Bull.Soc.Math. Banja Luka
ISSN 0354-5792 (p), ISSN 1986-521x (o)
Vol. 17 (2010), 33-42

The vulnerability of complete k-ary trees[1]

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Abstract

Computer or communication networks are so designed that they do not easily get disrupted under external attack and, moreover, these are easily reconstructible if they do get disrupted. These desirable properties of networks can be measured by various graph parameters like toughness, integrity, scattering number, tenacity and rupture degree. The complete k-ary trees are widely used in systems ranging from large supercomputers to small embedded systems-on-a-chip. In this paper, we determine these vulnerability parameters of the complete k-ary trees, thus settle a conjecture stated in [1]: *The rupture degree of the complete binary tree T with height h is $r(T) = \frac{2^{h+1}-4}{3}$ while h is odd and $\frac{2^{h+1}-2}{3}$ while h is even.* And give a counterexample for Theorem in [2]: *The minimum integrity of tree T with order $n \geq 3$ and maximum degree $\triangle \geq 2$ is $I(T) = \lfloor \frac{n-2}{\Delta-1} \rfloor + 1$ while $r(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor$ and $\lfloor \frac{n-2}{\Delta-1} \rfloor + 2$ while $r(\frac{n-2}{\Delta-1}) \geq \lfloor \frac{n-2}{\Delta-1} \rfloor$.*

AMS Mathematics Subject Classification (2010): 05C40, 68M10, 68R10

Key words and phrases: integrity, tenacity, rupture degree, complete k-ary tree

1 Introduction

In an analysis of the vulnerability of networks to disruption, three important quantities (there may be others) are (1) the number of elements that are not functioning, (2) the number of remaining connected subnetworks and (3) the size of a largest remaining group within which mutual communication can still occur. Based on these quantities, a number of graph parameters, such as connectivity, toughness, scattering number, integrity, tenacity and their edge-analogues, have been proposed for measuring the vulnerability of networks. We denote the number of components of a graph G by $\omega(G)$ and the order of the largest component of G by $m(G)$.

Connectivity is a parameter based on quantity (1). The connectivity of an incomplete graph G is defined as $\kappa(G) = min\{|X| : X \subset V(G), \omega(G-X) > 1\}$ and the connectivity of a complete graph K_n is defined as $n-1$.

[1]This paper is partially supported by NSFC (10861009) and NMFC (09QH02)

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Both the toughness and the scattering number take account of quantities (1) and (2). The toughness and scattering number of an incomplete connected graph G are defined in [3] and [5] as $\tau(G) = \min\{\frac{|X|}{\omega(G-X)} : X \subset V(G), \omega(G-X) > 1\}$ and $s(G) = \max\{\omega(G-X) - |X| : X \subset V(G), \omega(G-X) > 1\}$ respectively. The toughness and scattering number of K_n are defined as $n-1$ and $2-n$, respectively. The scattering number is called the additive dual of toughness. Although these two parameters share some similarities in their definitions, they differ in showing the vulnerability of networks.

The integrity of graphs is based on quantities (1) and (3). The integrity of a graph G is defined in [4] as $I(G) = \min\{|X| + m(G-X) : X \subset V(G)\}$.

The tenacity of graphs takes account of all three quantities. The tenacity of an incomplete connected graph G is defined in [6] as $T(G) = \min\{\frac{|X|+m(G-X)}{\omega(G-X)} : X \subset V(G), \omega(G-X) > 1\}$ and the tenacity of K_n is defined as n. Clearly, of all the above parameters, tenacity is the most appropriate for measuring the vulnerability of networks. It is natural to consider the additive dual of tenacity. We call this parameter the rupture degree of graphs. Formally, the rupture degree of an incomplete connected graph G is defined in [7] as $r(G) = \max\{\omega(G-X) - |X| - m(G-X) : X \subset V(G), \omega(G-X) > 1\}$ and the rupture degree of K_n is defined as $1-n$. Similarly to the relation between the toughness and scattering number, the rupture degree and tenacity also differ in showing the vulnerability of networks.

