SOME NOTES ON QUASI-ANTIORDERS AND COEQUALITY RELATIONS

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Abstract

It is known that each quasi-antiorder on anti-ordered set X induces coequality q on X such that X/q is an anti-ordered set. The converse of this statement also holds: Each coequality q on a set X such that X/q is anti-ordered set induces a quasi-antiorder on X. In this paper we give proofs that the families of all coequality relations q on X and the family of all quasi-antiorder relation on set X are complete lattices.

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1 Introduction and preliminary

This short investigation, in Bishop’s constructive mathematics in the sense of well-known books [2], [4], [6] and [18] and Bogdanić, Romano and Vinčić’s paper [3], Jojić and Romano’s paper [6], and Romano’s papers [7]-[16], is continuation of forthcoming the second author’s papers [17]. Bishop’s constructive mathematics is developed on Constructive Logic - logic without the Law of Excluded Middle \( P \lor \neg P \). Let us note that in the Constructive Logic the ‘Double Negation Law’ \( \neg\neg P \Rightarrow P \) does not hold, but the following implication \( P \Rightarrow \neg\neg P \) does even in the Minimal Logic. Since the Constructive Logic is a part of the Classical Logic, these results, in the Constructive mathematics, are compatible with suitable results in the Classical mathematics. Let us recall that the following deduction principle \( A \lor B, \neg B \vdash A \) is acceptable in the Constructive Logic.

Let \( (X, =, \neq) \) be a set, where the relation \( \neq \) is a binary relation on \( X \), called diversity on \( X \), which satisfies the following properties:

\[
\neg(x \neq x), \ x \neq y \Rightarrow y \neq x, \ x \neq y \land y = z \Rightarrow x \neq z.
\]

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Following Heyting, if the following implication $x \neq z \implies x \neq y \lor y \neq z$ holds, the diversity $\neq$ is called apartness. Let $x$ be an element of $X$ and $A$ a subset of $X$. We write $x \ll A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in X : x \ll A\}$.

In $X \times X$ the equality and diversity are defined by $(x, y) = (u, v) \iff x = u \land y = v$, $(x, y) \neq (u, v) \iff x \neq u \lor y \neq v$, and equality and diversity relations in power-set $\wp(X \times X)$ of $X \times X$ by

$$\alpha =_2 \beta \iff (\forall (x, y) \in X \times X)((x, y) \in \alpha \iff (x, y) \in \beta),$$

$$\alpha \neq_2 \beta \iff (\exists x, y \in X)((x, y) \in \alpha \land (x, y) \ll \beta) \lor (\exists x, y \in X)((x, y) \in \beta \land (x, y) \ll \alpha).$$

Let us note that the diversity relation $\neq_2$ is not an apartness relation in general case.

**Example I:** (1) The relation $\neg(=)$ is an apartness on the set $\mathbb{Z}$ of integers.

(2) The relation $q$, defined on the set $\mathbb{Q} \cap \mathbb{N}$ by

$$(f, g) \in q \iff (\exists k \in \mathbb{N})(\exists n \in \mathbb{N})(m \geq n \implies |f(m) - g(m)| > k^{-1}),$$

is an apartness relation. ♦

A relation $q$ on $X$ is a coequality relation ([7]-[9]) on $X$ if and only if it is consistent, symmetric and cotransitive:

$$q \subseteq \neq, \quad q = q^{-1}, \quad q \subseteq q \ast q,$$

where ”$\ast$” is the operation of relations $\alpha \subseteq X \times X$ and $\beta \subseteq X \times X$, called filled product ([8], [9], [12]-[14]) of relations $\alpha$ and $\beta$, are relation on $X$ defined by

$$(a, c) \in \beta \ast \alpha \iff (\forall b \in X)((a, b) \in \alpha \lor (b, c) \in \beta).$$

