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A Construction of Anti-ordered group by an Anti-ordered Semigroup with Apartness [1]

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Abstract

This investigation is in the Bishop's constructive algebra. A commutative anti-ordered semigroup $((S, =, \neq), \cdot, \alpha)$ can embedded in an anti-ordered group if the anti-order relation α is close with the semigroup operation. Quasi-antiorder relation plays an important role in this embedding.

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1 Introduction and preliminaries

This investigation is in the Bishop's constructive mathematics in sense of the following books: [1, 2, 3, 4] and [10]. Undefined notion and notations we refer to our articles [5, 6, 7, 8] and [9].

Let $((S, =, \neq), \cdot, \alpha)$ be an anti-ordered commutative semigroup ([6, 7]), where α is an anti-order on semigroup S. For relation α we say ([6, 7, 9]) that it is an anti-order relation on semigroup $((S, =, \neq), \cdot, \alpha)$ is holds $\alpha \subseteq \neq$, $\alpha \subseteq \alpha * \alpha$ (where the operation $'*'$ between relations α and β is defined by $(u, v) \in \beta * \alpha \Longleftrightarrow (\forall t)((u, t) \in \alpha \vee (t, v) \in \beta))$, $\neq = \alpha \cup \alpha^{-1}$ and α is compatible with the semigroup operation on S in the following sense $(xay, xby) \in \alpha \Longrightarrow (a, b) \in \alpha$ (for any $a, b, x, y \in S$). A coequality relation q on S is called anti-congruence on $(S, \cdot)$ if $(xa, xb) \in q$ and $(ax, bx) \in q$ implies $(a, b) \in q$ for every $a, b, x \in S$ ([5, 8]). We call α is *close* with the operation on S if $(a, b) \in \alpha$ implies $(ax, bx) \in \alpha$ and $(xa, xb) \in \alpha$.

A relation σ on S is called ([7, 9]) quasi-antiorder relation if $\sigma \subseteq \alpha$, $\sigma \subseteq \sigma * \sigma$ and σ is compatible with the semigroup operation.

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Let $((S, =, \neq), \cdot, \alpha)$ and $((T, =, \neq), \cdot, \beta)$ be anti-ordered semigroups. Let $f : S \longrightarrow T$ be a mapping from S into T. f is called isotone if $(\forall a, b \in S)((a, b) \in \alpha \Longrightarrow (f(a), f(b)) \in \beta)$ holds. f is called reverse isotone if $(\forall a, b \in S)((f(a), f(b)) \in \beta \Longrightarrow ((a, b) \in \alpha)$ holds. f is called a homomorphism if it is strongly extensional and satisfies $f(ab) = f(a)f(b)$ for all $a, b \in S$. f is called an isomorphism if it is onto, homomorphism, injective and embedding isotone and reverse isotone strongly extensional mapping. Two anti-ordered semigroups are called isomorphic if there exists an isomorphism between them, S is embedded in T if, by definition, S is isomorphic to a subset of T, i.e. if there exists a mapping $f : S \longrightarrow T$ which is strongly extensional injective and embedding isotone and reverse isotone homomorphism.

2 The theorem

The result of this paper is the following theorem:

Theorem 2.1 *Let $((S, =, \neq), \cdot, \alpha)$ be a commutative anti-ordered semigroup with apartness such that α is closed for the semigroup operation. Then we can construct an anti-ordered group G that there exists a strongly extensional isotone and reverse isotone mapping from S into G.*

Proof. Let $((S, =, \neq), \cdot, \alpha)$ be an anti-ordered commutative semigroup where the relation α is closed for the semigroup operation.

(I) The set $(S \times S, =_2, \neq_2)$, where equality $' =_2'$ and coequality $' \neq_2'$ given by

$$(a, b) =_2 (x, y) \Longleftrightarrow a = x \wedge b = y, \ \ (a, b) \neq_2 (x, y) \Longleftrightarrow a \neq x \vee b \neq y,$$

with the multiplication $'\circ'$ on $S \times S$ defined by

$$\circ : (S \times S) \times (S \times S) \ni ((a, b), (c, d)) \longmapsto (ac, bd) \in S \times S,$$

is a semigroup. Indeed:

(1) The operation $'\circ'$ is well defined:

$(a, b) =_2 (x, y) \wedge (c, d) =_2 (u, v) \Longleftrightarrow a = x \wedge b = y \wedge c = u \wedge d = v$
$\Longrightarrow ac = xu \wedge bd = yv$
$\Longrightarrow (ac, bd) =_2 (xu, yv)$
$\Longleftrightarrow (a, b) \circ (c, d) =_2 (x, y) \circ (u, v);$