A complete k-ary trees are widely used in systems ranging from large supercomputers to small embedded systems-on-a-chip. Thus Measuring the vulnerability of these networks is an important and interesting problem. *A complete k-ary tree* is a k-ary tree such that each non-leaf vertex has exactly k child vertices and all leaf vertices have identical path length. It is clear that a complete k-ary tree of height h has $\frac{k^{h+1}-1}{k-1}$ vertices.

Terminology and notation not defined in this paper can be found in [8]. A set $X \subseteq V(G)$ is *a cut-set* of G, if either $G-X$ is disconnected or $G-X$ has only one vertex. And we use $G[S]$ to denote the subgraph of G induced by S.

2 Toughness and scattering number of complete k-ary trees

In this section, we shall determine toughness and scattering number of complete k-ary trees. First introduce some useful lemmas.

Lemma 2.1 *Let $X \subset V(T)$ be a cut-set of tree T and the maximum degree of T is $\triangle$. Then $|X| \geq \frac{\omega(T-X)-1}{\triangle-1}$.*

Proof: For a cut-set X of tree T, we contruct a cut-set $X^* \subset V(T)$ such that $|X^*| = |X| = x$ and X^* has more vertices of degree $\triangle$ as possible. Denote that $X^* = \{u_1, u_2, \ldots, u_x\}$. Since the difference of $\omega(T-X^*) - \omega(T-X^* + u_k)$ is at most $\triangle -1$ for every $u_k \in X^*$, then $\omega(T-X) \leq \omega(T-X^*) \leq (\triangle -1)|X^*|+1 = (\triangle -1)|X|+1$. Thus $|X| \geq \frac{\omega(T-X)-1}{\triangle -1}$.

In fact, assume that $\omega(T-X^{'}) \geq (\triangle -1)|X^{'}|+2$ for a cut-set $X^{'}$. Then we will get that $\omega(T-u_i-u_j) \geq 2\triangle$ for any two vertices $u_i, u_j \in X^{'}$. This is a contradiction to the definition of T is a tree. $\square$

Lemma 2.2 *Let $X \subset V(T)$ be cut-set of a complete k-ary tree T with height h. Then*

$$\omega(T-X) \leq \begin{cases} \frac{k}{k^2-1}(k^{h+1}-1), & \textit{if } h \textit{ is odd;} \\ \frac{1}{k^2-1}(k^{h+2}-1), & \textit{if } h \textit{ is even.} \end{cases}$$

Proof: Let the root vertex u_{01} be at level 0 and by $S_i = \{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{ik^i}\}$ denote the set of vertices of T at any one level i. For a cut-set $X \subset V(T)$, we let $X_i = X \cap S_i$ for $i = 0, 1, 2, \ldots, h$. The following we distinguish two cases to prove the lemma. h is odd. Since $T[S_0 \cup S_1] \cup \ldots \cup T[S_{h-1} \cup S_h]$ is a spanning tree of T, then $\omega(T-X) \leq \sum\limits_{i=0,2,\ldots,h-1} \omega(T[S_i \cup S_{i+1}] - X_i \cup X_{i+1})$. At the same time, since $T[S_i \cup S_{i+1}] = k^i K_{1,k}$, then $\omega(T[S_i \cup S_{i+1}] - X_i \cup X_{i+1}) \leq k^{i+1}$. Therefore

$$\omega(T-X) \leq k + k^3 + k^5 + \ldots + k^h = \frac{k}{k^2-1}(k^{h+1}-1)$$

h is even. Since $T[S_0] \cup T[S_1 \cup S_2] \cup \ldots \cup T[S_{h-1} \cup S_h]$ is a spanning tree of T, similar to Case 1, we get

$$\omega(T-X) \leq 1 + k^2 + k^4 + \ldots + k^h = \frac{1}{k^2-1}(k^{h+2}-1)$$

$\square$

Theorem 2.1 *Let T be a complete k-ary tree with height h. Then toughness $\tau(T) = \frac{1}{k+1}$.*