For further study of coequality relation we suggest to read articles [8], [11], [13]-[16] (Specially, in articles [10], [12], [13] and [14], the author researches coequality relations compatible with the algebraic operations.) In article [7] and [8], problems of the existence of compatible equality and coequality relations on set with apartness are discussed. In article [9], the author has proved the following: If $e$ is an equivalence on set $X$, then there exists the maximal coequality relation $q$ on $X$ compatible with $e$ in the following sense:

$$e \circ q \subseteq q \text{ and } q \circ e \subseteq q.$$
Also, we can ([8],[11]) construct the factor-set \( X/q = \{aq : a \in X\} \): If \( q \) is a coequality relation on a set \( X \), then \( X/q \) is a set with:

\[
aq = _1 bq \iff (a,b) \trianglerighteq q, \quad aq \neq _1 bq \iff (a,b) \in q.
\]

It is easily to check that \( X/q \cong X/(q^C,q) \), besides, it is clear that the mapping \( \pi : X \longrightarrow X/q \), defined by \( \pi(x) = xq \), is a strongly extensional surjective function.

Subset \( C(x) = \{y \in X : y \neq x\} \) satisfies the following implication:

\[
y \in C(x) \land z \in X \implies y \neq z \lor z \in C(x).
\]

It is called a principal strongly extensional subset of \( X \) such that \( x \bowtie C(x) \). Following this special case, for a subset \( A \) of \( X \), we say that it is a strongly extensional subset of \( X \) if and only if the following implication

\[
x \in A \land y \in X \implies x \neq y \lor y \in A
\]

holds.

**Examples II:** (1) ([7]) Let \( T \) be a set and \( J \) be a subfamily of \( \wp(T) \) such that

\[
\emptyset \in J, \quad A \subseteq B \land B \in J \implies A \in J, \quad A \cap B \in J \implies A \in J \lor B \in J.
\]

If \( (X_t)_{t \in T} \) is a family of sets, then the relation \( q \) on \( \prod_{t \in T} X_t(\neq \emptyset) \), defined by \( (f,g) \in q \iff \{s \in T : (f(s) = g(s)) \in J \} \), is a coequality relation on the Cartesian product \( \prod X_t \).

(2) A ring \( R \) is a local ring if for each \( r \in R \), either \( r \) or \( 1 - r \) is a unit, and let \( M \) be a module over \( R \). The relation \( q \) on \( M \), defined by \( (x,y) \in q \) if there exists a homomorphism \( f : M \longrightarrow R \) such that \( f(x-y) \) is a unit, is a coequality relation on \( M \).

(3) ([11]) Let \( T \) be a strongly extensional consistent subset of semigroup \( S \), i.e. let \( (\forall x,y \in S)(xy \in T \implies x \in T \land y \in T) \) holds. Then, relation \( q \) on semigroup \( S \), defined by \( (a,b) \in q \) if and only if \( a \neq b \land (a \in T \lor b \in T) \), is a coequality relation on \( S \) and compatible with semigroup operation in the following sense \( (\forall x,y,a,b \in S)((xay,xbg) \in q \implies (a,b) \in q) \).

(4) Let \( (R,.,\neq,+,0,.,1) \) be a commutative ring. A subset \( Q \) of \( R \) is a coideal of \( R \) if and only if

\[
0 \bowtie Q, \quad -x \in Q \implies x \in Q, \quad x + y \in Q \implies x \in Q \land y \in Q, \quad xy \in Q \implies x \in Q \land y \in Q.
\]

Coideals of commutative ring with apartness were first defined and studied by Ruitenburg 1982 in his dissertation. After that, coideals (anti-ideals) are studied by A.S. Troelstra and D. van Dalen in their monograph [18]. This author proved, in 1988, if \( Q \) is a coideal of a ring \( R \), then the relation \( q \) on \( R \), defined
by \((x, y) \in q \iff x - y \in Q\), satisfies the following properties:

(a) \(q\) is a coequality relation on \(R\);

(b) \((\forall x, y, u, v \in R)((x + u, y + v) \in q \implies (x, y) \in q \lor (u, v) \in q)\);

(c) \((\forall x, y, u, v \in R)((xu, yv) \in q \implies (x, y) \in q \lor (u, v) \in q)\).

