Bull.Soc.Math. Banja Luka
ISSN 0354-5792
Vol. 16(2009), 11-20

The Average Lower Independence Number On Graph Operations

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Abstract

Reliability and invulnerability of interconnection networks are both primarily important. When investigating the resistance of a communication network to disruption of operation after the failure of certain stations or communication links, several vulnerability measures are used and a communication network can be modeled as a graph. If we think of a graph as modeling a network, the average lower independence number of a graph is one measure of graph vulnerability and it is defined by $i_{av}(G) = \frac{1}{|V(G)|}\sum_{v\in V(G)} i_v(G)$, where $i_v(G)$ is the minimum cardinality of a maximal independent set of G that contains v. In this paper, we defined and examined this parameter and considered the average lower independence number of composition graphs of paths and cycles, join graphs and coronas $G \circ K_n$ are calculated.

Keywords: Connectivity, network design and communication, average lower independence number, composition graph, join graphs, coronas.
AMS subject classification (2000): 05C40, 68R10, 68M10.

1. Introduction

Computer or communication networks are so designed that they do not easily get disrupted under external attack and, moreover, these are

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easily reconstructible if they do get disrupted. These desirable properties of networks can be measured by various parameters like connectivity, toughness, integrity, domination and its variations [1, 3, 4, 6, 9]. The average lower independence number of a graph is a new parameter to measure the vulnerability of networks. This parameter is closely related to the problem of finding large independent sets in graphs.
In a graph $G = (V(G), E(G))$, a subset $S \subseteq V$ of vertices is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The dominating number $\gamma(G)$ is the minimum cardinality of a dominating set. The independent domination number (also called the lower independence number) $i(G)$ of G is the minimum cardinality of a set that is both independent and dominating.
Henning introduced the concept of average independence. For a vertex v of a graph G, the lower independence number, denoted by $i_v(G)$, is the minimum cardinality of a maximal independent set of G that contains v. The average lower independence number of G, denoted by $i_{av}(G)$, is the value $i_{av}(G) = \frac{1}{|V(G)|}\sum_{v \in V(G)} i_v(G)$ [2, 7]. It is clear that $i(G) = min\{i_v(G)|v \in V(G)\}$ and so $i(G) \leq i_{av}(G)$.
Throughout this paper of any graph G,$\kappa(G)$, $\alpha(G)$ and $\beta(G)$, respectively, denote the connectivity, covering number and independence number of G .
As in examples, we consider the two graphs, both of which have same edges and vertices. For the connectivity and the independence number of two graphs G_1 and , G_2 the equalities are $k(G_1) = k(G_2)$ and $\beta(G_1) = \beta(G_2)$. According to these statements, we can not say that which graph is more reliable. Then, for measuring the reliability, we compute average lower independence number of these two graphs. If $i_{av}(G_1) < i_{av}(G_2)$, we can say that graph G_1 is more reliable than graph G_2 .
In Section 2, known results on the average lower independence number are given. In Section 3, we give some results for the average lower independence number of composition graphs of paths and cycles, join graphs and coronas $G \circ K_n$ are calculated.

2. Main Results

In this section, we will review some of the known result on average lower independence number.

Theorem 2.1. [2, 7] *For every vertex v in a graph,*

$$a)\ i(G) \leq i_v(G) \leq \beta(G)$$

$$b)\ i(G) \le i_{av}(G) \le \beta(G)$$

Theorem 2.2. *[7] For any graph G of order n with independent domination number i and independence number β ,*

$$i_{av}(G) \le \beta - \frac{i(\beta - i)}{n}$$

Theorem 2.3. *[7] If T is a tree of order $n \ge 2$, then*

$$i_{av}(G) \le n - 2 + \frac{2}{n}$$

3. Composite, Join Graphs and Coronas

In this section, the average lower independence number of composition graphs of paths and cycles, join graphs and coronas $G \circ K_n$ are calculated.