Proof: Let u_{01} be the root vertex and $X \subset V(T)$ be a cut-set of complete k-ary tree T. If $X = \{u_{01}\}$, then $\frac{|X|}{\omega(T-X)} = \frac{1}{k} > \frac{1}{k+1}$. Otherwise, By Lemma 2.1, since $|X| \geq \frac{\omega(T-X)-1}{\triangle -1}$ and $\omega(T-X) \geq \triangle = k+1$, then $\frac{|X|}{\omega(T-X)} \geq \frac{\frac{\omega(T-X)-1}{\triangle-1}}{\omega(T-X)} \geq \frac{\frac{\omega(T-X)}{\triangle}}{\omega(T-X)} = \frac{1}{\triangle} = \frac{1}{k+1}$. Thus we have $\tau(T) \geq \frac{1}{k+1}$. On the other hand, we choose $X_0 = \{u\}$ such that $d_T(u) = \triangle$. Then by the definition of toughness we have $\tau(T) \leq \frac{|X_0|}{\omega(T-X_0)} = \frac{1}{\triangle} = \frac{1}{k+1}$.

Therefore the toughness of T is $\tau(T) = \frac{1}{k+1}$. $\square$

Theorem 2.2 *Let T be a complete k-ary tree with height h. Then scattering number of T is*

$$s(T) = \begin{cases} \frac{1}{k+1}(k^{h+1} - 1), & \textit{if } h \textit{ is odd;} \\ \frac{1}{k+1}(k^{h+1} + 1), & \textit{if } h \textit{ is even.} \end{cases}$$

Proof: Let X be a cut-set of complete k-ary tree T. By Lemma 2.1 and the maximum degree of T is $\triangle = k+1$, we have $|X| \geq \frac{\omega(T-X)-1}{\triangle - 1} = \frac{\omega(T-X)-1}{k}$. Combine this with $\omega(T-X) - |X|$ is an integer and Lemma 2.2 we have

$$\omega(T-X) - |X| \leq \frac{(k-1)\omega(T-X)+1}{k} \leq \begin{cases} \frac{1}{k+1}(k^{h+1} - 1), & \text{if } h \text{ is odd;} \\ \frac{1}{k+1}(k^{h+1} + 1), & \text{if } h \text{ is even.} \end{cases}$$

Thus we have

$$s(T) \leq \begin{cases} \frac{1}{k+1}(k^{h+1} - 1), & \text{if } h \text{ is odd;} \\ \frac{1}{k+1}(k^{h+1} + 1), & \text{if } h \text{ is even.} \end{cases}$$

On the other hand, let S_i be the vertex set of a complete k-ary tree T at any one level i. When h is odd, we choose $X_1 = S_0 \cup S_2 \cup S_4 \ldots \cup S_{h-1}$. Then $\omega(T - X_1) - |X_1| = \frac{1}{k+1}(k^{h+1} - 1)$; When h is even, we choose $X_2 = S_1 \cup S_3 \cup S_5 \ldots \cup S_{h-1}$. And then $\omega(T - X_2) - |X_2| = \frac{1}{k+1}(k^{h+1} + 1)$. Thus by the definition of scattering number we get

$$s(T) \geq \begin{cases} \frac{1}{k+1}(k^{h+1} - 1), & \text{if } h \text{ is odd;} \\ \frac{1}{k+1}(k^{h+1} + 1), & \text{if } h \text{ is even.} \end{cases}$$

Therefore, the theorem is completed. □

In this Theorem, if $k = 2$ we get

Colorallary 2.1 *Let T be a complete binary tree with height h. Then scattering number of T is*

$$s(T) = \begin{cases} \frac{2^{h+1}-1}{3}, & \textit{if } h \textit{ is odd;} \\ \frac{2^{h+1}+1}{3}, & \textit{if } h \textit{ is even.} \end{cases}$$

3 Integrity of complete k-ary trees

In this section we determine the integrity of complete k-ary trees. Thus give a counterexample for the Theorem in [2]. First give a useful definition.

Definition 3.1 *For a subset $X \subset V(G)$, denote $Sc(X) = |X| + m(G - X)$. If $Sc(X) = I(G)$, we call X is a $I-$ set of graph G.*

Clearly, the integrity of G is $I(G) = \min\{Sc(X) | X \subset V(G)\}$.