A relation \(q\) on \(R\), which satisfies the property (a)-(c), is called anticongruence on \(R\) ([4]) or coequality relation compatible with ring operations. If \(q\) is an anti-congruence on a ring \(R\), then the set \(Q = \{x \in R : (x, 0) \in q\}\) is a coideal of \(R\).

As in [12],[13], [14] and [15] a relation \(\alpha\) on \(X\) is antiorder on \(X\) if and only if
\[
\alpha \not\subseteq \neq, \alpha \subseteq \alpha \ast \alpha, \neq \not\subseteq \alpha \cup \alpha^{-1} \text{ (linearity)}. 
\]

Let \(g\) be a strongly extensional mapping of anti-ordered set from \((X, =, \neq, \alpha)\) into \((Y, =, \neq, \beta)\). For \(g\) we say that it is:

(i) isotone if \((\forall a, b \in X)((a, b) \in \alpha \implies (g(a), g(b)) \in \beta)\) holds;

(ii) reverse isotone if \((\forall a, b \in X)((g(a), g(b)) \in \beta \implies (a, b) \in \alpha)\) holds.

A relation \(\sigma\) on \(X\) is a quasi-antiorder ([11]-[16]) on \(X\) if
\[
\sigma \subseteq (\alpha \subseteq) \neq, \sigma \subseteq \sigma \ast \sigma. 
\]

It is clear that each coequality relation \(q\) on set \(X\) is a quasi-antiorder relation on \(X\), and the apartness is a trivial anti-order relation on \(X\). It is easy to check that if \(\sigma\) is a quasi-antiorder on \(X\), then ([10]) the relation \(q = \sigma \cup \sigma^{-1}\) is a coequality relation on \(X\). The notion of quasi-antiorder is defined for first time in article [8], and the notion of anti-order relation is defined for the first time in article [10]. Those relations and their properties are investigated by Baroni in [1], Bogdanić, Jojić and Romano in [3], Jojić and Romano in [6], and van Plato in [19] also.

**Examples III:** Let \(a\) and \(b\) be elements of semigroup \((S, =, \neq, \cdot)\). Then ([11]), the set \(C(a) = \{x \in S : x \cong SaS\}\) is a consistent subset of \(S\) such that :
- \(a \in C(a)\);
- \(C(a) \not= \emptyset \implies 1 \in C(a)\);
- Let \(a\) be an invertible element of \(S\). Then \(C(a) = \emptyset\);
- \((\forall x, y \in S)(C(a) \subseteq C(xy))\);
- \(C(a) \cup C(b) \subseteq C(ab)\).

Let \(a\) be an arbitrary element of a semigroup \(S\) with apartness. The consistent subset \(C(a)\) is called a principal consistent subset of \(S\) generated by \(a\). We introduce relation \(f\), defined by \((a, b) \in f \iff b \in C(a)\). The relation \(f\) has the
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following properties ([11, Theorem 7]):
- \( f \) is a consistent relation ;
- \( (a, b) \in f \implies (\forall x, y \in S)((xy, b) \in f) \);
- \( (a, b) \in f \implies (\forall n \in \mathbb{N})((a^n, b) \in f) \);
- \( (\forall x, y \in S)((a, xby) \in f \implies (a, b) \in f) ;
- \( (\forall x, y \in S)((a, xby) \in f) \).