$(a, b) \circ (c, d) \neq_2 (x, y) \circ (u, v) \Longleftrightarrow (ac, bd) \neq_2 (xu, yv)$
$\Longleftrightarrow ac \neq xu \vee bd \neq yv$
$\Longrightarrow a \neq x \vee c \neq u \vee b \neq y \vee d \neq v$
$\Longleftrightarrow (a, b) \neq_2 (x, y) \vee (c, d) \neq_2 (u, v).$

(2) The operation $'\circ'$ is associative:

$$((a, b) \circ (x, y)) \circ (u, v) =_2 (ax, by) \circ (u, v) =_2 (axu, byv) =_2 (a, b) \circ ((x, y) \circ (u, v)).$$

(3) Let us defined relation γ on $((S \times S, =_2, \neq_2), \circ)$ by

$$((a,b),(c,d)) \in \gamma \Longleftrightarrow (a,c) \in \alpha \vee (d,b) \in \alpha.$$

Then
(3.1) $((a,b),(c,d)) \in \gamma \Longleftrightarrow (a,c) \in \alpha \vee (d,b) \in \alpha$
$\Longrightarrow a \neq c \vee d \neq b$
$\Longrightarrow (a,b) \neq_2 (c,d)$;

(3, 2) $a \neq c \vee d \neq b \Longrightarrow (a,c) \in \alpha \vee (c,a) \in \alpha \vee (d,b) \in \alpha \vee (b,d) \in \alpha$
$\Longrightarrow ((a,b),(c,d)) \in \gamma \vee ((c,d),(a,b)) \in \gamma$;

(3.3) $((a,b),(c,d)) \in \gamma \Longleftrightarrow (a,c) \in \alpha \vee (d,b) \in \alpha$
$\Longrightarrow (a,x) \in \alpha \vee (x,c) \in \alpha \vee (d,y) \in \alpha \vee (y,b) \in \alpha$
$\Longrightarrow ((a,b),(x,y)) \in \gamma \vee ((x,y),(c,d)) \in \gamma$.

(3.4) $((a,b) \circ (u,v)),((c,d) \circ (u,v)) \in \gamma \Longleftrightarrow ((au,bv),(cu,dv)) \in \gamma$
$\Longleftrightarrow (au,cu) \in \alpha \vee (dv,bv) \in \alpha$
$\Longrightarrow (a,c) \in \alpha \vee (d,b) \in \alpha$
$\Longleftrightarrow ((a,b),(c,d)) \in \gamma$.

(3.5) $((u,v) \circ (a,b),(u,v) \circ (c,d)) \in \gamma \Longrightarrow ((a,b),(c,d)) \in \gamma$
(Analogously to (3.4))

So, the relation γ is an anti-order on $(S \times S, =_2, \neq_2)$.

(II) Let σ be the relation on $((S \times S, =_2, \neq_2), \circ, \gamma)$ defined as follows

$$((a,b),(c,d)) \in \sigma \Longleftrightarrow (ad,bc) \in \alpha.$$

σ is a quasi-anriorder relation on $((S \times S, =_2, \neq_2), \circ, \gamma)$. In fact:

(1) $((a,b),(c,d)) \in \sigma \Longleftrightarrow (ad,bc) \in \alpha$
$\Longrightarrow (ad,cd) \in \alpha \vee (cd,bc) \in \alpha$
$\Longrightarrow (a,c) \in \alpha \vee (d,b) \in \alpha$
$\Longrightarrow ((a,b),(c,d)) \in \gamma$;

(2) $((a,b),(e,f)) \in \sigma \Longleftrightarrow (af,be) \in \alpha$
$\Longrightarrow (afdc,bedc) \in \alpha$ (because α is closed for the semigroup operation)
$\Longrightarrow (afdc,fcbc) \in \alpha \vee (fcbc,bedc) \in \alpha$
$\Longrightarrow (ad,bc) \in \alpha \vee (cf,de) \in \alpha$
$\Longrightarrow ((a,b),(c,d)) \in \sigma \vee ((c,d),(e,f)) \in \sigma$;

(3) $((a,b) \circ (e,f),(c,d) \circ (e,f)) \in \sigma \Longleftrightarrow ((ae,bf),(ce,df)) \in \sigma$
$\Longleftrightarrow (aedf,bfce) \in \alpha$
$\Longrightarrow (ad,bc) \in \alpha$
$\Longleftrightarrow ((a,b),(c,d)) \in \sigma$;

(4)$((e,f) \circ (a,b),(e,f) \circ (c,d)) \in \sigma \Longrightarrow ((a,b),(c,d)) \in \sigma$ (Analogously to (3))

Finally, $((S \times S, =_2, \neq_2), \circ$ is anti-ordered semigroup under the anti-order γ and the relation σ is an quasi-antiorder on $((S \times S, =_2, \neq_2), \circ)$.

