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SOME SPECIFIC CLASSES OF RELATIONS, A REVIEW

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ABSTRACT. In this paper, the concepts of several new classes of relations on sets are presented introduced by this author in the previous five years. The following classes of relations have been introduced and partly described in several his articles: the class of quasi-regular, the class of quasi-conjugative, the class of quasi-normal and the class of normally conjugative relations.

1. Introduction

For a set X, we call α a binary relation on X if $\alpha \subseteq X^2$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x,z) \in X^2 : (\exists y \in X)((x,y) \in \alpha, (y,z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $\mathcal{B}(X)$, with composition, is a monoid (semigroup with identity). Namely, $\triangle_X = \{(x,x) : x \in X\}$ is its identity element. For a binary relation α on a set X, define $\alpha^{-1} = \{(x,y) \in X^2 : (y,x) \in \alpha\}$ and $\alpha^c = X^2 \setminus \alpha$. Thus $(\alpha^c)^{-1} = (\alpha^{-1})^c$ holds.

Let A be a subset of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a,y) \in \alpha)\}, \ \ \alpha A = \{x \in X : (\exists b \in A)((x,b) \in \alpha)\}.$$

It is easy to see that $A\alpha = \alpha^{-1}A$ holds. Specially, we put $a\alpha$ instead of $\{a\}\alpha$ and αb instead of $\alpha\{b\}$.

The fundamental works of K.A.Zareckii [**22**], B.M.Schein [**19**] and others on regular relations motivated several mathematicians to investigate similar classes of relations, obtained by putting α^{-1}, α^c or $(\alpha^c)^{-1}$ in place of one or both α's on the right side of the regularity equation

$$\alpha = \alpha \circ \beta \circ \alpha$$

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(where β is some relation). The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated:

DEFINITION 1.1. ([**3**]) The relation $\alpha \in \mathcal{B}(X)$ is called a *conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

DEFINITION 1.2. ([**3**]) The relation $\alpha \in \mathcal{B}(X)$ is called a *dually conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

In our texts [**13, 16**] we introduced ([**13**], Definition 2.1) and analyzed ([**13**], Theorem 2.3) the concepts of bi-conjugative relation and finitely bi-conjugative relations ([**16**], Definition 3.2 and Theorem 3.1).

DEFINITION 1.3. ([**5**]) The relation $\alpha \in \mathcal{B}(X)$ is called a *normal* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^c)^{-1} \circ \beta \circ \alpha.$$

DEFINITION 1.4. ([**4**]) The relation $\alpha \in \mathcal{B}(X)$ is called a *dually normal* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^c)^{-1} \circ \beta \circ \alpha.$$

In our texts [**12, 21**] we introduced ([**12**], Definition 4.2) and analyzed ([**12**], Theorem 4.4) the concepts of bi-normal relation and finitely bi-normal relations ([**21**], Definition 3.2 and Theorem 3.3).

Put $\alpha^1 = \alpha$. The previous definitions give equality

$$\alpha = (\alpha^a)^i \circ \beta \circ (\alpha^b)^j$$

for some $\beta \in \mathcal{B}(X)$ where $i, j \in \{-1, 1\}$ and $a, b \in \{1, c\}$.

In the past five years the author (independently or in cooperation with other authors) has been exploring all other possibilities. In this text we briefly expose the results of our researches [**7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 20, 21**]. The concepts of the various relations on sets that we have introduced in the reports on the aforementioned research are outlined.

Notions and notations which are not explicitly exposed but are used in this article, can be found in book [**6**] and articles [**1, 2, 3, 4, 5, 19, 22, 23**].

2. Concepts of new relations

2.1. Quasi-regular relation. In article [**9**] we introduced the class of *quasi-regular* relations and the class of *dually quasi-regular* relations.

DEFINITION 2.1. ([**9**], Definition 2.2) A relation $\alpha \in \mathcal{B}(X)$ is called a *quasi-regular* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^c \circ \beta \circ \alpha.$$

DEFINITION 2.2. ([**9**]: page 132) A relation $\alpha \in \mathcal{B}(X)$ is called a *dually quasi-regular* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^c.$$

The class of bi-quasiregular relations is described by the following definition:

Definition 2.3 ([**12**], Definition 4.1). A relation $\alpha \in \mathcal{B}(X)$ is called a *bi-quasiregular* relation if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^c \circ \beta \circ \alpha^c.$$

In addition, some basic characteristics of such constructed relationships have been proved. See, for example, the Theorems 21 in the text [**9**] and Theorem 4. in the article [**12**]. These theorems describe the necessary and sufficient conditions for a relation to be a quasi-regular relation (bi-quasiregular relation). In addition to these new classes of relations, the concept of the final extension of a dually quasi-regular relation was introduced in our article [**15**] (Definition 3.2). In the text [**20**] we introduced the concept of the final extension of bi-quasiregular relation. The necessary and sufficient conditions have been constructed to make a relation is a finitely extensional of a quasi-regular relation (Theorem 3.1) and bi-quasiregular relation (Theorem 2.1).

2.2. Quasi-conjugative relations. In the article [**8**] we introduced and analyzed the class of quasi-conjugative relations ([**8**]: Definition 2.2) and the class of dually quasi-conjugative relations ([**8**]: Remark 2.1). The properties of finite extensions of these relations were discussed in [**10**].

Definition 2.4. ([**8**], Definition 2.2) A relation $\alpha \in \mathcal{B}(X)$ is called a *quasi-conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^c.$$

Definition 2.5. ([**8**], Remark 2.1) A relation $\alpha \in \mathcal{B}(X)$ is called a *dually quasi-conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^c \circ \beta \circ \alpha^{-1}.$$

2.3. Quasi-normal relations.

Definition 2.6. ([**14**], Definition 2.1 (a)) A relation $\alpha \in \mathcal{B}(X)$ is called a *quasi-normal* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^c \circ \beta \circ (\alpha^c)^{-1}.$$

Definition 2.7. ([**14**], Definition 2.1 (b)) A relation $\alpha \in \mathcal{B}(X)$ is called a *dually quasi-regular* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^c)^{-1} \circ \beta \circ \alpha^c.$$

In [**17**], a finitely extension of a dual quasi normal relation was analyzed.

2.4. Normally conjugative relations.

Definition 2.8. ([**18**], Definition 2.2) A relation $\alpha \in \mathcal{B}(X)$ is called a *normally conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ (\alpha^{-1})^c.$$

DEFINITION 2.9. ([**18**], Definition 2.3) A relation $\alpha \in \mathcal{B}(X)$ is called a *dually normal conjugative* if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^{-1})^c \circ \beta \circ \alpha^{-1}.$$

In [**17**], a finitely extension of a normally conjugative relation was introduced (Definition 4.2) and analyzed (Theorem 41 and Theorem 4.2).

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