We can construct the cotransitive relation \( c(f) = \bigcap^n f \) as cotransitive fulfillment of the relation \( f \) ([8]-[11],[15]). As consequences of these assertions we have the following results. The relation \( c(f) \) satisfies the following properties:
- \( c(f) \) is a quasi-antiorder on \( S \);
- \( (\forall x, y \in S)((a, xay) \Rightarrow c(f)) \);
- \( (\forall n \in \mathbb{N})((a, a^n) \Rightarrow c(f)) \);
- \( (\forall x, y \in S)((a, b) \in c(f) \Rightarrow (xay, b) \in c(f)) \);
- \( (\forall n \in \mathbb{N})((a, b) \in c(f) \Rightarrow (an, b) \in c(f)) \);
- \( (\forall x, y \in S)((a, xby) \in c(f) \Rightarrow (a, b) \in c(f)). \)

For a given anti-ordered set \((X, =, \neq, \alpha)\) is essential to know if there exists a coequality relation \( q \) on \( X \) such that \( X/q \) is an anti-ordered set. This plays an important role for studying the structure of anti-ordered sets. The following question is natural: If \((X, =, \neq, \alpha)\) is an anti-ordered set and \( q \) a coequality relation on \( X \), is the factor-set \( X/q \) anti-ordered set? Naturally, anti-order on \( X/q \) should be the relation \( \Theta \) on \( X/q \) defined by means of the anti-order \( \alpha \) on \( X \) such that \( \Theta = \{(xq, yq) \in X/q : (x, y) \in \alpha\} \), but it is not held in general case. The following question appears: Is there coequality relation \( q \) on \( X \) for which \( X/q \) is an anti-ordered set such that the natural mapping \( \pi : X \twoheadrightarrow X/q \) is reverse isotone? The concept of quasi-antiorder relation was introduced by this author in his papers [8] and [9]-[16] (Particularly, in articles [10] and [14], the author investigated anti-ordered algebraic systems with apartness.). According to Lemma 0 in [12], if \((X, =, \neq)\) is a set and \( \sigma \) is a quasi-antiorder on \( X \), then ([12, Lemma 1]) the relation \( q \) on \( X \), defined by \( q = \sigma \cup \sigma^{-1} \), is a coequality relation on \( X \), and the set \( X/q \) is an anti-ordered set under anti-order \( \Theta \) defined by \( (xq, yq) \in \Theta \iff (x, y) \in \sigma \). So, according to results in [12] and [13], each quasi-antiorder \( \sigma \) on an ordered set \( X \) under anti-order \( \alpha \) induces an coequality relation \( q = \sigma \cup \sigma^{-1} \) on \( X \) such that \( X/q \) is an anti-ordered set under \( \Theta \). (For a further study of quasi-antiorders on anti-ordered set we refer to papers [12], [13] and forthcoming the author’s paper [17].) In paper [14] we proved that the converse of this statement also holds. If \((X, =, \neq, \alpha)\) is an anti-ordered set and \( q \) coequality relation on \( X \), and if there exists an order relation \( \Theta_1 \) on \( X/q \) such that the \((X/q, =_{1}, \neq_{1}, \Theta_1)\) is an anti-ordered and the mapping \( \pi : X \twoheadrightarrow X/q \) is reverse isotone (so-called regular coequality), then there exists a quasi-antiorder \( \sigma \) on \( X \) such that \( q = \sigma \cup \sigma^{-1} \). So, each regular coequality \( q \) on a set \((X, =, \neq, \alpha)\)
induces a quasi-antiorder on \( X \). Besides, connections between the family of all quasi-antiorders on \( X \), the family of coequality relations on \( X \), and the family of all regular coequality relations \( q \) on \( X \) are given.

**Lemma 1.1** Let \( \tau \) be a quasi-antiorder on set \( X \). Then \( x \tau (\tau x) \) is a strongly extensional subset of \( X \), such that \( x \bowtie x \tau x \), for each \( x \in X \). Besides, the following implication \((x, z) \in \tau \Rightarrow x \tau \cup \tau z = X \) holds for each \( x, z \) of \( X \).

**Proof:** From \( \tau \subseteq \not\exists \) it follows \( x \bowtie x \tau \). Let \( yz \in \tau \) holds, and let \( z \) be an arbitrary element of \( X \). Thus, \((x, y) \in \tau \) and \((x, z) \in \tau \cap (z, y) \in \tau \). So, we have \( z \in x \tau \cup y \not\bowtie z \). Therefore, \( x \tau \) is a strongly extensional subset of \( X \) such that \( x \bowtie x \tau \).

