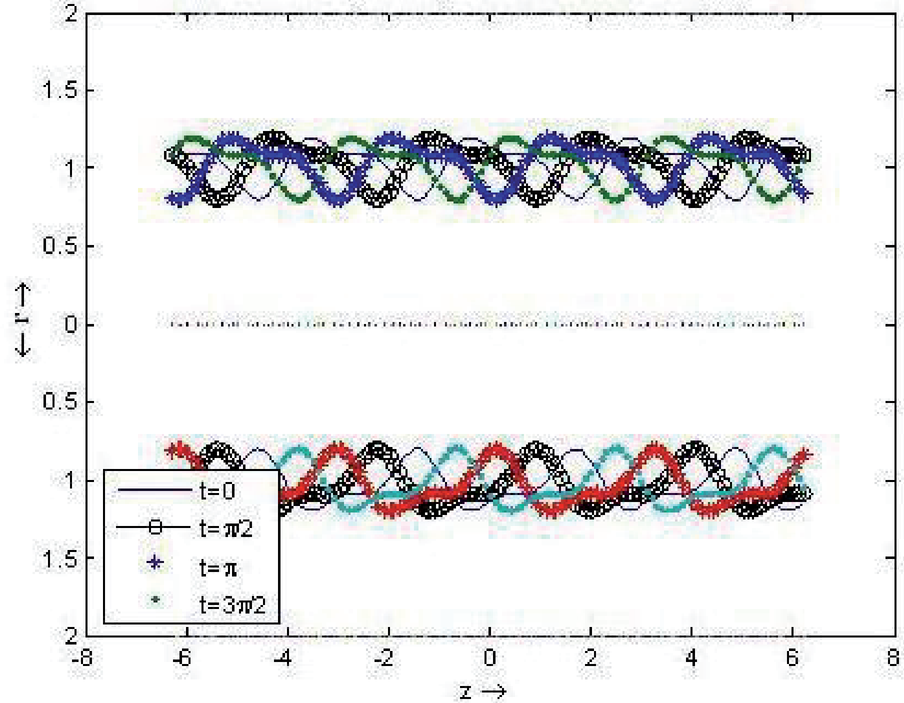

Figure 2
Assuming A(t)=0 and z=0

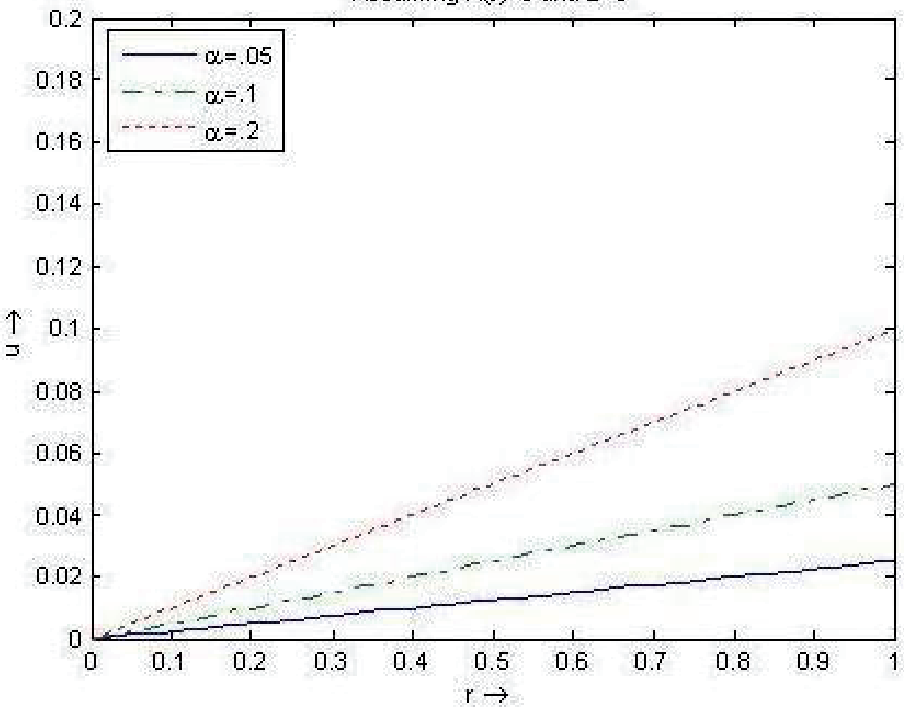

Figure 3
Assuming A(t)=0 and z=0

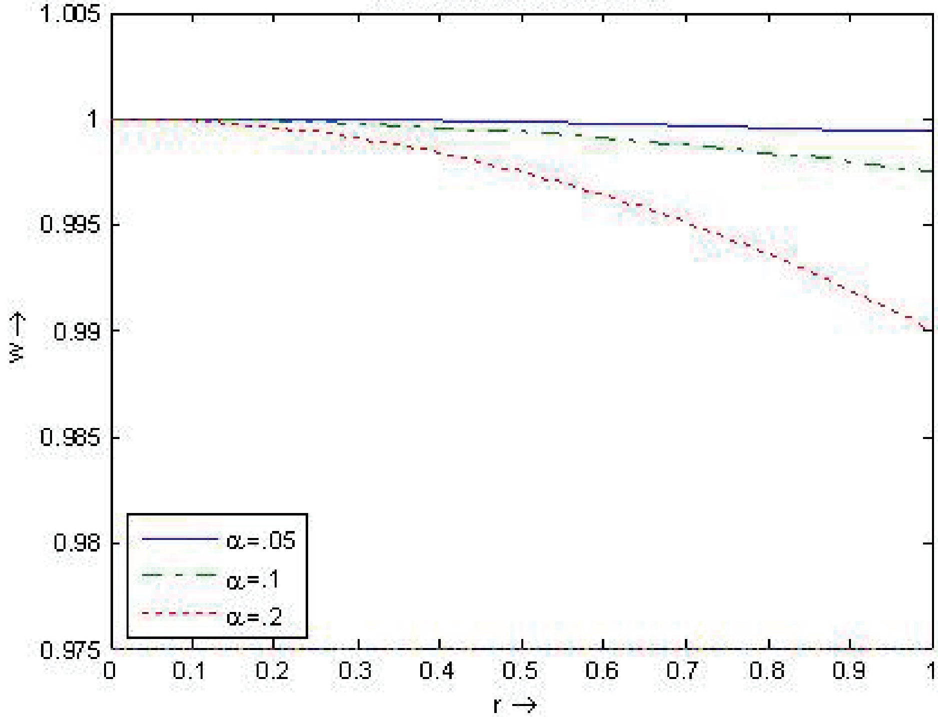