Theorem 3.1 *Let T be a complete k-ary tree with height h. Then integrity of T is*

$$I(T) = \begin{cases} \frac{(2k-1)k^{\frac{h-1}{2}}-1}{k-1}, & \text{if } h \text{ is odd;} \\ \frac{k^{\frac{h+2}{2}}-1}{k-1}, & \text{if } h \text{ is even.} \end{cases}$$

Proof: For convenient to natation, denote $S_i = \{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{ik^i}\}$ be the set of vertices of the complete k-ary tree T at any one level i. If h is even, we choose a subset $X_e = S_{\frac{h}{2}} \subset V(G)$. Then

$Sc(X_e) = |X_e| + m(T - X_e) = k^{\frac{h}{2}} + \max\{\frac{k^{\frac{h}{2}}-1}{k-1}, \frac{k^{\frac{h}{2}}-1}{k-1}\} = \frac{k^{\frac{h+2}{2}}-1}{k-1}$.

If h is odd, we choose $X_o = S_{\frac{h-1}{2}}$. Then

$Sc(X_o) = |X_o| + m(T - X_o) = k^{\frac{h-1}{2}} + \max\{\frac{k^{\frac{h-1}{2}}-1}{k-1}, \frac{k^{\frac{h+1}{2}}-1}{k-1}\} = \frac{(2k-1)k^{\frac{h-1}{2}}-1}{k-1}$.

The following we distinguish two cases h is even and h is odd to prove X_e and X_o are $I-$ set of the complete k-ary tree T, respectively. This means for any subset $X \subset V(G)$ that $Sc(X) \geq Sc(X_e)$ while h is even and $Sc(X) \geq Sc(X_o)$ while h is odd. Clearly, if $\omega(T - X) = 1$, then $Sc(X) = \frac{k^{h+1}-1}{k-1} > \max\{Sc(X_e), Sc(X_o)\}$. Thus we assume that $\omega(T - X) \geq 2$.

Case 1. h is even.

Let $X \subset V(T)$ and suppose $X = S_{p_1} \bigcup \ldots \bigcup S_{p_n}$ for $0 \leq p_1 < \ldots < p_n < h$. Then

$Sc(X) = k^{p_1} + k^{p_2} + \ldots + k^{p_n} + \max\{\frac{k^{p_1}-1}{k-1}, \frac{k^{p_2-p_1-1}-1}{k-1}, \ldots, \frac{k^{p_n-p_{n-1}-1}-1}{k-1}, \frac{k^{h-p_n}-1}{k-1}\}$

First we show that $Sc(X) \geq Sc(X_e) = \frac{k^{\frac{h+2}{2}}-1}{k-1}$ holds for $n = 1$. Clearly, $Sc(X) = |S_{p_1}| + m(T - S_{p_1}) = k^{p_1} + \max\{\frac{k^{p_1}-1}{k-1}, \frac{k^{h-p_1}-1}{k-1}\}$. In fact, while $p_1 \geq \frac{h}{2}$, $Sc(X) = k^{p_1} + \frac{k^{p_1}-1}{k-1} = \frac{k^{p_1+1}-1}{k-1} \geq \frac{k^{\frac{h+2}{2}}-1}{k-1}$. While $p_1 < \frac{h}{2}$, $Sc(X) = k^{p_1} + \frac{k^{h-p_1}-1}{k-1}$, since function $f(x) = k^x + \frac{k^{h-x}-1}{k-1}$ is decreasing in interval $(-\infty, \frac{h}{2})$, then $Sc(X) > k^{\frac{h}{2}} + \frac{k^{h-\frac{h}{2}}-1}{k-1} = \frac{k^{\frac{h+2}{2}}-1}{k-1}$.

Next we by four subcases to show that $Sc(X) \geq \frac{k^{\frac{h+2}{2}}-1}{k-1}$ for some $n(n \geq 2)$.

Subcase 1.1 If $p_1 \geq \frac{h}{2}$. Then

$$\begin{aligned} Sc(X) &= k^{p_1} + k^{p_2} + \ldots + k^{p_n} + m(T - X) \geq k^{p_1} + k^{p_2} + \ldots + k^{p_n} + \frac{k^{p_1}-1}{k-1} \\ &> k^{p_1} + \frac{k^{p_1}-1}{k-1} \geq \frac{k^{\frac{h+2}{2}}-1}{k-1}. \end{aligned}$$

Subcase 1.2 If $p_1 < \frac{h}{2}$, $p_n > \frac{h}{2}$. Then there exist a $p_j \geq \frac{h}{2} + 1 (2 \leq j \leq n)$. Thus

$$Sc(X) = k^{p_1} + k^{p_2} + \ldots + k^{p_n} + m(T - X) > k^{p_j} \geq k^{\frac{h}{2}+1} \geq \frac{k^{\frac{h+2}{2}}-1}{k-1}.$$