Definition 3.1. [5] The composition $G_1[G_2]$ of two graphs G_1 and G_2 has its vertex set $V(G_1) \times V(G_2)$, with (u_1, u_2) adjacent to (v_1, v_2) if either u_1 is adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2. The composition is also known as the lexicographic product. This operation is not commutative.

Lemma 3.1. Let $k \in Z^+$, $m \in Z^+$ and $m = 3k + 1$, then

$\lceil \frac{m}{3} \rceil + 2.\lfloor \frac{m}{3} \rfloor = m$

P r o o f. Let $m = 3k + 1$

$\lceil \frac{3k+1}{3} \rceil + 2.\lfloor \frac{3k+1}{3} \rfloor = \lceil k + \frac{1}{3} \rceil + 2.\lfloor k + \frac{1}{3} \rfloor = k + 1 + 2.k = 3k + 1 = m$

Then, the proof is completed ■

Lemma 3.2. Let $k \in Z^+$, $m \in Z^+$ and $m = 3k + 2$, then

$2.\lceil \frac{m}{3} \rceil + \lfloor \frac{m}{3} \rfloor = m$

Proof. The proof follows directly from *lemma 3.1* ■

Theorem 3.1. *Let m and n be two positive integers with $m \geq 2$, $n \geq 3$, then*

$$i_{av}(P_m[C_n]) = \begin{cases} \frac{\lceil \frac{n}{3} \rceil.(\frac{m^2+m}{3}+\lfloor \frac{m}{3} \rfloor)}{m} & , \quad m = 2 \ (mod3) \\ \lceil \frac{n}{3} \rceil.(\frac{m+2}{3}) & , \quad otherwise \end{cases}$$

Proof. Graph $P_m[C_n]$ has $m.n$ vertices. $P_m[C_n]$ consists of totally m levels including disjoint graphs C_n . Due to $P_m[C_n]$, every vertices in one level is adjacent to all vertices in another levels above and below. When computing the average lower independence number of graph $P_m[C_n]$, we have to consider the vertices in graph C_n for every level. There are three cases according to the number of vertices.
Case1. If $m = 3t, t \in Z^+$, then
(i) For $s = t-1$, let v be a vertex in the level *1. , 4. , ... , 3s+1.*The number of levels providing this condition is $\frac{m}{3}$. The total number of vertices in *Case 1(i)* is $\frac{m}{3}.n$. The levels which are in connection with other levels in $P_m[C_n]$ above and below are as follows.

$$1 : 2, 3 : 4 : 5, 6 : 7 : 8, ..., 3s : 3s + 1 : 3s + 2, m$$

It is easy to see that there are $\frac{m-3}{3}$ triple level groups. For $v \in$ *Case1(i)*, if v is a vertex of a triple level groups, because of including C_n in every level in $P_m[C_n]$ and being $i_v(C_n) = \lceil \frac{n}{3} \rceil$,then $i_v(P_m[C_n]) = \lceil \frac{n}{3} \rceil$. Hence, for $\frac{m-3}{3}$ triple level groups, we have,

$$i_v(P_m[C_n]) = (\frac{m-3}{3}).\lceil \frac{n}{3} \rceil$$

Except these triple levels groups, the maximal independent set which has minimum cardinality must include $2.\lceil \frac{n}{3} \rceil$ vertices for the first and mth level. Thus, for $v \in$ *Case1(i)*, we have

$$i_v(P_m[C_n]) = (\frac{m-3}{3}).\lceil \frac{n}{3} \rceil + 2.\lceil \frac{n}{3} \rceil = \lceil \frac{n}{3} \rceil.(\frac{m+3}{3})$$

(ii) For $s = t-1$, let vertex v be in the level *2., 5., . . . , 3s+2.* Again, the levels which are in connection with other levels in $P_m[C_n]$ above and below are as follows.