Figure 4
Assuming r=0 ,z=0 and λ =1

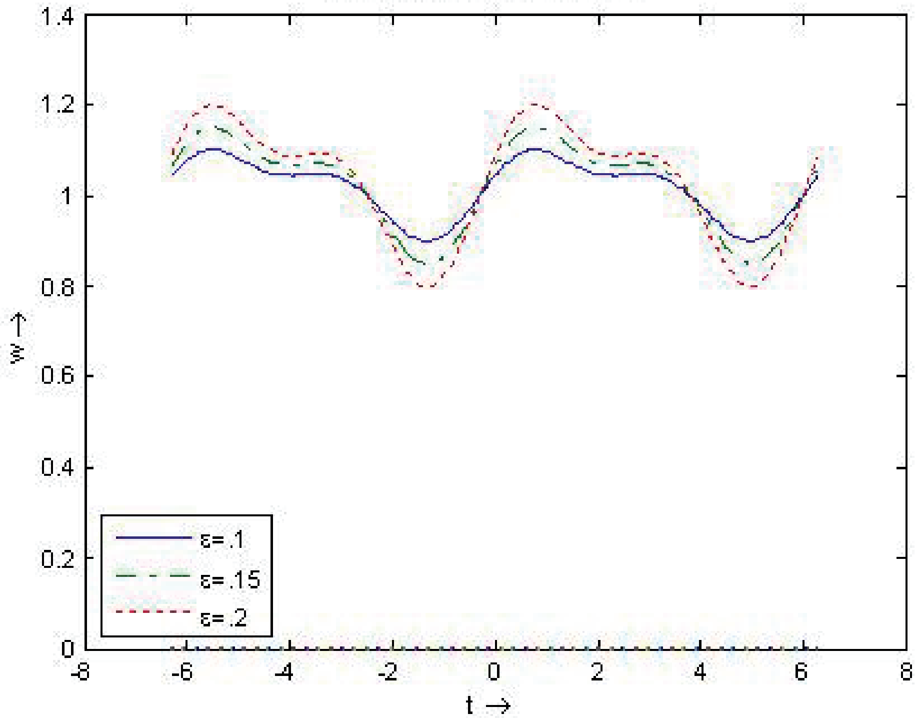

Figure 5
The axial velocity on various time
When r=0 and α =.1

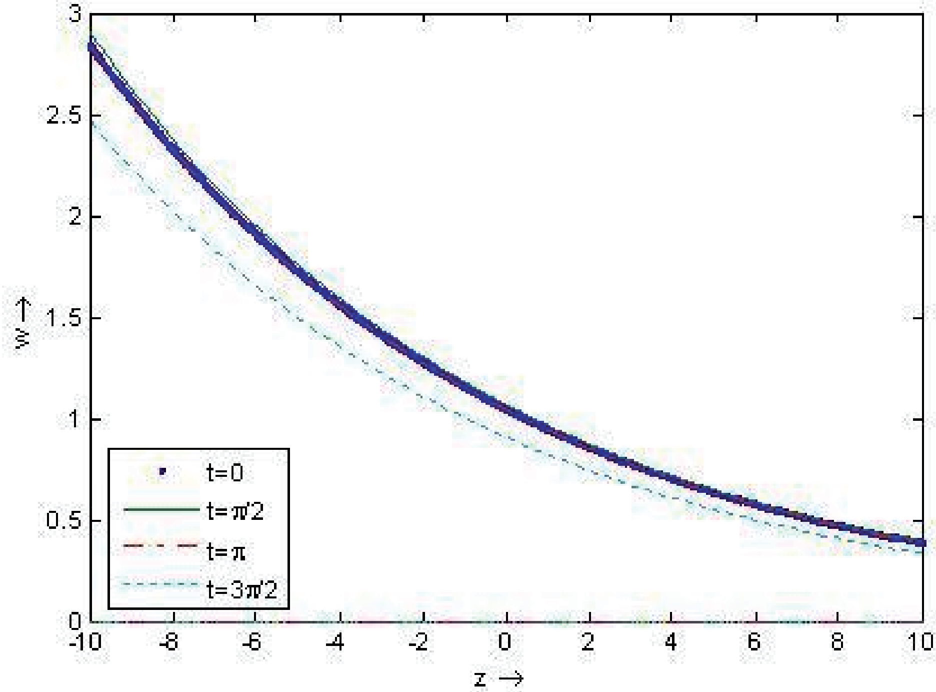