The proof that \( \tau x \) is a strongly extensional subset of \( X \) such that \( x \bowtie \tau x \) is analogous. Besides, the following implication \((x, z) \in \tau \Rightarrow x \tau \cup \tau z = X \) holds for each \( x, y \) of \( X \). Indeed, if \((x, z) \in \tau \) and \( y \) is an arbitrary element of \( X \), then \((\forall y \in X)((x, y) \in \tau \cap (y, z) \in \tau) \). Thus, \( X = x \tau \cup \tau z \). \( \square \)

Let \( \tau \) be a quasi-antiorder on set \( X \). Then for every pair \((x, z)\) of \( \tau \) there exists a pair \((A_x, B_z)\) of strongly extensional subsets of \( X \) such that \( x \bowtie A_x \land z \bowtie B_z \) and \( X = A_x \cup B_z \) and \( x \in B_z \land z \in A_x \).

**Example IV:** If \( A \) is a strongly extensional subset of \( X \), then the relation \( \sigma \) on \( X \), defined by \((x, y) \in \sigma \iff x \in A \land x \not\bowtie y \), is a quasi-antiorder relation on \( X \).

**Proof:** It is clear that \( \sigma \) is a consistent relation on \( X \). Assume \((x, z) \in \sigma \) and let \( y \) be an arbitrary element of \( X \). Then, \( x \in A \land x \not\bowtie y \). Thus, \( x \not\bowtie y \lor y \not\bowtie z \).

If \( x \not\bowtie y \) and \( x \in A \), then \((x, y) \in \sigma \). If \( y \not\bowtie z \) and \( x \in A \), by strongly extensionality of \( A \), we have \( y \not\bowtie z \) and \( x \in A \) and \( x \not\bowtie y \lor y \in A \). In the case of \( y \not\bowtie z \land x \in A \land x \not\bowtie y \) we have again \((x, y) \in \sigma \); in the case of \( y \not\bowtie z \) and \( x \in A \) and \( y \in A \) we have \((y, z) \in \sigma \). So, the relation \( \sigma \) is a cotransitive relation.

Therefore, relation \( \sigma \) is a quasi-antiorder relation on \( X \). Further on, we have:

\[
x \in A \Rightarrow x\sigma = C(x), \quad \neg(x \in A) \Rightarrow x\sigma = \emptyset;
\]

\[
y \in A \Rightarrow y\sigma = C(y) \cap A, \quad y \bowtie A \Rightarrow y\sigma = A. \checkmark
\]

### 2 Main Results

In the following proposition we give a connection between the family \( \mathcal{Z}(X) \) of all quasi-antiorders on set \( X \) and the family \( \mathcal{Q}(X) \) of all coequality relation on \( X \).

For a set \((X, \bowtie, \not\bowtie, \alpha)\) by \( \mathfrak{R}(X, \alpha) \) we denote the family of all regular coequality relations \( q \) on \( X \) with respect to \( \alpha \), and by \( \mathcal{Z}(X, \alpha) \) denotes the family of all quasi-antiorder relation on \( X \) included in \( \alpha \).

Let us note that families \( \mathcal{Z}(X) \), \( \mathcal{Z}(X, \alpha) \) and \( \mathcal{Q}(X) \) are complete lattices. Indeed, in the following two theorems we give proofs for those facts:
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**Theorem 2.1** If $\{\tau_k\}_{k \in J}$ is a family of quasi-antiorders on a set $(X,=,\neq)$, then $\bigcup_{k \in J} \tau_k$ and $c(\bigcap_{k \in J} \tau_k)$ are quasi-antiorders in $X$. So, the families $\Im(X)$ and $\Im(X,\alpha)$ are complete lattices.