$$1 : 2 : 3, 4 : 5 : 6, 7 : 8 : 9, ..., 3s + 1 : 3s + 2 : m$$

The number of these triple level groups in graph $P_m[C_n]$ is $\frac{m}{3}$. In view of the circumstances, for a vertex v in this case, the maximal independent set which has minimum cardinality has $\lceil \frac{n}{3} \rceil$ vertices, so we have

$$i_v(P_m[C_n]) = \lceil \frac{n}{3} \rceil . \frac{m}{3}$$

(iii) For $s = t - 1$, let vertex v be in the levels of *3.,6., . . .,3s+3.* The formation of the levels which are in connection with other levels in $P_m[C_n]$ above and below are as follows.

$$1, 2 : 3 : 4, 5 : 6 : 7, 8 : 9 : 10, ..., 3s - 1 : 3s : 3s + 1, 3s + 2 : m$$

The proof is similar to *Case 1(i)*.For each vertices v, the value of $i_v(P_m[C_n])$ is,

$$i_v(P_m[C_n]) = \lceil \frac{n}{3} \rceil . (\frac{m+3}{3})$$

When $m = 3t$, by *Case1(i),Case1(ii)* and *Case1(iii)*

$$i_{av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} i_v(G)$$

$$i_{av}(P_m[C_n]) = \frac{\frac{m}{3}.n.\lceil \frac{n}{3} \rceil . \frac{m}{3} + 2.\frac{m}{3}.n.(\frac{m+3}{3}).\lceil \frac{n}{3} \rceil}{m.n}$$

$$= \lceil \frac{n}{3} \rceil . (\frac{m+2}{3})$$

Case2. If $m = 3t + 1, t \in Z^+$, then
(i) For $s = t$, let v be a vertex in the level *1. , 4. , . . . , 3s+1.* There are $\lceil \frac{m}{3} \rceil$ levels. The levels which are in connection with other levels in $P_m[C_n]$ above and below are as follows.

$$1 : 2, 3 : 4 : 5, 6 : 7 : 8, ..., 3s - 3 : 3s - 2 : 3s - 1, 3s : 3s + 1$$

It is easy to see there are $\frac{m-4}{3}$ triple level groups except for first two levels and last two levels. The proof is similar to *Case 1(i)*.For each vertices v, the value of $i_v(P_m[C_n])$ is,

$$i_v(P_m[C_n]) = \lceil \frac{n}{3} \rceil . (\frac{m+2}{3})$$

(ii) For $s = t$, let vertex v be in the level *2., 5., . . . , 3s+2.* There are $\lfloor \frac{m}{3} \rfloor$ levels. The levels which are in connection with other levels in $P_m[C_n]$ above and below are as follows.

$$1 : 2 : 3, 4 : 5 : 6, 7 : 8 : 9, , 3s - 2 : 3s - 1 : 3s, 3s + 1$$

For *Case2(ii)*, the number of triple level groups is $\frac{m-1}{3}$. The proof is similar to *Case 1(i)*. For each vertices v, we have

$$i_v(P_m[C_n]) = \lceil \frac{n}{3} \rceil.(\frac{m+2}{3})$$

(iii) For $s = t$, let vertex v be in the levels of *3.,6., . . . ,3s*. The formation of the levels which are in connection with other levels in $P_m[C_n]$ above and below are as follows.

$$1, 2 : 3 : 4, 5 : 6 : 7, 8 : 9 : 10, , 3s - 1 : 3s : 3s + 1$$

The proof is now similar to *Case2(ii)*.For each vertices v, we have

$$i_v(P_m[C_n]) = \lceil \frac{n}{3} \rceil.(\frac{m+2}{3})$$

When $m = 3t + 1$, by *Case2(i),Case2(ii)* and *Case2(iii)*

$$i_{av}(P_m[C_n]) = \frac{\lceil \frac{m}{3} \rceil.n.\lceil \frac{n}{3} \rceil.(\frac{m+2}{3})+2.\lfloor \frac{m}{3} \rfloor}{m.n}$$