Figure 6
The pressure on boundary when t is fixed and
p_0=2,α=.2, λ =1

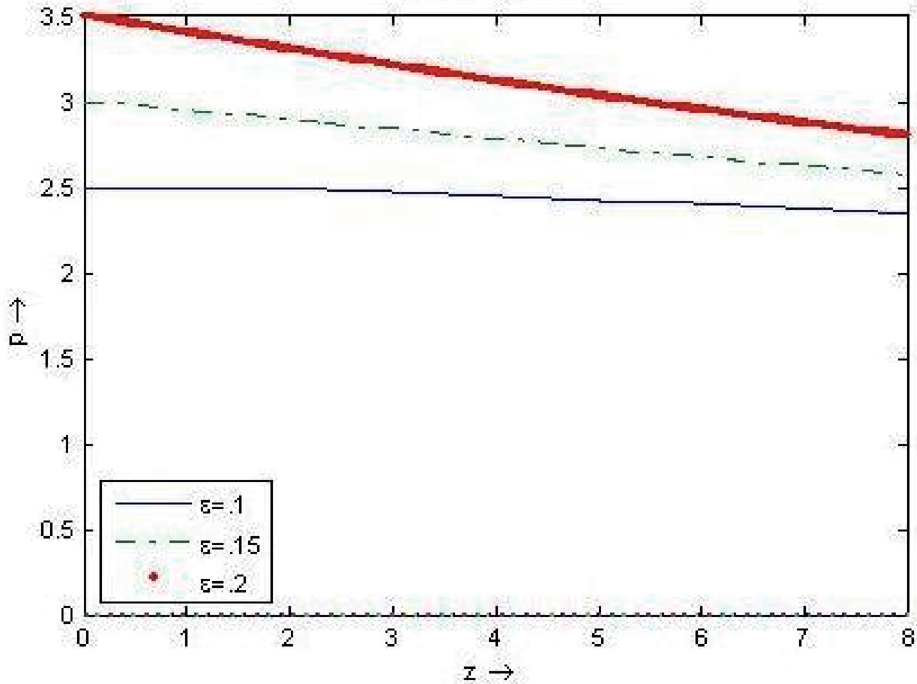

Figure 7
The pressure on axial direction and
$p_0=2, \alpha=.1, \varepsilon=.1$