Subcase 1.3 If $p_1 < \frac{h}{2}$, $p_n = \frac{h}{2}$. Then

$$Sc(X) = k^{p_1} + \ldots + k^{p_n} + m(T - X) > k^{p_n} + \frac{k^{\frac{h}{2}} - 1}{k - 1} = \frac{k^{\frac{h+2}{2}} - 1}{k - 1}.$$

Subcase 1.4 If $p_1 < \frac{h}{2}$, $p_n < \frac{h}{2}$. Then

$$Sc(X) = k^{p_1} + \ldots + k^{p_n} + m(T - X) \geq k^{p_1} + \ldots + k^{p_n} + \frac{k^{\frac{h+2}{2}} - 1}{k - 1} > \frac{k^{\frac{h+2}{2}} - 1}{k - 1}.$$

Case 2. h is odd.

It is similar to case 1 we have
$Sc(X) = k^{p_1} + k^{p_2} + \ldots + k^{p_n} + \max\{\frac{k^{p_1}-1}{k-1}, \frac{k^{p_2-p_1-1}-1}{k-1}, \ldots, \frac{k^{p_n-p_{n-1}-1}-1}{k-1}, \frac{k^{h-p_n}-1}{k-1}\}$

First we show that $Sc(X) \geq Sc(X_o) = \frac{(2k-1)k^{\frac{h-1}{2}}-1}{k-1}$ holds for $n = 1$. Clearly, $Sc(X) = |S_{p_1}| + m(T - S_{p_1}) = k^{p_1} + \max\{\frac{k^{p_1}-1}{k-1}, \frac{k^{h-p_1}-1}{k-1}\}$. In fact, when $p_1 \leq \frac{h-1}{2}$, $Sc(X) = k^{p_1} + \frac{k^{h-p_1}-1}{k-1}$, since function $f(x) = k^x + \frac{k^{h-x}-1}{k-1}$ is decreasing in interval $(-\infty, \frac{h-1}{2}]$, then $Sc(X) > k^{\frac{h-1}{2}} + \frac{k^{h-\frac{h-1}{2}}-1}{k-1} = \frac{(2k-1)k^{\frac{h-1}{2}}-1}{k-1}$. When $p_1 \geq \frac{h+1}{2}$, $Sc(X) = k^{p_1} + \frac{k^{p_1}-1}{k-1} \geq \frac{k^2 k^{\frac{h-1}{2}}-1}{k-1} \geq \frac{(2k-1)k^{\frac{h-1}{2}}-1}{k-1}$.

Next we by four subcases to show that $Sc(X) \geq \frac{(2k-1)k^{\frac{h-1}{2}}-1}{k-1}$ for some $n (n \geq 2)$.

Subcase 2.1 If $p_1 \geq \frac{h+1}{2}$. Then

$$\begin{aligned} Sc(X) &= k^{p_1} + k^{p_2} + \ldots + k^{p_n} + m(T - X) \geq k^{p_1} + k^{p_2} + \ldots + k^{p_n} + \frac{k^{p_1} - 1}{k - 1} \\ &> k^{p_1} + \frac{k^{p_1} - 1}{k - 1} \geq \frac{(2k - 1)k^{\frac{h-1}{2}} - 1}{k - 1}. \end{aligned}$$

Subcase 2.2 If $p_1 < \frac{h+1}{2}$, $p_n > \frac{h+1}{2}$. Then there exist a $p_j \geq \frac{h+1}{2} + 1 (2 \leq j \leq n)$. Then

$$\begin{aligned} Sc(X) =& k^{p_1} + k^{p_2} + \ldots + k^{p_n} + m(T - X) > k^{p_j} \geq k^{\frac{h+1}{2}+1} = k^2 k^{\frac{h-1}{2}} \\ &\geq \frac{k^2 k^{\frac{h-1}{2}} - 1}{k - 1} \geq \frac{(2k - 1)k^{\frac{h-1}{2}} - 1}{k - 1}. \end{aligned}$$