$$= \frac{n.\lceil \frac{n}{3} \rceil.(\frac{m+2}{3}).(\lceil \frac{m}{3} \rceil+2.\lfloor \frac{m}{3} \rfloor)}{m.n} \Rightarrow (By\ Lemma3.1)$$

$$= \lceil \frac{n}{3} \rceil.(\frac{m+2}{3})$$

Case3. If $m = 3t + 2, t \in Z^+$, then
(i) For $s = t$, let v be a vertex in the level *1. , 4. , ... , 3s+1*. The number of levels is $\lceil \frac{m}{3} \rceil$. The formation of the levels which are in connection with other levels in $P_m[C_n]$ above and below are as follows.

$$1 : 2, 3 : 4 : 5, 6 : 7 : 8, ..., 3s : 3s + 1 : 3s + 2$$

It is easily seen that there are $\frac{m-2}{3}$ triple level groups except for first two levels. The proof is now similar to *Case2(ii)*. For each vertices v, we have

$$i_v(P_m[C_n]) = \lceil \frac{n}{3} \rceil.(\frac{m+1}{3})$$

(ii) For $s = t$, let vertex v be in the level *2., 5.,. . . , 3s+2*. The formation of the levels is as follows,

$$1 : 2 : 3, 4 : 5 : 6, 7 : 8 : 9, ..., 3s - 2 : 3s - 1 : 3s, 3s + 1 : 3s + 2$$

The proof is now similar to *Case3(i)*.For each vertices v, we have

$$i_v(P_m[C_n]) = \lceil\frac{n}{3}\rceil.(\frac{m+1}{3})$$

(iii) For $s = t$, let vertex v be in the levels of *3.,6., . . . , 3s.* The number of levels is $\lfloor\frac{m}{3}\rfloor$. These levels are connected with other levels are as follows,

$$1, 2 : 3 : 4, 5 : 6 : 7, 8 : 9 : 10, , 3s - 1 : 3s : 3s + 1, 3s + 2$$

For *Case3(iii)*, the number of triple level groups is $\frac{m-2}{3}$. The proof is now similar to *Case2(i)*. For each vertices v, we have

$$i_v(P_m[C_n]) = \lceil\frac{n}{3}\rceil.(\frac{m+4}{3})$$

When $m = 3t + 2$, by *Case3(i),Case3(ii)* and *Case3(iii)*

$$i_{av}(P_m[C_n]) = \frac{2.\lceil\frac{m}{3}\rceil.n.[\lceil\frac{n}{3}\rceil.(\frac{m+1}{3})]+n.\lfloor\frac{m}{3}\rfloor.[\lceil\frac{n}{3}\rceil.(\frac{m+4}{3})]}{m.n}$$

$$=\frac{n.\lceil\frac{n}{3}\rceil.(\frac{m+1}{3}).(2.\lceil\frac{m}{3}\rceil+\lfloor\frac{m}{3}\rfloor)+n.\lfloor\frac{m}{3}\rfloor.\lceil\frac{n}{3}\rceil}{m.n} \Rightarrow (By\ Lemma3.2)$$

$$=\frac{\lceil\frac{n}{3}\rceil.(\frac{m^2+m}{3}+\lfloor\frac{m}{3}\rfloor)}{m}$$

By *Case1,Case2* and *Case3* the proof is completed. ■

Theorem 3.2. *Let G_1 and G_2 be two connected graphs of order m and n, respectively, then*

$$i_{av}(G_1 + G_2) = \frac{i_{av}(G_1).m + i_{av}(G_2).n}{m + n}$$

Proof. We have two cases for the proof.