(III) By the Lemma 1 in the paper [7], and by Theorem 3 in the paper [5], the relation $q = \sigma \cup \sigma^{-1}$ is an anticongruence on $((S \times S, =_2, \neq_2), \circ)$ and the factor-set $(S \times S)/q$ with multiplication $'\otimes'$ and the anti-order $'\Theta'$ given below

$$((a, b))q \otimes ((c, d))q =_1 ((a, b) \circ (c, d))q =_1 ((ac, bd))q,$$

$$(((a, b))q, ((c, d))q) \in \Theta \Longleftrightarrow ((a, b), (c, d)) \in \sigma.$$

is an anti-ordered semigroup. Further, we have that $(((S \times S)/q, =_1, \neq_1), \otimes)$ is a commutative anti-ordered group. Indeed:

(1)The first, we have

$$((a, b))q \otimes ((c, d))q =_1 ((ac, bd))q =_1 ((ca, db))q =_1 ((c, d))q \otimes ((a, b))q$$

(2) We will prove that $((a, a))q =_1 ((b, b))q$ for any $a, b \in S$. let $a, b \in S$ be arbitrary elements of S. Since $ab = ba$, we have $(ab, ba) \bowtie \alpha$, i.e. $((a, a), (n, b)) \bowtie \sigma$ and $((b, b), (a, a)) \bowtie \sigma$. Thus, $((a, a,))q =_1 ((b, b))q$

(3) Let $((c, d))q$ be an arbitrary element of $(S \times S)/q$ and a be an arbitrary element of S. Then, $((a, a))q \otimes ((c, d))q =_1 ((ac, ad))q =_1 ((c, d))q$. We prove that $((ac, ad), (c, d)) \bowtie q$ Since $((ad)c, c(ad)) \bowtie q$ and $(c(ad), d(ac)) \bowtie q$ because the semigroup S is a commutative semigroup, we have $((c, d), (ac, ad)) \bowtie \sigma$ and $((ac, ad), (c, d)) \bowtie \sigma$. Thus, $((ac, ad), (c, d)) \bowtie q$

(4) Let $((c, d))q$ be an arbitrary element of $(S \times S)/q$. then, $(((c, d))q)^{-1} =_1 ((d, c))q$. Indeed, we have

$$((c, d)q \otimes ((d, c))q =_1 ((cd, dc))q =_1 ((a, a))q$$

(IV) At he end, we prove that S is embeddable in $(((S \times S)/q, =_1, \neq_1), \otimes)$. We consider the mapping

$$\varphi : S \ni a \longmapsto ((a^2, a))q \in (S \times S)/q$$

Then:
(1) The first, from the equality

$$\varphi(ab) =_1 ((abab, ab))q =_1 (((ab)^2, ab))q =_1 ((a^2, a))q \otimes ((b^2, b))q =_1 \varphi(a) \otimes \varphi(b)$$

we conclude that the mapping φ is well defined. the second, let $\varphi(a) =_1 \varphi(b)$ for some $a, b \in S$. This means $((a^2, a))q \neq_1 ((b^2, b))q$. Thus, we have $((a^2, a), (b^2, b)) \in q$ and $(a^2b, ab^2) \in \alpha$ or $(b^2a, ba^2) \in \alpha$. Therefore, we conclude that φ is a ctrongly extensional mapping because we have $a \neq b$ from the both cases.

(2)Let $a, b \in S$ such that $(a, b) \in \alpha$. Then, $(a^2b, ab^2) \in \alpha$ because the relation α is closed for the semigroup operation. Thus, $((a^2, a), (b^2, b)) \in \sigma$ and $(((a^2, a))q, ((b^2, b))q) \in \Theta$. Opposite, we have
$(((a^2, a))q, ((b^2, b))q) \in \Theta \Longleftrightarrow ((a^2, a), (b^2, b)) \in \sigma$

$\Longleftrightarrow (a^2b, ab^2) \in \alpha$
$\Longrightarrow (a, b) \in \alpha$
because α is compatible with the semigroup operation. Therefore, the mapping φ is isotone and reverse isotone homomorphism. □

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