Subcase 2.3 If $p_1 < \frac{h+1}{2}$, $p_n = \frac{h+1}{2}$. Then

$$\begin{aligned} Sc(X) =& k^{p_1} + \ldots + k^{p_n} + m(T - X) > k^{p_n} + \frac{k^{h-p_n} - 1}{k - 1} = \frac{k^{\frac{h-1}{2}}(k^2 - k + 1) - 1}{k - 1} \\ &\geq \frac{(2k - 1)k^{\frac{h-1}{2}} - 1}{k - 1}. \end{aligned}$$

Subcase 2.4 If $p_1 < \frac{h+1}{2}$, $p_n < \frac{h+1}{2}$. Then

$$Sc(X) \geq k^{p_1} + \ldots + k^{p_n} + \frac{k^{h-p_n} - 1}{k - 1} > k^{p_n} + \frac{k^{h-p_n} - 1}{k - 1}.$$

Since function $f(x) = k^x + \frac{k^{h-x}-1}{k-1}$ is decreasing in interval $(-\infty, \frac{h-1}{2}]$, then $Sc(X) > k^{\frac{h-1}{2}} + \frac{k^{h-\frac{h-1}{2}}-1}{k-1} = \frac{(2k-1)k^{\frac{h-1}{2}}-1}{k-1}$.

Therefore, X_e and X_o are $I-$ set of complete k-ary tree T while h is even and h is odd, respectively. And thus we get

$$I(T) = \begin{cases} \frac{(2k-1)k^{\frac{h-1}{2}}-1}{k-1}, & \text{if } h \text{ is odd;} \\ \frac{k^{\frac{h+2}{2}}-1}{k-1}, & \text{if } h \text{ is even.} \end{cases}$$

□

In this Theorem, let $k = 2$ we get an interesting corollary.

Colorallary 3.1 *Let T be a complete binary tree with height h. Then integrity of T is*

$$I(T) = \begin{cases} 3 \cdot 2^{\frac{h+1}{2}-1} - 1, & \textit{if } h \textit{ is odd;} \\ 2^{\frac{h+2}{2}} - 1, & \textit{if } h \textit{ is even.} \end{cases}$$

By this corollary, it is clear that the complete binary tree is a counterexample for the conclusion of following Theorem.

Theorem 3.2 ([2]) *Let T be a tree with order $n \geq 3$ and maximum degree $\triangle \geq 2$. Then the minimum integrity of T is*

$$\min_{T \in \mathcal{T}[n,\Delta]} I(T) = \begin{cases} \lfloor \frac{n-2}{\Delta-1} \rfloor + 1, & \textit{if } r(\frac{n-2}{\Delta-1}) < \lfloor \frac{n-2}{\Delta-1} \rfloor; \\ \lfloor \frac{n-2}{\Delta-1} \rfloor + 2, & \textit{if } r(\frac{n-2}{\Delta-1}) \geq \lfloor \frac{n-2}{\Delta-1} \rfloor, \end{cases}$$

where $r(\frac{n-2}{\Delta-1})$ denotes the remainder of $n-2$ divided by $\Delta - 1$.

4 Rupture degree and tenacity of complete k-ary trees

In this section we determine the rupture degree and tenacity of complete k-ary trees, thus completing the proof of the conjecture which is stated in [1].

Theorem 4.1 *Let T be a complete k-ary tree with height h. Then rupture degree of T is*

$$r(T) = \begin{cases} \frac{k^{h+1}-k-2}{k+1}, & \textit{if } h \textit{ is odd;} \\ \frac{k^{h+1}-k}{k+1}, & \textit{if } h \textit{ is even.} \end{cases}$$

Proof: First let S_i be the vertices set of T at any one level i. If h is odd, we choose $X_1 = S_0 \cup S_2 \cup S_4 \ldots \cup S_{h-1}$. Then $\omega(T - X_1) - |X_1| - m(T - X_1) = \frac{k^{h+1}-k-2}{k+1}$; If h is even, we choose $X_2 = S_1 \cup S_3 \cup S_5 \ldots \cup S_{h-1}$. Then

$\omega(T-X_2)-|X_2|-m(T-X_2)=\frac{k^{h+1}-k}{k+1}$. Thus by the definition of rupture degree we get

$$r(T)\geq\begin{cases}\frac{k^{h+1}-k-2}{k+1}, & \text{if } h \text{ is odd;}\\ \frac{k^{h+1}-k}{k+1}, & \text{if } h \text{ is even.}\end{cases}$$

The following we distinguish two cases to complete another half proof of this theorem.