Case1. Let $v \in V(G_1)$. From the definition of operation $G_1 + G_2$, a maximal set of minimum cardinality including v can not include any of the vertices of G_2. The average lower independence number of G_1,

$$i_{av}(G_1) = \frac{1}{m}\sum_{v\in V(G_1)} i_v(G_1)$$

$$i_{av}(G_1).m = \sum_{v\in V(G_1)} i_v(G_1)$$

Case2. Let $v \in V(G_2)$. Then, clearly, this case is similar to Case1. Then, it' s easy to see that,

$$i_{av}(G_2) = \frac{1}{n}\sum_{v\in V(G_2)} i_v(G_2)$$

$$i_{av}(G_2).n = \sum_{v\in V(G_2)} i_v(G_2)$$

By *Cases 1 and 2*, obviously we have,

$$\sum_{v\in V(G_1+G_2)} i_v(G_1+G_2) = \sum_{v\in V(G_1)} i_v(G_1) + \sum_{v\in V(G_2)} i_v(G_2)$$

$$\sum_{v\in V(G_1+G_2)} i_v(G_1+G_2) = i_{av}(G_1).m + i_{av}(G_2).n$$

$$\frac{\sum_{v\in V(G_1+G_2)} i_v(G_1+G_2)}{m+n} = \frac{i_{av}(G_1).m + i_{av}(G_2).n}{m+n}$$

$$i_{av}(G_1+G_2) = \frac{i_{av}(G_1).m + i_{av}(G_2).n}{m+n}$$

Then the proof is completed. ∎

Result 3.1. Let G_1 and G_2 be two connected graphs of order m and n, respectively. Then,

$$i_{av}(G_1+G_2) \le \frac{\beta(G_1).m + \beta(G_2).n}{m+n}$$

Proof. From *Theorem 2.1 (b)*, we have $i_{av}(G) \le \beta(G)$. Then, it 's easy to see that,

$$i_{av}(G_1+G_2) \le \frac{\beta(G_1).m + \beta(G_2).n}{m+n}$$

Then the proof is completed. ∎

Definition 3.2. [8] The corona $G_1 \circ G_2$ is obtained by taking one copy of G_1 and $|G_1|$ copies of G_2, and by joining each vertex of G_2 the ith copy of G_2 to the ith vertex of G_1, $i = 1, 2, ..., |G_1|$.

Theorem 3.3. *Let K_n be a complete graph of order n. Then for any graph G of order m,*

$$i_{av}(G \circ K_n) = m$$

P r o o f. Let v be any vertex of graph G. When v is joined graph K_n, then we have m graphs K_{n+1} . We have two cases according to the vertices of K_{n+1}.

Case1. Let $v \in V(G)$. Then, the maximal independent set of minimum cardinality includes the vertex v itself and *m-1* vertices from graphs K_{n+1} which are not adjacent to v. Each of these *m-1* vertices belongs to each of graph G. For a vertex v of graph G, we have

$$i_v(G \circ K_n) = m$$

Case2. Let $v \in V(K_n)$. The vertex v is adjacent to all vertices of K_{n+1}. Then, the maximal independent set of minimum cardinality includes the only vertex v itself. Hence, for a vertex v of graph K_n, we have

$$i_v(G \circ K_n) = m$$

By *Cases 1 and 2*, we have

$$i_{av}(G \circ K_n) = \frac{1}{|V(G \circ K_n)|} \sum_{v \in V(G \circ K_n)} i_v(G \circ K_n)$$

$$= \frac{1}{m+m.n}(m.m + m.m.n)$$

$$= \text{m}$$

Then, the proof is completed. ■

4. Conclusion

If we want to design a communications network, we wish that it is as impossible as stable. Then, we model any communication network by a connected graph. In graph theory, we have many stability measures are called as connectivity, toughness, integrity, domination and its variations. In this paper, we introduce and study the concept of average lower independence number in graphs, a concept closely related to the problem of finding large independent sets in graphs. In the design of two networks having the same number of processors, if we want to choose the more stable one from these, we take their graph models and it is enough to choose the model whose the average lower independence number is smaller.

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Received by the editors on October 23, 2009