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O JEDNOJ ALGEBARSKOJ NEJEDNAKOSTI

(About one algebraic inequality)

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Sažetak. U radu su data tri dokaza jednog uopštenja nejednakosti iz [1]:

$$\frac{x}{kx+y+z}+\frac{y}{x+ky+z}+\frac{z}{x+y+kz} \le \frac{3}{k+2},$$

gdje su x, y, z, k pozitivni brojevi i $k > 1$.

Ključne riječi: pozitivni brojevi, nejednakost, aritmetičko - geometrijska nejednakost, uopštenje, konkavna funkcija, Jensenova nejednakost.

Abstract. In this paper three proofs are given for the generalization of an algebraic inequality in [1]:

$$\frac{x}{kx+y+z}+\frac{y}{x+ky+z}+\frac{z}{x+y+kz} \le \frac{3}{k+2},$$

where x, y, z, k be the positive numbers and $k > 1$.

Key words: positive numbers, inequality, arithmetic – geometric inequality, generalization, concave function, Jensen's inequality.

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U [1], na str. 204, nalazi se sljedeća nejednakost za pozitivne brojeve x, y, z:

$$\frac{x}{2x+y+z}+\frac{y}{x+2y+z}+\frac{z}{x+y+2z} \le \frac{3}{4}. \tag{1}$$

Ovdje dajemo tri dokaza njenog uopštenja (generalizacije):

$$\frac{x}{kx+y+z}+\frac{y}{x+ky+z}+\frac{z}{x+y+kz} \leq \frac{3}{k+2}, k>1. \tag{2}$$

Dokaz 1. Ako stavimo

$$kx+y+z=A,\ x+ky+z=B \text{ i } x+y+kz=C,$$

poslije sabiranja ovih jednakosti dobijamo $(k+2)(x+y+z)=A+B+C$,

tj.

$$x+y+z=\frac{1}{k+2}(A+B+C).$$

Sada je $x=\frac{1}{k^2+k-2}((k+1)A-B-C, y=\frac{1}{k^2+k-2}(-A+(k+1)B-C)$ i

$z=\frac{1}{k^2+k-2}(-A-B+(k+1)C)$.

Tada imamo

$$M=\frac{x}{kx+y+z}+\frac{y}{x+ky+z}+\frac{z}{x+y+kz}=\frac{1}{k^2+k-2}(3(k+1)-\frac{B+C}{A}-\frac{C+A}{B}-\frac{A+B}{C})$$
$$=\frac{1}{(k+2)(k-1)}\left(3k+3-\left(\frac{B}{A}+\frac{C}{B}+\frac{A}{C}\right)-\left(\frac{A}{B}+\frac{B}{C}+\frac{C}{A}\right)\right).$$

Na osnovu aritmetičko-geometrijske nejednakosti dobijamo

$$\frac{B}{A}+\frac{C}{B}+\frac{A}{C} \geq 3\sqrt[3]{\frac{B}{A}\cdot\frac{C}{B}\cdot\frac{A}{C}}=3 \text{ i } \frac{A}{B}+\frac{B}{C}+\frac{C}{A} \geq 3,$$

pa je

$$M \leq \frac{1}{(k+2)(k-1)}(3k+3-3-3)=\frac{3}{k+2}, \text{ za } k>1,$$

tj. (2) važi.

Napomena. Specijalno, iz nejednakosti (2), za $k=2$, slijedi (1).

Dokaz 2. Postavljena nejednakost (2) je ekvivalentna sa

$$\frac{kx+y+z-y-z}{kx+y+z}+\frac{x+ky+z-x-z}{x+ky+z}+\frac{x+y+kz-x-y}{x+y+kz} \leq \frac{3}{k+2},$$

tj. sa

$$\frac{y+z}{kx+y+z}+\frac{x+z}{x+ky+z}+\frac{x+y}{x+y+kz} \geq \frac{6}{k+2}, k>1. \tag{3}$$

U [2] autor je dokazao sljedeću nejednakost za pozitivne brojeve *A*, *B*, *C*, *X*, *Y*, *Z*:

$$\frac{A^2}{X}+\frac{B^2}{Y}+\frac{C^2}{Z} \geq \frac{(A+B+C)^2}{X+Y+Z}. \tag{4}$$

Ako u (4) stavimo

$X=(y+z)(kx+y+z), Y=(x+z)(x+ky+z), Z=(x+y)(x+y+kz)$,

$A=y+z, B=x+z$ i $C=x+y$, dobijamo

$$\frac{(y+z)^2}{(y+z)(kx+y+z)}+\frac{(x+z)^2}{(x+z)(x+ky+z)}+\frac{(x+y)^2}{(x+y)(x+y+kz)} \geq \frac{(2(x+y+z))^2}{(2k+2)(xy+yz+zx)+2(x^2+y^2+z^2)}$$

$$= \frac{2(x+y+z)^2}{(x+y+z)^2+(k-1)(xy+yz+zx)},$$

tj.

$$\frac{y+z}{kx+y+z}+\frac{x+z}{x+ky+z}+\frac{x+y}{x+y+kz} \geq \frac{2(x+y+z)^2}{(x+y+z)^2+(k-1)(xy+yz+zx)}. \tag{5}$$

S obzirom da je

$$xy+yz+zx \leq \frac{1}{3}(x+y+z)^2$$

ekvivalentno sa tačnom nejednakošću

$$(x-y)^2+(y-z)^2+(z-x)^2 \geq 0,$$

iz (5) proizlazi

$$\frac{y+z}{kx+y+z}+\frac{x+z}{x+ky+z}+\frac{x+y}{x+y+kz} \geq \frac{2(x+y+z)^2}{(x+y+z)^2+\frac{k-1}{3}(x+y+z)^2}$$

$$= \frac{6}{k+2}, \text{ tj. (3).}$$

Ovim je dokaz 2 okončan.

Dokaz 3. Neka je $x+y+z=m$. Zadana nejednakost (2) postaje

$$\frac{x}{(k-1)x+m}+\frac{y}{(k-1)y+m}+\frac{z}{(k-1)z+m} \leq \frac{3}{k+2}. \tag{6}$$

Posmatrajmo funkciju

$$f(a) = \frac{a}{(k-1)a+m}, (a > 0, k > 1).$$

Njen drugi izvod je

$$f''(a) = \frac{-2m(k-1)}{((k-1)a+m)^3} < 0, \text{ zbog } a > 0 \text{ i } k > 1.$$

Funkcija f je konkavna, pa primjenom Jensenove nejednakosti za konkavne funkcije dobijamo

$$\frac{f(x)+f(y)+f(z)}{3} \leq f(\frac{x+y+z}{3}),$$

tj.

$$\frac{x}{(k-1)x+m} + \frac{y}{(k-1)y+m} + \frac{z}{(k-1)z+m} = f(x) + f(y) + f(z) \leq 3f\left(\frac{x+y+z}{3}\right) = 3f(\frac{m}{3})$$

$$= \frac{3}{k+2},$$

što predstavlja nejednakost (6), a samim tim i traženu nejednakost (2).

LITERATURA

[1] Cvetkovski, Z.: *Inequalities – Theorems, Technique and Select Problems.* Springer – Verlag, Heidelberg/Dordrecht/New York, 2012.

[2] Milošević, D.: *Jedna nejednakost i njena primena*. Tangenta (Beograd), 55(2008/2009 – 3), 8 – 10.