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STIRLING'S[1] FORMULA AND ITS APPLICATION

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Abstract: In this paper are given the proof of important Stirling's formula and several hers interesting applications.

Key words and phrases: Stirling's formula, Taylor series, limit of sequence, Weierstrass criterium, Wallis formula.

1. The proof of the Stirling's formula

Stirling's formula has the form:

$$n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\cdot e^{\frac{\theta_n}{12n}};\quad (0<\theta_n<1). \tag{1}$$

We begin with the Taylor series expansions:

$$ln(1\pm x)=\pm x-\frac{x^2}{2}\pm\frac{x^3}{3}+\frac{x^4}{4}\pm\frac{x^5}{5}+\dots,\text{ for } x\in(-1,1).$$

Combining these two, we obtain the Taylor series expansion:

$$ln\frac{1+x}{1-x}=2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\dots+\frac{2}{2m+1}x^{2m+1}+\dots,$$

again for $x\in(-1,1)$. In particular, for $x=\frac{1}{2n+1}$, where $n\in\boldsymbol{N}$, we have:

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[1] James Stirling (1692.- 177.), Scottish mathematician.

$$ln\frac{n+1}{n}=\frac{2}{2n+1}+\frac{2}{3(2n+1)^3}+\frac{2}{5(2n+1)^5}+\dots,$$

which can be written as:

$$\left(n+\frac{1}{2}\right)ln\frac{n+1}{n}=1+\frac{1}{3(2n+1)^2}+\frac{1}{5(2n+1)^4}+\dots \ .$$

The right-hand side is greater than 1. It can be bounded from above as follows:

$$1+\frac{1}{3(2n+1)^2}+\frac{1}{5(2n+1)^4}+\dots<1+\frac{1}{3}\sum_{k=1}^{\infty}\frac{1}{(2n+1)^{2k}}$$

$$=1+\frac{1}{3(2n+1)^2}\cdot\frac{1}{1-\frac{1}{(2n+1)^2}}$$

$$=1+\frac{1}{12n(n+1)} \ .$$

So using Taylor series we have obtained the double inequality:

$$1<\left(n+\frac{1}{2}\right)ln\frac{n+1}{n}<1+\frac{1}{12n(n+1)}$$

or

$$1<ln\left(\frac{n+1}{n}\right)^{n+\frac{1}{2}}<1+\frac{1}{12n(n+1)} \ .$$

This transforms by exponentiating and dividing by e into:

$$1<\frac{1}{e}\left(\frac{n+1}{n}\right)^{n+\frac{1}{2}}<e^{\frac{1}{12n(n+1)}} \ .$$

To bring this closer to Stirling's formula, note that the term in the middle is equal to:

$$\frac{e^{-n-1}(n+1)^{n+1}\left((n+1)!\right)^{-1}\sqrt{n+1}}{e^{-n}n^n(n!)^{-1}\sqrt{n}}=\frac{x_{n+1}}{x_n} \ ,$$

where $x_n = e^{-n}n^n n!\sqrt{n}$, a number that we want to prove is equal to $\sqrt{2\pi}e^{-\frac{\theta_n}{12n}}$ with $0<\theta_n<1$. In order to prove this, we write the above double inequality as:

$$1\le \frac{x_n}{x_{n+1}} \le \frac{e^{\frac{1}{12n}}}{e^{\frac{1}{12(n+1)}}} .$$

We deduce that the sequence x_n is positive and decreasing, while the sequence $e^{-\frac{1}{12n}}x_n$ is increasing. Because $e^{-\frac{1}{12n}}$ converges to 1, and because $(x_n)_{n\in N}$ converges by the Weierstrass[2] criterion, both x_n and $e^{-\frac{1}{12n}}x_n$ must converge to the same limit L. We claim that $L=\sqrt{2\pi}$. Before proving this, note that:

$$e^{-\frac{1}{12n}}x_n < L < x_n ,$$

so by the intermediate value property there exists $\theta_n \in (0,1)$ such that $L=e^{-\frac{\theta_n}{12n}}x_n$, i.e. $x_n = e^{\frac{\theta_n}{12n}}L$.