Case 1. h is even.

By Lemmas 2.1 and 2.2, $|X|\geq\frac{\omega(T-X)-1}{k}$ and $\omega(T-X)\leq\frac{1}{k^2-1}(k^{h+2}-1)$ for any cut-set X. Combining these with $m(T-X)\geq 1$ and $k>1$ together, we get $\omega(T-X)-|X|-m(T-X)\leq\frac{(k-1)\omega(T-X)-k+1}{k}\leq\frac{k^{h+1}-k}{k+1}$. Thus $r(T)\leq\frac{k^{h+1}-k}{k+1}$.

Case 2. h is odd.

By Lemmas 2.1 and 2.2, it is clear that $|X|\geq\frac{\omega(T-X)-1}{k}$ and $2\leq\omega(T-X)\leq\frac{k}{k^2-1}(k^{h+1}-1)$ for any cut-set X. Now we consider the value of $\omega(T-X)-|X|-m(T-X)$ in cases $|X|=\frac{\omega(T-X)-1}{k}$ and $|X|>\frac{\omega(T-X)-1}{k}$.

Subcase 2.1 If $|X|=\frac{\omega(T-X)-1}{k}$. Then we can find that

$$m(T-X)\geq\frac{|V(T)|-|X|}{\omega(T-X)}=\frac{\frac{k^{h+1}-1}{k-1}-\frac{\omega(T-X)-1}{k}}{\omega(T-X)}\geq 1+\frac{k^2-1}{k^2(k^{h+1}-1)}\geq 2$$

Thus we get

$$\begin{aligned}\omega(T-X)-|X|-m(T-X)&\leq\frac{(k-1)\omega(T-X)-2k+1}{k}\\ &\leq\frac{k^{h+1}-1}{k+1}+\frac{1-2k}{k}\\ &<\frac{k^{h+1}-1}{k+1}-1=\frac{k^{h+1}-k-2}{k+1}.\end{aligned}$$

Subcase 2.2 If $|X|>\frac{\omega(T-X)-1}{k}$. This means that $\omega(T-X)\leq k|X|$ for any cut-set X. Combine this with $m(T-X)\geq 1$ we get that

$$\omega(T-X)-|X|-m(T-X)\leq\frac{(k-1)\omega(T-X)-k}{k}\leq\frac{k^{h+1}-k-2}{k+1}.$$

Therefore,

$$r(T)\leq\begin{cases}\frac{k^{h+1}-k-2}{k+1}, & \text{if } h \text{ is odd;}\\ \frac{k^{h+1}-k}{k+1}, & \text{if } h \text{ is even.}\end{cases}$$

Thus complete the proof of Theorem. □

In this Theorem, let $k=2$ we get the following corollary, and thus prove the conjecture which is stated in [1].

Colorallary 4.1 *Let T be a complete binary tree with height h. Then rupture degree of T is*

$$r(T) = \begin{cases} \frac{2^{h+1}-4}{3}, & \text{if } h \text{ is odd;} \\ \frac{2^{h+1}-2}{3}, & \text{if } h \text{ is even.} \end{cases}$$

Similar to rupture degree, we can determine the tenacity of complete k-ary tree and complete binary tree as follows immediately.

Theorem 4.2 *Let T be a complete k-ary tree with height h. Then tenacity of T is*

$$T(T) = \begin{cases} \frac{k^{h+1}+k^2-2}{k(k^{h+1}-1)}, & \text{if } h \text{ is odd;} \\ \frac{k^{h+1}+k^2-k-1}{k^{h+2}-1}, & \text{if } h \text{ is even.} \end{cases}$$

Colorallary 4.2 *Let T be a complete binary tree with height h. Then tenacity of T is*

$$T(T) = \begin{cases} \frac{2^h+1}{2^{h+1}-1}, & \text{if } h \text{ is odd;} \\ \frac{2^{h+1}+1}{2^{h+2}-1}, & \text{if } h \text{ is even.} \end{cases}$$

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Received by the editors June 25, 2010