**Proof:** (1) Let $\{\tau_k\}_{k \in J}$ be a family of quasi-antiorders on a set $(X,=,\neq)$ and let $x, z$ be an arbitrary elements of $X$ such that $(x,z) \in \bigcup_{k \in J} \tau_k$. Then, there exists $k$ in $J$ such that $(x,z) \in \tau_k$. Hence, for every $y \in X$ we have $(x,y) \in \tau_k \lor (y,z) \in \tau_k$. So, $(x,y) \in \bigcup_{k \in J} \tau_k \lor (y,z) \in \bigcup_{k \in J} \tau_k$. On the other hand, for every $k$ in $J$ holds $\tau_k \subseteq \neq$. From this we have $\bigcup_{k \in J} \tau_k \subseteq \neq$. So, we can put $\text{sup} \{\tau_k : k \in J\} = \bigcup_{k \in J} \tau_k$.

(2) Let $R (\subseteq \neq)$ be a relation on a set $(X,=,\neq)$. Then for an inhabited family of quasi-antiorders under $R$ there exists the biggest quasi-antiorder relation under $R$. That relation is exactly the relation $c(R)$. In fact:

By (1), there exists the biggest quasi-antiorder relation on $X$ under $R$. Let $Q_R$ be the inhabited family of all quasi-antiorder relation on $X$ under $R$. With $(R)$ we denote the biggest quasi-antiorder relation $\bigcup Q_R$ on $X$ under $R$. On the other hand, the fulfillment $c(R) = \bigcap_{n \in N} n R$ of the relation $R$ is a cotransitive relation on set $X$ under $R$. Therefore, $c(R) \subseteq (R)$ holds.

We need to show that $(R) \subseteq c(R)$. Let $\tau (\subseteq (R) = \bigcup Q_R)$ be a quasi-antiorder relation in $X$ under $R$. Firstly, we have $\tau \subseteq R = \text{sup} R$. Assume $(x,z) \in \tau$. Then, out of $(\forall y \in X)((x,y) \in \tau \lor (y,z) \in \tau)$ we conclude that for every $y$ in $X$ holds $(x,y) \in R \lor (y,z) \in R$, i.e. holds $(x,z) \in R^+ = \text{sup} R$. So, we have $\tau \subseteq \text{sup} R$. Now, we will suppose that $\tau \subseteq n R$, and suppose that $(x,z) \in \tau$. Then, $(\forall y \in X)((x,y) \in \tau \lor (y,z) \in \tau)$ implies that $(x,y) \in R \lor (y,z) \in n R$ holds for every $y \in X$. Therefore, $(x,z) \in n^+ R$. So, we have $\tau \subseteq n^+ R$. Thus, by induction, we have $\tau \subseteq \bigcap n R$. Let us remember that $\tau$ is an arbitrary quasi-antiorder on $X$ under $R$. Hence, we proved that $(R) \subseteq \bigcup Q_R \subseteq c(R)$. If $\{\tau_k : k \in J\}$ is a family of quasi-antiorders on a set $(X,=,\neq)$, then $c(\bigcap_{k \in J} \tau_k)$ is a quasi-antiorder in $X$, and we can set $\text{inf} \{\tau_k : k \in J\} = c(\bigcap_{k \in J} \tau_k)$.

**Theorem 2.2** Let $(X,=,\neq)$ be a set with aparness. The family $q(X)$ is a complete lattice.

**Proof:** If $\{q_k : k \in \Lambda\}$ is a family of coequality relations on $X$, then $\bigcup q_k$ and $c(\bigcap q_k)$ are coequality relations on $X$ such that $(\forall k \in \Lambda)(q_k \subseteq q_k)$ and $(\forall k \in \Lambda)(c(\bigcap q_k) \subseteq q_k)$. Since $\bigcup q_k$ is the minimal extension of every $q_k$ we can put $\text{sup} \{q_k : k \in \Lambda\} = \bigcup q_k$, and since $c(\bigcap q_k)$ is the maximal coequality relation under $\bigcap q_k (\subseteq q_k)$ we can set $\text{inf} \{q_k : k \in \Lambda\} = c(\bigcap q_k)$.

**References**


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