The only thing left is the computation of the limit L. For this we employ the Wallis[3] formula:

$$\lim_{n\to\infty}\left[\frac{2\cdot 4\cdot 6\cdot ...\cdot 2n}{1\cdot 3\cdot 5\cdot ...\cdot (2n-1)}\right]^2\cdot\frac{1}{n}=\pi .$$

We rewrite this limit as:

$$\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n)!}\cdot\frac{1}{\sqrt{n}}=\sqrt{\pi} .$$

Substituting $n!$ and $(2n)!$ be the formula found above gives:

$$\lim_{n\to\infty}\frac{nL^2\left(\frac{n}{e}\right)^{2n}\cdot e^{\frac{2\theta_n}{12n}}\cdot 2^{2n}}{\sqrt{2n}L\left(\frac{2n}{e}\right)^{2n}\cdot e^{\frac{\theta_{2n}}{24n}}}\cdot\frac{1}{\sqrt{n}}=\lim_{n\to\infty}\frac{1}{\sqrt{2}}Le^{\frac{4\theta_n-\theta_{2n}}{24n}}=\sqrt{\pi} .$$

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[2] Karl Theodor Withelm Weierstrass (1815. - 1897.), german mathematician
[3] John Wallis (1616. - 1703.), english mathematician

Hence $L=\sqrt{2\pi}$, and Stirling's formula (1) is proved.

2. The application of the Stirling's formula

Example 1. Prove that the sequence $a_n=\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}$; $n\geq 1$, $n\in\boldsymbol{N}$ is convergent and find its limit.

Solution. It uses Stirling's formula (1):

$$n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\cdot e^{\frac{\theta_n}{12n}}\text{, with } 0<\theta_n<1.$$

Taking the n-th root and passing to the limit, we obtain:

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=e.$$

We also deduce that

$$\lim_{n\to\infty}\frac{n+1}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{n}{\sqrt[n]{n!}}=e.$$

Therefore

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n=\lim_{n\to\infty}\left(\sqrt[n(n+1)]{\frac{((n+1)!)^n}{(n!)^{n+1}}}\right)^n=$$

$$=\lim_{n\to\infty}\left(\sqrt[n(n+1)]{\frac{(n+1)^n}{(n!)}}\right)^n=\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{n+1}{\sqrt[n]{n!}}}\right)^n=\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n]{n!}}\right)^{\frac{n}{n+1}}=$$

$$=\left(\lim_{n\to\infty}\frac{n+1}{\sqrt[n]{n!}}\right)^{\lim\limits_{n\to\infty}\frac{n}{n+1}}=e.$$

Taking the n-th root and passing to the limit, we obtain:

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}=1,$$

and hence

$$\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}}=\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1=0\,.$$

Thus, if we set

$$b_n=\left(1+\frac{a_n}{\sqrt[n]{n!}}\right)^{\frac{\sqrt[n]{n!}}{a_n}},$$

then $\lim_{n\to\infty} b_n=e$. From the equality:

$$\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n=b_n^{a_n\cdot\frac{n}{\sqrt[n]{n!}}},$$

we obtain

$$a_n=\ln\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n\cdot(\ln b_n)^{-1}\cdot\left(\frac{n}{\sqrt[n]{n!}}\right)^{-1}.$$

The right-hand side is a product of three sequences that eonverge, respectively, to $1=\ln e, 1=\ln e$ and $\frac{1}{e}$. Therefore, the sequence $(a_n)_{n\in\mathbf{N}}$ converges to the limit $\frac{1}{e}$. q.e.d.

Example 2. Prove that

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n\sqrt[2n]{n}}{e}\right)=\frac{\ln(2\pi)}{2e}\,.$$

Solution. Using the double inequality with reregard to Stirling's formula proved of the H. Robbins in [5], we have:

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\cdot e^{\frac{1}{12n+1}}<n!<\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\cdot e^{\frac{1}{12n}};\ n\in\mathbf{N}$$

or

$$\sqrt[2n]{2\pi n}\cdot\frac{n}{e}\cdot e^{\frac{1}{12n^2+n}}<\sqrt[n]{n!}<\sqrt[2n]{2\pi n}\cdot\frac{n}{e}\cdot e^{\frac{1}{12n^2}}$$

$$\Leftrightarrow\frac{n\sqrt[2n]{n}}{e}\left(\sqrt[2n]{2\pi}\cdot e^{\frac{1}{12n^2+n}}-1\right)<\sqrt[n]{n!}-\frac{n\sqrt[2n]{n}}{e}<\frac{n\sqrt[2n]{n}}{e}\left(\sqrt[2n]{2\pi}\cdot e^{\frac{1}{12n^2}}-1\right)$$

$$\Leftrightarrow\frac{n\sqrt[2n]{n}}{e}\left(e^{\frac{\ln(2\pi)}{2n}+\frac{1}{12n^2+n}}-1\right)<\sqrt[n]{n!}-\frac{n\sqrt[2n]{n}}{e}<\frac{n\sqrt[2n]{n}}{e}\left(e^{\frac{\ln(2\pi)}{2n}+\frac{1}{12n^2}}-1\right).$$

Because

$$\lim_{n\to\infty}\frac{n\sqrt[2n]{n}}{e}\left(e^{\frac{\ln(2\pi)}{2n}+\frac{1}{12n^2+n}}-1\right)$$

$$=\lim_{n\to\infty}\frac{e^{\frac{\ln(2\pi)}{2n}+\frac{1}{12n^2+n}}-1}{\frac{\ln(2\pi)}{2n}+\frac{1}{12n^2+n}}\cdot\frac{n\sqrt[2n]{n}}{e}\left(\frac{\ln(2\pi)}{2n}+\frac{1}{12n^2+n}\right)$$
$$=\lim_{n\to\infty}\frac{n\sqrt[2n]{n}}{e}\left(\frac{\ln(2\pi)}{2n}+\frac{1}{12n^2+n}\right)$$

$$=\lim_{n\to\infty}\sqrt[2n]{n}\left(\frac{\ln(2\pi)}{2e}+\frac{1}{e(12n+1)}\right)=\frac{\ln(2\pi)}{2e}$$

and analogously

$$\lim_{n\to\infty}\frac{n\sqrt[2n]{n}}{e}\left(e^{\frac{\ln(2\pi)}{2e}+\frac{1}{12n^2}}-1\right)=\frac{\ln(2\pi)}{2e}.$$

Using now the known criterion for the convergence of a sequence and its limit, we get:

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n\sqrt[2n]{n}}{e}\right)=\frac{\ln(2\pi)}{2e}\text{. q.e.d.}$$

Example 3. Compute

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n}{e}-\frac{\ln n}{2e}\right).$$

Solution. We will use the result of the Example 2:

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n\sqrt[2n]{n}}{e}\right)=\frac{\ln(2\pi)}{2e}. \tag{2}$$

We have further by (2):

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n}{e}\right)=\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n\sqrt[2n]{n}}{e}\right)+\lim_{n\to\infty}\left(\frac{n\sqrt[2n]{n}}{e}-\frac{n}{e}\right)=$$

$$=\frac{ln(2\pi)}{2e}+\lim_{n\to\infty}\frac{n}{e}\left(e^{\frac{ln\,n}{2n}}-1\right)=\frac{ln(2\pi)}{2e}+\lim_{n\to\infty}\frac{e^{\frac{ln\,n}{2n}}-1}{\frac{ln\,n}{2n}}\cdot\frac{n}{e}\cdot\frac{ln\,n}{2n}=$$

$$=\frac{ln(2\pi)}{2e}+\lim_{n\to\infty}\frac{ln\,n}{2e}\left(\text{because }\lim_{n\to\infty}\frac{e^{\frac{ln\,n}{2n}}-1}{\frac{ln\,n}{2n}}=1\right)\text{, i.e.}$$

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n}{e}\right)=\frac{ln(2\pi)}{2e}+\lim_{n\to\infty}\frac{ln\,n}{2e}$$

and from here:

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n}{e}\right)-\lim_{n\to\infty}\frac{ln\,n}{2e}=\frac{ln(2\pi)}{2e}\text{, i.e.}$$

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}-\frac{n}{e}-\frac{ln\,n}{2e}\right)=\frac{ln(2\pi)}{2e}\text{. q.e.d.}$$

Remark. In the mathematical literary works the approximation:

$$n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{3}$$

has too the name the Stirling's formula. We have in [4], page 62 the example 81. of the inequality (with the proof):

$$\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n \quad (n\in\boldsymbol{N}) \quad . \tag{4}$$

The proof follows using the mathematical induction and the inequality $\left(1+\frac{1}{n}\right)^n<e$.

But, from (3) and (4) we get:

$$\left(\frac{n}{e}\right)^n<\sqrt{2\pi n}\left(\frac{n}{e}\right)^n<e\left(\frac{n}{2}\right)^n$$

$$\Leftrightarrow 1<\sqrt{2\pi n}<\frac{e^{n+1}}{2^n}\text{, i.e.}$$

$$1<2\pi n<\left(\frac{e^2}{4}\right)^n\cdot e^2,$$

what is exact because $2\pi\approx 6,28$; $e^2\approx 7,387524$ and $\frac{e^2}{4}\approx 1,846881$.

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